A Theory of Production, Matching, and Distribution

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Abstract

This paper develops a search-theoretic model of the labor market in which heterogeneous firms compete directly to hire unemployed workers. This process of direct competition simultaneously determines both the expected match output and workers' effective bargaining power. The framework delivers a *unified* aggregate production and matching technology, and firms are paid both *productivity rents* and *matching rents*. Both the curvature of the endogenous production technology and the distribution of output are influenced by properties of the underlying firm productivity distribution, particularly the *tail index* (a measure of tail fatness). For example, if the firm productivity distribution is Pareto, the labor share is decreasing in its tail index if the value of matching rents is not too high.

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1 Introduction

A standard approach in macroeconomics is to assume a specific aggregate production function and competitive factor markets. The distribution of output between workers and firms is determined by the production technology since factors are paid their marginal products. In search-theoretic models, by contrast, there is a frictional process of matching workers and firms and wage determination is explicitly modelled. In Diamond-Mortensen-Pissarides (DMP) models, for example, wages are determined by generalized Nash bargaining and the distribution of output is governed by workers' bargaining power – a parameter.[1]

This paper develops a framework that incorporates frictional unemployment and non-competitive wage determination – as in the search-theoretic approach – but features a tractable aggregate production and matching function. In this sense, the paper is similar in spirit to Lagos (2006), which derives an aggregate production function in a DMP style search-theoretic model. The key novelty of the present model – in contrast to models such as Lagos (2006) featuring Nash bargained wages – is that both the production *and* the distribution of output are determined by a process of direct competition between firms to hire workers. This unified approach to production, matching, and distribution enables us to examine how changes in the model's primitives simultaneously affect aggregate outcomes such as unemployment, output per match, factor income shares, and the *endogenous* effective bargaining power of workers.

In the model, workers are "sellers" of labor and firms are "buyers" that compete to hire workers. Similarly to the theory of competing auctions developed in Peters and Severinov (1997), workers hold second-price auctions with reservation wages ("reserve prices") and firms pay an entry cost to approach workers.[2] After approaching a worker, firms independently draw match-specific productivities ("valuations") from a common distribution. The highest productivity firm targeting a worker hires that worker and pays a wage equal to the second-highest productivity (or the reservation wage if no other firms compete).

This process of direct competition delivers three key outcomes: *production*, *matching*, and *distribution*. First, it endogenizes the average match output: greater competition between firms increases the expected match output because

[1] See Pissarides (2000) and Mortensen and Pissarides (1994).

[2] Peters and Severinov (1997) builds on Wolinsky (1988) and McAfee (1993).

workers can be more selective. Second, it endogenizes the unemployment rate: greater competition increases the matching probability for workers. Third, it endogenizes the distribution of output between workers and firms: greater competition between firms to hire workers increases labor's income share and the effective bargaining power of workers.

In the spirit of Houthakker (1955), Jones (2005), and Lagos (2006), the aggregate production function that arises in this environment inherits its properties from a primitive underlying distribution of firm productivities. In the present model, however, production and matching are interdependent processes and we therefore obtain not a standard aggregate production function but a *unified* aggregate production and matching function that incorporates the matching frictions. Moreover, since wages are determined through an auction mechanism, both the production technology *and* the distribution of output are influenced by the distribution of firm productivities. In the limit as unemployment goes to zero, the output elasticity with respect to vacancies – a measure of the curvature of the production technology – and both the labor share and workers' effective bargaining power – two measures of the distribution of output – depend only on the extreme value *tail index* of the distribution of firm productivities.

The link between production and distribution also ensures that a generalized version of the Hosios (1990) condition holds endogenously. Firms are paid both *matching rents,* which arise from the frictional matching process, and *productivity rents,* which arise from the heterogeneity in firm productivities. Since firms are paid their social marginal value – i.e. their contribution to both the number of matches and the expected output per match – the economy is always constrained efficient.[3] This contrasts with DMP style models with Nash bargaining where constrained efficiency arises only in a knife-edge case.

Although most of the paper's results hold for any well-behaved distribution of firm productivities, I consider the Pareto distribution as a lead example. In this case, the unified aggregate production and matching function is quite tractable and it delivers the Cobb-Douglas aggregate production function as a special limiting case.[4] The Pareto distribution yields tractable and empirically relevant

[3]Constrained efficiency is a common feature of directed or competitive search economies (Shimer, 1996; Moen, 1997). Albrecht, Gautier, and Vroman (2014) establishes constrained efficiency in a competing auctions environment similar to the present one.

[4]The connection between the Pareto distribution and the Cobb-Douglas aggregate production function has been known since Houthakker (1955), Jones (2005), and Lagos (2006).

expressions for factor income shares, and it allows us to determine the effects of changes in the firm productivity distribution on aggregate outcomes. For example, I show that the labor share is decreasing in the tail index of the Pareto distribution if the value of matching rents is not too high. That is, fatter tails of the firm productivity distribution imply a lower labor share.

Outline. Section 2 discusses related literature. Section 3 presents the model. Section 4 characterizes the equilibrium. Section 5 presents the theory's predictions for aggregate outcomes. Section 6 discusses efficiency. Section 7 presents the lead example and Section 8 concludes. All proofs are in the Appendix.

2 Related Literature

This paper is closely related to Lagos (2006), which derives a tractable aggregate production function – the Cobb-Douglas and more general C.E.S. function – in a DMP style environment with labor market frictions. While similar in spirit, there are some key differences between Lagos (2006) and this paper. First, Lagos' main focus is on how total factor productivity (TFP) is endogenously determined by labor market conditions. Second, the fact that wages are determined by auctions here instead of generalized Nash bargaining means that production and distribution are intimately linked since both are influenced by the underlying distribution of firm productivities. Third, the link between production and distribution ensures that a generalized version of the Hosios condition holds endogenously and the economy is always constrained efficient. Finally, this paper considers a static environment, while Lagos' model is dynamic.

This paper is also related to Jones (2005), which derives a global production function that is asymptotically Cobb-Douglas in the long run as the total number of ideas becomes large; and Houthakker (1955), which derives a Cobb-Douglas aggregate production function using a Leontief local production technology and a Pareto distribution of capacities.[5] There are two main differences between

[5] Levhari (1968) shows that Houthakker's result can be generalized to a C.E.S. aggregate production function with $\sigma < 1$; and Growiec (2013) generalizes some of Jones' aggregation results by providing ideas-based microfoundations for normalized C.E.S. production functions (see Klump, McAdam, and Willman (2012)). Alternative approaches are found in Dupuy (2012), which uses a tasks assignment model similar to Rosen (1978) to derive a C.E.S. aggregate production function, and Eden (2017), which examines how the curvature of the aggregate production technology depends on the degree of misallocation at the micro level.

my approach and these papers. First, labor market frictions and unemployment are central to this paper. Second, I develop joint microfoundations for both production *and* distribution by modelling wage determination explicitly.[6]

This paper contributes to the competing auctions literature, including Peters and Severinov (1997) and later work by Albrecht et al. (2014); Albrecht, Gautier, and Vroman (2016); Kim and Kircher (2015); and Lester, Visschers, and Wolthoff (2015). This paper is also related to the wider literature on directed and competitive search surveyed in Guerrieri, Julien, Kircher, and Wright (2016).[7] In particular, Julien et al. (2000) models workers as sellers who "auction" their labor in a directed search model similar to Burdett et al. (2001). The large economy version of Julien et al. (2000) is a special case of the present model where there is no heterogeneity. Also closely related is Shi (2001), which presents a directed search model with two-sided ex ante heterogeneity. Shi (2001) shows that the labor share for each machine type is increasing in its market tightness.

This paper is complementary to Shi (2002), Shi (2006), and Shimer (2005). Shi (2002) develops a directed search model with two-sided heterogeneity: there are two types of firms and two types of workers. Shi (2006) considers a directed search environment with many types of workers and homogeneous firms. Shimer (2005) features two-sided heterogeneity with many types of workers and firms. Workers apply to firms who choose the most productive applicant, and match output is a function of both worker and firm type. A key difference between this paper and Shi (2002), Shi (2006), and Shimer (2005) is the use of a continuous distribution of match-specific productivities instead of a finite number of worker/firm types.[8]

Finally, this paper complements others that apply extreme value theory to economics.[9] In particular, the asymptotic results regarding factor shares and the extreme value tail index are closely related to results found in Gabaix, Laibson, Li, Li, Resnick, and de Vries (2016) regarding the asymptotic value of markups.

[6]In addition, both Jones (2005) and Growiec (2013) derive the aggregate production function by taking the convex hull of the set of available technologies, not by aggregating across heterogeneous firms. See also Growiec (2008).

[7]Early contributions include Montgomery (1991), Peters (1991), Shimer (1996), Moen (1997), Acemoglu and Shimer (1999), Julien, Kennes, and King (2000), Burdett, Shi, and Wright (2001), and Shi (2001).

[8]Menzio and Shi (2011) also uses a continuous distribution of match-specific productivities in a directed search model with bilateral meetings. The authors show that match-specific productivity is quantitatively important in accounting for the volatility of labor market variables.

[9]For example, Kortum (1997); Eaton and Kortum (1999, 2002); Bernard, Eaton, Jensen, and Kortum (2003); Gabaix and Landier (2008); Growiec (2013); and Oberfield (2013).

3 Model

The model is static. There are two kinds of risk-neutral agents: workers and firms. There is a continuum of ex ante identical workers of measure L and a continuum of ex ante identical firms. All workers are initially unemployed: $L = U$. The measure of firms who decide to enter (or "vacancies") is V and the ratio of such firms to workers is $\theta \equiv V/U$, the *labor market tightness.*

The timing is as follows. First, workers choose reservation wages. Second, firms make an entry decision and pay the entry cost. Third, firms approach workers. Fourth, firms draw match-specific productivities. Fifth, the second-price auctions take place. Finally, firms earn profits and wages are paid.

Workers are "sellers" of a single unit of labor and firms are "buyers". Workers post second-price auctions and choose a reservation wage ("reserve price"), taking into account the effect on firms' entry decisions.[10] While workers may in principle choose different reservation wages, we focus on symmetric equilibria where all workers choose the *same* reservation wage w_R for a given market tightness θ.

Firms observe the reservation wage w_R and, if they choose to enter, pay an entry cost C to search for a worker. After paying the cost C, each firm can approach a single worker. Since firms are uncoordinated and workers are ex-ante identical, I assume that firms approach workers independently and at random. Meetings may involve either one firm and one worker (one-on-one or bilateral), or many firms and one worker (many-on-one or multilateral). The number of firms approaching any given worker is a Poisson random variable with parameter θ.[11]

Firms learn their productivities *ex post.* After approaching a single worker, each firm independently draws a match-specific productivity x from a common distribution with cdf G. A firm with productivity x can produce x units of output, with price normalized to one, using a single unit of labor.[12]

[10] Second-price auctions are an *optimal mechanism* if each worker ("seller") could choose a mechanism that maximized their expected "revenue". McAfee (1993) proved that in any setting with independent private valuations and competing sellers, it is an equilibrium outcome for sellers to hold identical second-price auctions with reserve prices equal to their own valuations. Note that the assumption of risk-neutrality is crucial for the optimality of auctions.

[11] Since firms approach each worker with equal probability, the matching process is *urn-ball.* The Poisson distribution arises endogenously because we are taking the limit of a binomial distribution – a standard result.

[12] In the example presented in Section 7, we interpret firms as owning a single unit of capital. For now, we leave this interpretation open since capital is not essential to the main results.

A firm's productivity draw represents its private "valuation" of a unit of labor since a firm with productivity x is willing to pay up to x to purchase a unit of labor. We assume that x is *private information.*[13] Since it is a weakly dominant strategy for buyers to bid their true valuations in second-price auctions, we assume that firms do so.

If no firms approach a given worker, he is unemployed. Unemployed workers receive the *value of non-market activity*, z, which represents both the value of leisure and the value of unemployment insurance benefits. By the Poisson distribution, unemployment occurs with probability $e^{-\theta}$, so $u(\theta) = e^{-\theta}$ is the *unemployment rate*. The matching probability for workers is therefore $m(\theta) = 1 - e^{-\theta}$ and the probability a firm successfully hires a worker is $q(\theta) \equiv m(\theta)/\theta$.[14]

In a one-on-one meeting, exactly one firm approaches a worker. The firm employs the worker and produces output equal to its productivity x. The worker is paid his reservation wage w_R in this case. In a many-on-one meeting, two or more firms approach a worker. The firm with the highest productivity employs the worker and produces output equal to its productivity. The worker's wage equals the second-highest productivity among the firms competing for that worker. Firms that are unsuccessful in hiring receive a payoff of zero.

Throughout the paper, we make the following assumptions.[15]

Assumption 1. *The distribution G is twice differentiable with density $g = G' > 0$, a finite mean, no mass points, and support $[x_0, \infty)$ where $x_0 \geq 0$.*

For simplicity, we assume that $z \leq x_0$ so that workers will always accept job offers. To ensure positive firm entry, we assume that $E_G(x)$ is not too low.

Assumption 2. *The distribution G satisfies $E_G(x) > z + C$ and $0 \leq z \leq x_0$.*

For some results, we assume that G is a *well-behaved* distribution.

[13] Note that the same outcomes would arise if there was Bertrand competition between firms and complete information. We choose to assume that firms' productivities are private information in order to align the model with the competing auctions literature.

[14] The Poisson meeting technology is *invariant,* a crucial property identified by Lester et al. (2015). Invariance implies *non-rivalry,* as defined in Eeckhout and Kircher (2010), and *joint concavity*, as defined in Cai, Gautier, and Wolthoff (2017).

[15] Assumption 1 could be relaxed to include distributions with bounded upper support, i.e. support $[x_0, x_{\max}]$ where $x_{\max} \in \mathbb{R}^+$. Definitions and other expressions that follow would need to be adjusted accordingly and property *(iv)* of Proposition 1 would no longer hold.

Definition 1. *For $x \in [x_0, \infty)$, the* generalized hazard rate $\varepsilon_G(x)$ *is defined by*

$$\varepsilon_G(x) \equiv \frac{xg(x)}{1 - G(x)}. \tag{1}$$

Definition 2. *A distribution G is* well-behaved *if and only if $\varepsilon_G(x)$ is weakly increasing, i.e. $\varepsilon'_G(x) \geq 0$ for all $x \in [x_0, \infty)$.*

The requirement that the generalized hazard rate is weakly increasing is a very mild condition that is satisfied by almost all standard distributions.[16,17] It is strictly weaker than both the increasing hazard rate condition and log-concavity.

4 Equilibrium

The expected value of a filled vacancy is given by $J = p_G(\theta) - w_G(\theta; w_R)$ where $p_G(\theta)$ is the expected match output and $w_G(\theta; w_R)$ is the expected wage when the reservation wage is w_R. For any given reservation wage $w_R \in \mathbb{R}^+$, the market tightness $\theta^*(w_R)$ must satisfy

$$q(\theta)(p_G(\theta) - w_G(\theta; w_R)) \leq C \tag{2}$$

and $\theta^*(w_R) \geq 0$, with complementary slackness.

Workers' reservation wage w_R^* maximizes their expected payoff, taking into account the effect on firm entry. That is,

$$w_R^* = \arg \max_{w_R \in [0,\infty)} \; m(\theta^*(w_R))w_G(\theta^*(w_R); w_R) + (1 - m(\theta^*(w_R)))z. \tag{3}$$

To eliminate any indeterminacy, we impose the following restriction on beliefs off the equilibrium path and restrict our attention to equilibria that satisfy this restriction. For all reservation wages w_R such that $J \geq C$, the market tightness $\theta^*(w_R)$ satisfies $q(\theta)J = C$, and for all w_R such that $J < C$ we have $\theta^*(w_R) = 0$.

[16] The elasticity $\varepsilon_G(x)$ is also sometimes called the *log hazard rate*.

[17] Banciu and Mirchandani (2013) provides a list of distributions that feature a weakly increasing generalized hazard rate. Examples include the Uniform, Exponential, Normal, Logistic, Laplace, Gumbel, Weibull, Gamma, Beta, Pareto, Chi, Lognormal, Cauchy, and F distributions. Additional conditions on the parameters are required to ensure that Assumption 1 is satisfied.

Definition 3. *An* equilibrium *is a reservation wage* $w_R^* \in \mathbb{R}^+$ *and a market tightness function* $\theta^* : \mathbb{R}^+ \to \mathbb{R}^+$ *such that, given the function* $\theta^*(.)$, w_R^* *satisfies (3), and for all* $w_R \in \mathbb{R}^+$ *the function* $\theta^*(.)$ *satisfies (2).*

In the Appendix, we show that for any distribution G, there exists a unique equilibrium with $w_R^* = z$ and $\theta^* > 0$ where $\theta^* \equiv \theta^*(w_R^*)$.[18]

5 Aggregate implications

We now turn to the model's predictions and describe how it offers a unified approach to thinking about production, matching, and distribution. This approach enables us to examine how changes in the model's primitives simultaneously affect aggregate outcomes such as unemployment, output, and the labor share, as well as the effective bargaining power of workers.

5.1 Production and matching

The process by which firms approach workers at random, draw match-specific productivities, and compete to hire workers, determines an *endogenous* distribution of output across potential workers, which depends on both the market tightness θ and the underlying distribution of firm productivities. Both production and matching outcomes are captured by this distribution.

5.1.1 Endogenous distribution of output across workers

Both the employment status of a given worker and his expected output are determined by the number of firms competing to hire that worker. To see this, suppose that n firms approach a given worker. The matching and production outcomes are as follows. *Matching:* if $n = 0$, the worker is unemployed; if $n \geq 1$, the worker is employed. *Production:* if $n = 0$, the worker produces zero output; if $n \geq 1$, the firm with the highest productivity hires the worker and match output is equal to the maximum of n draws from the distribution G.

[18] The fact that $w_R^* = z$ is consistent with the result of McAfee (1993) discussed in Section 3 since a worker's own "valuation" of his labor equals the value of non-market activity z. A closely related result is derived in Albrecht, Gautier, and Vroman (2012), which amends an earlier result in Peters and Severinov (1997).

We denote the cdf of the endogenous distribution of output across workers by $H_G(x;\theta)$. To derive this distribution, let $H_G(x|n) = G(x)^n$, the cdf of the worker's output conditional on n firms approaching. To obtain the cdf $H_G(x;\theta)$, the conditional cdf is weighted by the Poisson probability that n firms approach:

$$H_G(x;\theta) = \sum_{n=0}^{\infty} \frac{\theta^n e^{-\theta}}{n!} G(x)^n = e^{-\theta(1-G(x))}. \tag{4}$$

The distribution $H_G(x;\theta)$ of output across all workers (including the unemployed) has continuous support $[x_0, \infty)$ plus a mass point at zero corresponding to unemployment. We can also define the endogenous distribution of output across *employed* workers by $H_G^e(x;\theta) \equiv (H_G(x;\theta) - e^{-\theta})/m(\theta)$.

We define output per capita $f_G(\theta)$ as the expected value of the distribution of output across *all* workers, i.e. $f_G(\theta) \equiv \int_{x_0}^{\infty} x \; dH_G(x;\theta)$. We can break down $f_G(.)$ into a "production technology" and a "matching technology" by writing $f_G(\theta) = m(\theta)p_G(\theta)$, where $p_G(\theta) \equiv \int_{x_0}^{\infty} x \; dH_G^e(x;\theta)$, the expected match output.

$$f_G(\theta) = \underbrace{(1 - e^{-\theta})}_{\text{matching technology } m(\theta)} \underbrace{\left(\int_{x_0}^{\infty} x \; dH_G^e(x;\theta)\right)}_{\text{production technology } p_G(\theta)} \tag{5}$$

The matching technology is a standard urn-ball function, $m(\theta) = 1 - e^{-\theta}$. However, the nature of the "production technology" $p_G(.)$, which determines the expected match output, is governed by the underlying distribution G of firm productivities: the parameters of the technology $p_G(.)$ will be inherited from that distribution. Since $f_G(.)$ is the natural outcome of the production and matching processes and it incorporates both, we focus on the properties of this function.

5.1.2 Unified aggregate production and matching function

Aggregate output is given by $Y = F_G(V, U)$ where F_G is a constant-returns-to-scale function defined by $F_G(V, U) \equiv f_G(\theta)U$. Similarly to a standard matching function $M(V, U)$ – but different from a standard aggregate production function – the factor inputs of F_G are the total measure of vacancies V and unemployed workers U, not employed labor or utilized capital. Similarly to a standard aggregate production function – but different from a standard matching function –

F_G represents the total *output* produced, not the total number of matches. Total output depends on both the number of matches and the expected match output.

We refer to F_G as an aggregate *production and matching* function since $f_G(\theta) = m(\theta)p_G(\theta)$ and it incorporates both a "production technology" $p_G(.)$ and a "matching technology" $m(.)$. Crucially, the functions $p_G(.)$ and $m(.)$ do not arise independently in this setting: instead, $f_G(.)$ is the natural outcome. We therefore refer to F_G as a *unified* aggregate production and matching function.

In the special case where G is degenerate and there is no firm heterogeneity, match output is constant and we have $f_G(\theta) = m(\theta)$ if $x_0 = 1$. The function $f_G(.)$ can therefore be seen as a "generalization" of $m(.)$ to an environment where the expected match output is endogenous and depends on the market tightness.

The unified nature of the aggregate production function affects the way in which marginal products should be interpreted. Since the relevant factor inputs are vacancies and unemployed workers, $f'_G(\theta)$ represents the marginal contribution to aggregate output of an extra vacancy, which may end up either filled or unfilled depending on whether or not the firm is successful in hiring. The marginal product of "labor", $f_G(\theta) - \theta f'_G(\theta)$, represents the effect on aggregate output of an extra *potential* worker, who may end up either employed or unemployed.

5.1.3 Properties of the unified production and matching function

Before we present Proposition 1, we first define the *tail index* – a measure of the degree of fatness of the tails of the distribution G (not a measure of dispersion). A higher tail index λ_G implies fatter tails. Note that if G is well-behaved by Definition 2, then it has tail index $\lambda_G \in [0, 1)$.[19]

Definition 4. *A distribution G with support $[x_0, \infty)$ has extreme value* tail index λ_G *if and only if*

$$\lim_{x \to \infty} \frac{d}{dx} \left(\varepsilon_G(x)^{-1} \, x \right) = \lambda_G \text{ for some } \lambda_G \in \mathbb{R}. \tag{6}$$

The first part of Proposition 1 states that for any distribution G that satisfies Assumption 1, the function $f_G(.)$ has some desirable features: it is twice differentiable, increasing, strictly concave, and it satisfies all the standard Inada

[19] For details, see the proof of Proposition 1 in the Appendix.

conditions except that $\lim_{\theta\to 0} f'_G(\theta)$ is finite.[20]

The second part of Proposition 1 concerns the *elasticity of substitution* $\sigma_G(\theta)$ between V and U. Intuitively, the elasticity of substitution reflects the relative ease of substitution between vacancies and unemployed workers. It is defined by

$$\sigma_G(\theta) \equiv -\left(\frac{\partial \log(F_V/F_U)}{\partial \log(V/U)}\right)^{-1} \tag{7}$$

where $F_V \equiv \partial F_G/\partial V$ and $F_U \equiv \partial F_G/\partial U$.

Proposition 1. *The function* $f_G : \mathbb{R}^+ \to \mathbb{R}^+$ *is given by*

$$f_G(\theta) = \int_{x_0}^{\infty} \theta e^{-\theta(1-G(x))} x g(x) dx. \tag{8}$$

For any distribution G, *(i)* $f'_G(\theta) > 0$; *(ii)* $f''_G(\theta) < 0$; *(iii)* $f_G(0) = 0$; *(iv)* $\lim_{\theta\to\infty} f_G(\theta) = +\infty$; *(v)* $\lim_{\theta\to\infty} f'_G(\theta) = 0$; *and (vi)* $\lim_{\theta\to 0} f'_G(\theta) = E_G(x) \geq x_0$. *If* G *is well-behaved, the elasticity of substitution* $\sigma_G(\theta)$ *is less than or equal to one, or, equivalently, the output elasticity* $\eta_f(\theta)$ *is decreasing in* θ. *In the limit as* $\theta \to \infty$, *we have* $\sigma_G(\theta) \to 1$ *and* $\eta_f(\theta) \to \lambda_G \in [0, 1)$, *the tail index of* G.

Since $f_G(.)$ can be interpreted as a generalization of $m(.)$, these results generalize standard properties often assumed to hold for the matching technology. For example, the result that the elasticity of substitution $\sigma_G(\theta)$ is always less than one is equivalent to the result that the *output elasticity* with respect to vacancies, defined by $\eta_f(\theta) \equiv f'_G(\theta)\theta/f_G(\theta)$, is decreasing in θ. This result nests as a special case the fact that the matching elasticity $\eta_m(\theta) \equiv m'(\theta)\theta/m(\theta)$ is decreasing in θ, a common assumption in search models.

The output elasticity $\eta_f(\theta)$ is a measure of *curvature* of the function $f_G(.)$. In the limit as $\theta \to \infty$, i.e. as unemployment goes to zero, the elasticity of substitution $\sigma_G(\theta)$ converges to one and the output elasticity $\eta_f(\theta)$ converges to λ_G.[21] In this way, the tail index λ_G – which measures the fatness of the tails

[20] The Inada condition $\lim_{\theta\to 0} f'(\theta) = \infty$ is a sufficient but not a necessary condition for the existence of a steady state equilibrium in most applications in macroeconomics. Generally, what is strictly necessary is that $\lim_{\theta\to 0} f'(\theta)$ is sufficiently large.

[21] This does *not* imply that the aggregate production function F_G converges asymptotically to a Cobb-Douglas function, but rather that the *value* of the variable elasticity is one when evaluated in the limit as $\theta \to \infty$. In Section 7, however, we will see that in the special case where G is Pareto, the aggregate production function is indeed asymptotically Cobb-Douglas.

of the distribution of firm productivities – is a key parameter that governs the curvature of the unified production and matching function $f_G(.)$.

5.2 Distribution

Since wages are determined by auctions, it is not just the endogenous production technology that is governed by properties of the distribution of firm productivities. The distribution of output between workers and firms is also influenced by properties of this distribution. In particular, we will see that the aggregate labor share and workers' effective bargaining power – two measures of how output is distributed – are also driven by the *tail index* of this distribution.

5.2.1 Productivity and matching Rents

To understand firm profits and wages in this model, we can decompose the value of a filled vacancy, $J = p_G(\theta) - w_G(\theta; z)$, into two components:

$$J = \underbrace{\pi_G(\theta)}_{\text{productivity rents}} + \underbrace{\mu(\theta)(x_0 - z)}_{\text{matching rents}} \tag{9}$$

where $\mu(\theta) \equiv \theta e^{-\theta}/m(\theta)$ is the *proportion of bilateral meetings* and $\pi_G(\theta)$ is the expected value of *productivity rents*, defined by

$$\pi_G(\theta) \equiv \int_{x_0}^{\infty} \varepsilon_G(x)^{-1} \; x \; dH_G^e(x; \theta), \tag{10}$$

where $H_G^e(x; \theta)$ is the endogenous distribution of match output across employed workers and $\varepsilon_G(x)$ is the generalized hazard rate given by (1).

Productivity rents arise because firms are heterogeneous. In many-on-one meetings, successful firms receive the difference between the highest and second highest productivity among the competing firms. The expected productivity rents available to firms are reflected in the first term on the right of (9).[22] This term disappears when there is no heterogeneity in firm productivities.

Matching rents arise because there may be a positive match surplus even for the *least* productive firm in the economy. Even if a firm has productivity close to the minimum value x_0, it is still possible to earn a profit if they are lucky

[22] Note that $\pi_G(\theta)$ already incorporates the probability that a meeting is many-on-one.

enough to be matched with a worker in a one-on-one meeting. The matching rents available to firms are reflected in the second term on the right of (9). The value of matching rents is $x_0 - z$ and the proportion of bilateral meetings is $\mu(\theta)$ where $\mu'(\theta) < 0$ and $\mu(\theta) \to 0$ as $\theta \to \infty$. The *value* of matching rents disappears if $z = x_0$, while the *probability* of obtaining these rents disappears as $\theta \to \infty$.

Two special cases. This general framework nests two important special cases. In competing auctions models such as Peters and Severinov (1997) and Albrecht et al. (2014), there are no matching rents. This is because sellers' valuations are assumed to be greater than or equal to the minimum buyers' valuation, i.e. there is no "gap" between z and x_0.[23] The value of matching rents, $x_0 - z$, therefore disappears and the expected payoff for a successful buyer is simply $J = \pi_G(\theta)$. In directed search models such as Julien et al. (2000), where all firms have the same productivity, there are no productivity rents. The value of a filled vacancy is $J = \mu(\theta)(x_0 - z)$ and a positive "gap", $x_0 - z$, is necessary to ensure firm entry.

5.2.2 Wages, labor share, and workers' effective bargaining power

Expected wages $w_G(\theta)$ is determined by the market tightness θ, the value of non-market activity z, and the distribution G.[24] Using (9) and the fact that $J = p_G(\theta) - w_G(\theta)$, the expected wage for employed workers is given by

$$w_G(\theta) = \underbrace{p_G(\theta)}_{\text{expected match output}} - \underbrace{\pi_G(\theta)}_{\text{productivity rents}} - \underbrace{\mu(\theta)(x_0 - z)}_{\text{matching rents}}. \tag{11}$$

The *aggregate labor share* $s_L(\theta; G)$ represents workers' share of the expected match output $p_G(\theta)$ and it is defined by $s_L(\theta; G) \equiv w_G(\theta)/p_G(\theta)$, so we have

$$s_L(\theta; G) = 1 - \underbrace{\frac{\pi_G(\theta)}{p_G(\theta)}}_{\text{productivity rents}} - \underbrace{\frac{\mu(\theta)(x_0 - z)}{p_G(\theta)}}_{\text{matching rents}}. \tag{12}$$

Here, productivity rents and matching rents are expressed as a share of the expected match output. In the special case where $x_0 = z$, there are no matching

[23] In Peters and Severinov (1997), sellers' valuation is zero and buyers' valuations are drawn from a distribution with support [0,1]. In Albrecht et al. (2014), both sellers' and buyers' valuations are drawn from distributions with support [0,1].

[24] For simplicity, we now suppress the dependence of $w_G(\theta; z)$ on z and simply write $w_G(\theta)$.

rents for firms and $s_L(\theta; G) = 1 - \pi_G(\theta)/p_G(\theta)$. In the special case where all firms have the same productivity, i.e. $\bar{p} = x_0$, there are no productivity rents and $s_L(\theta; G) = 1 - \mu(\theta)(x_0 - z)/x_0$.

The *effective bargaining power* of workers $\beta_G(\theta)$ is defined as the endogenous value of workers' bargaining power β that would yield the same average wages as the current model if wages were determined by generalized Nash bargaining instead of auctions. Since expected wages under Nash bargaining is $w_G^N(\theta) = \beta(p_G(\theta) - z) + z$, equating $w_G^N(\theta)$ and (11) yields

$$\beta_G(\theta) = 1 - \underbrace{\frac{\pi_G(\theta)}{p_G(\theta) - z}}_{\text{productivity rents}} - \underbrace{\frac{\mu(\theta)(x_0 - z)}{p_G(\theta) - z}}_{\text{matching rents}}. \tag{13}$$

When $z = 0$, we have $\beta_G(\theta) = s_L(\theta; G)$. In the special case where $x_0 = z$, there are no matching rents for firms and $1 - \beta_G(\theta) = \pi_G(\theta)/(p_G(\theta) - z)$. In the special case where all firms have the same productivity, i.e. $\bar{p} = x_0$, there are no productivity rents and $1 - \beta_G(\theta) = \mu(\theta)$.

Proposition 2. *If G is well-behaved, expected productivity rents as a share of output, $\pi_G(\theta)/p_G(\theta)$, is decreasing in θ, and both labor's share $s_L(\theta; G)$ and workers' effective bargaining power $\beta_G(\theta)$ are increasing in θ. In the limit as $\theta \to \infty$, we have $\pi_G(\theta)/p_G(\theta) \to \lambda_G$, $s_L(\theta; G) \to 1 - \lambda_G$, and $\beta_G(\theta) \to 1 - \lambda_G$.*

Greater competition to hire workers reduces the expected productivity rents available to firms, as a share of match output, by reducing the profit share in multilateral meetings. In the limit as $C \to 0$ and $\theta^* \to \infty$, the expected matching rents go to zero (since $\mu(\theta) \to 0$) but the value of productivity rents as a share of output goes to λ_G, the tail index of G. Intuitively, the tail index is important because successful firms in multilateral meetings receive the difference between the highest and second-highest productivity, $x^1 - x^2$, and fatter tails increase the expected value of the highest productivity x^1 by relatively more than the second-highest productivity x^2.

Greater competition to hire workers increases both labor's income share and workers' effective bargaining power because $\pi_G(\theta)/p_G(\theta)$ is decreasing in θ, and also $\mu(\theta)/p_G(\theta)$ is decreasing in θ. However, even in the limit as $C \to 0$ and $\theta^* \to \infty$ neither labor's share nor workers' effective bargaining power approaches one unless the tail index $\lambda_G = 0$. In the limit as $\theta \to \infty$, both the labor share and workers' effective bargaining power converge to $1 - \lambda_G$.[25]

In this way, both the production *and* the distribution of output are influenced by the nature of the underlying firm productivity distribution. In particular, the tail index λ_G is a key determinant of both the curvature of the production and matching technology $f_G(.)$ and the distribution of output. This link between production and distribution arises because both are determined simultaneously through the process of direct competition between firms.

5.3 Comparative statics

Proposition 3 presents some comparative statics results.

Proposition 3. *If G is well-behaved, (i) the market tightness θ^* is decreasing in both the value of non-market activity z and the cost of entry C; (ii) unemployment u^* is increasing in both z and C; (iii) output per match p^* is decreasing in both z and C; (iv) output per capita y^* is decreasing in both z and C; (v) labor share s_L^* is decreasing in C and increasing in z; and (vi) workers' effective bargaining power β^* is decreasing in both C and z.*

If the cost of entry C increases, there is less firm entry and therefore a lower equilibrium market tightness θ^*. As a result, equilibrium unemployment u^* is higher since $u'(\theta) < 0$ and equilibrium output per capita y^* is lower since $f_G'(\theta) > 0$ by Proposition 1. At the same time, the equilibrium labor share s_L^* and workers' bargaining power β^* decrease by Proposition 2. Equilibrium output per match p^* is also lower since $p_G'(\theta) > 0$.[26] Intuitively, this is because greater competition to hire workers allows workers to be more "selective". Since workers are hired by the most productive firm who approaches them, a greater number of competing firms increases the expected value of the *highest* productivity among those firms.

If the value of non-market activity z increases, workers' reservation wage increases and entry is less attractive for firms, decreasing the market tightness θ^*. As a result, unemployment u^* is higher and both y^* and p^* are lower. There are two effects on the labor share s_L^*. The direct effect of an increase in z is

[25] It is possible to reconcile this result with Theorem 2 in Gabaix et al. (2016) regarding the asymptotic value of markups when the number of competing firms is large.

[26] See Lemma 2 in the Appendix for a proof that $p_G'(\theta) > 0$ if G is well-behaved.

that the labor share increases because firms must pay higher wages in bilateral meetings, while the indirect effect is that θ^* decreases, which has a negative effect on labor's share by Proposition 2. The direct effect always dominates: labor's share is *increasing* in z. Both the direct and indirect effects on workers' effective bargaining power are negative and thus β^* is *decreasing* in z.

6 Efficiency

The social planner's objective is to choose the market tightness θ^P that maximizes total output plus the total value of non-market activity minus the total costs of entry. Dividing by U, this is equivalent to maximizing

$$\Lambda(\theta) = f_G(\theta) + (1 - m(\theta))z - C\theta. \tag{14}$$

The social planners' choice is "constrained" in the sense that it is subject to the constraints of both the matching frictions *and* the production technology.

In the Appendix, we prove that for any distribution G, the economy is constrained efficient. While constrained efficiency is a common feature of directed and competitive search economies, this result extends those regarding the constrained efficiency of both competing auctions environments such as Albrecht et al. (2014), where only productivity rents exist, and directed search models such as Julien et al. (2000), where only matching rents exist.

Generalized Hosios Condition. One consequence of the fact that the expected match output $p_G(\theta)$ is endogenous here is that the standard Hosios condition does not apply. A generalized version of the Hosios (1990) condition is necessary to achieve constrained efficiency in this environment.[27] Since this condition holds endogenously here, the economy is *always* constrained efficient.

Since $f_G(\theta) = m(\theta)p_G(\theta)$, we have $\eta_f(\theta) = \eta_m(\theta) + \eta_p(\theta)$, where $\eta_f(\theta)$ is the elasticity of $f_G(.)$ with respect to θ, $\eta_m(\theta)$ is the elasticity of $m(.)$ with respect to θ, and $\eta_p(\theta)$ is the elasticity of $p_G(.)$ with respect to θ. Rearranging the first-order

[27] For a detailed discussion of efficiency in such environments, see Julien and Mangin (2016).

condition for (14), constrained efficiency holds only if the equilibrium θ^* satisfies

$$\eta_f(\theta) = \frac{C\theta}{f_G(\theta)} + \frac{\eta_m(\theta) z}{p_G(\theta)}. \tag{15}$$

In the present model, condition (15) holds endogenously for any z and any distribution G. The economy is always constrained efficient (i.e. it is not a knife-edge condition). Notice that condition (15) is a generalization of the standard Hosios (1990) condition. In the special case where G is degenerate and expected match output is constant, we have $p_G(\theta) = x_0$ and $\eta_f(\theta) = \eta_m(\theta)$. Substituting into (15) and rearranging, we recover the standard Hosios condition.

7 Example: Pareto distribution with capital

We now turn to a specific distribution G : the Pareto distribution. This distribution is frequently used to represent the distribution of firm productivities and it arises naturally in many contexts.[28] In the present model, it yields the widely used Cobb-Douglas aggregate production function as an asymptotic result and it delivers tractable and empirically relevant expressions for factor shares.

Let $G(x) = 1 - \left(\frac{x}{x_0}\right)^{-1/\lambda}$ for $x \in [x_0, \infty)$ and $G(x) = 0$ otherwise. The Pareto distribution is *well-behaved* in the sense of Definition 2. In fact, it is the unique distribution G such that $\varepsilon_G(x)$ is constant, i.e. $\varepsilon_G(x)^{-1} = \lambda$ for all $x \in [x_0, \infty)$. The parameter $\lambda > 0$ is the *tail index* of G and $x_0 \geq 0$ is the minimum firm productivity. To ensure that Assumptions 1 and 2 are satisfied, we impose $\lambda < 1$, $x_0/(1-\lambda) > z + C$, and $0 \leq z \leq x_0$.[29]

Suppose that each firm acquires a single machine, or unit of capital, upon entry. We can interpret the cost of entry C as incorporating the cost of purchasing a machine. Total capital K is given by firms' demand, so $K = V$. Since all workers are initially unemployed, $L = U$ and thus $\theta = K/L$.

7.1 Production, matching, and distribution

If G is Pareto, the endogenous distribution $H(x;\theta)$ is a truncated Fréchet extreme value distribution with a mass point. The mass point at zero with

[28] See Gabaix (2009) and Gabaix (2016) for an overview of the use of power laws in economics.

[29] Note that $E_G(x) = x_0/(1-\lambda)$ and G has a finite variance if and only if $\lambda < 1/2$.

probability mass $u(\theta) = e^{-\theta}$ represents unemployment.

$$H(x;\theta) = \begin{cases} e^{-\theta x^{-1/\lambda}} & \text{if } x \in [x_0, \infty) \\ e^{-\theta} & \text{if } x \in [0, x_0) \end{cases} \tag{16}$$

We now introduce an important function that is a "generalization" of the Gamma function defined by $\Gamma(s) \equiv \int_0^\infty t^{s-1}e^{-t}\,dt$ for $s \in \mathbb{R}^+$.

Definition 5. *For any s, $x \in \mathbb{R}^+$, the Lower Incomplete Gamma function is*

$$\gamma(s,x) \equiv \int_0^x t^{s-1}e^{-t}\,dt. \tag{17}$$

For the Pareto distribution, output per capita $f(\theta)$ is given by

$$f(\theta) = x_0\theta^\lambda\gamma(1-\lambda,\theta). \tag{18}$$

Setting $\lambda = 0$ in (18) corresponds to the special case where match output is constant, $\bar{p} = x_0$. Since $\gamma(1,\theta) = 1 - e^{-\theta}$, we have $f(\theta) = m(\theta)\bar{p}$, i.e. the *matching* function multiplied by a constant. On the other hand, the asymptotic *production* function arises in the limit as $C \to 0$ and $\theta^* \to \infty$. This function is $f(\theta) = A\theta^\lambda$ where $A = x_0\Gamma(1-\lambda)$ since $\lim_{\theta\to\infty}\gamma(1-\lambda,\theta) = \Gamma(1-\lambda)$ and therefore $f(\theta) \sim A\theta^\lambda$ in the "frictionless" limit where $\theta \to \infty$.

Since each firm hires one unit of capital, $\theta = K/L$ and aggregate output is

$$Y = x_0\gamma(1-\lambda,\theta)K^\lambda L^{1-\lambda}. \tag{19}$$

Importantly, this function is *not* Cobb-Douglas since the term $\gamma(1-\lambda,\theta)$ is not constant but depends on $\theta = K/L$. Consistent with Proposition 1, the elasticity of substitution between capital and labor is strictly less than one. However, this function is *asymptotically* Cobb-Douglas in the limit as $\theta \to \infty$.[30]

To determine firm profits and expected wages, the Pareto distribution has the useful property that $\pi(\theta) = \lambda p(\theta)$, i.e. a firm's expected productivity rents are *linear* in match output. Substituting into (11) yields

$$w(\theta) = \underbrace{(1-\lambda)p(\theta)}_{\text{expected output minus productivity rents}} - \underbrace{\mu(\theta)(x_0 - z)}_{\text{matching rents}}. \tag{20}$$

[30] In the Appendix, we derive an exact Cobb-Douglas function that holds for *finite* θ.

The aggregate capital share $s_K(\theta) \equiv 1 - s_L(\theta)$ is given by

$$s_K(\theta) = \underbrace{\lambda}_{\text{productivity rents}} + \underbrace{\frac{\mu(\theta)(x_0 - z)}{p(\theta)}}_{\text{matching rents}}. \tag{21}$$

and firms' effective bargaining power is given by

$$1 - \beta(\theta) = \underbrace{\frac{\lambda p(\theta)}{p(\theta) - z}}_{\text{productivity rents}} + \underbrace{\frac{\mu(\theta)(x_0 - z)}{p(\theta) - z}}_{\text{matching rents}}. \tag{22}$$

Consistent with Proposition 2, we have $s_K(\theta) \to \lambda$ and $1 - \beta(\theta) \to \lambda$ in the limit as $C \to 0$ and $\theta^* \to \infty$. Since $\mu(\theta) \to 0$ exponentially fast, we have $s_K(\theta) \approx \lambda$ and $1 - \beta(\theta) \approx \lambda$ when unemployment is low.

7.2 Comparative statics

The Pareto distribution allows us to obtain some new results regarding the effects on aggregate outcomes of changes in the underlying distribution G of firm productivities, i.e. changes in the tail index λ and the minimum productivity x_0. Note that an increase in λ is an increase in tail fatness, *not* a mean-preserving spread, since both the mean and the variance are increasing in λ.

Proposition 4 states that an increase in the tail index λ leads to an increase in the market tightness θ^* and therefore a decrease in unemployment. At the same time, an increase in λ leads to an increase in both output per capita y^* and output per match p^*. The effect on output is due to both the indirect effect through θ^* and also the direct effect of the tail index λ on the *curvature* of the production technology. Since workers are hired by the most productive firms, the "selection" process features diminishing marginal returns – which leads to concavity of the production technology – but a higher tail index λ *lessens* the extent of the diminishing returns. This is because fatter tails make it more likely that an additional firm will yield a significantly higher productivity draw.

Proposition 4. *If G is Pareto, (i) the market tightness θ^* is increasing in the tail index λ and the minimum productivity x_0; (ii) unemployment u^* is decreasing in both λ and x_0; (iii) output per match p^* is increasing in both λ and x_0; (iv) output per capita y^* is increasing in both λ and x_0; and (v) labor share s_L^* is* *decreasing in λ if the value of matching rents is not too high, i.e. if*

$$\frac{x_0 - z}{x_0} < \frac{1-\lambda}{2-\lambda}. \tag{23}$$

When λ increases, there are two opposing effects on factor shares. Productivity rents, as a share of output, are clearly *increasing* in λ because $\pi(\theta)/p(\theta) = \lambda$. However, matching rents, as a share of output, are *decreasing* in λ.

$$\frac{ds_K^*}{d\lambda} = \underbrace{\frac{d}{d\lambda}(\lambda)}_{>0 \text{ effect on productivity rents}} \underbrace{+(x_0 - z)\frac{d}{d\lambda}\frac{\mu(\theta^*)}{p(\theta^*)}}_{<0 \text{ effect on matching rents}}. \tag{24}$$

First, the market tightness θ^* is increasing in λ, and $\mu(\theta)/p(\theta)$ is decreasing in θ since $\mu'(\theta) < 0$ and $p'(\theta) > 0$, so the indirect effect on matching rents is negative. Intuitively, a higher value of λ induces greater competition to hire workers, which both decreases the proportion of bilateral meetings and increases the expected match output. The direct effect on matching rents is also negative. An increase in λ directly increases the expected match output but not the fixed value of matching rents, $x_0 - z$, therefore matching rents fall as a *share* of match output. Overall, the net effect depends on the relative size of the opposing effects on productivity and matching rents. Condition (23) states that the effect on productivity rents dominates if the value of matching rents is not too high.[31]

The effect of an increase in λ on firms' effective bargaining power is ambiguous. The Appendix provides a sufficient condition under which workers' effective bargaining power is decreasing in the tail index λ.

The Appendix also provides a condition that is both necessary and sufficient for labor share to be increasing in the minimum productivity x_0. This condition is sufficient for workers' effective bargaining power to be increasing in x_0.

7.3 Numerical example

The link between the firm productivity distribution and the model's aggregate implications provides a novel way to calibrate the model. In contrast with a DMP style model where wages are determined by Nash bargaining, the model enables us

[31] If G is Pareto and $x_0 = 1$, the sufficient condition (23) in Proposition 4 is consistent with Assumption 2 if and only if $C < \frac{1}{(1-\lambda)(2-\lambda)}$. Since $\lambda \in (0,1)$, $C < 1/2$ will suffice.

to bypass the need to determine the value of non-market activity z and workers' bargaining power β. This is a useful feature of the model because there is no consensus about the values of these key parameters in the search literature.

The tail index parameter λ can be determined by using data on productivity dispersion. Consistent with the approach found in Mangin and Sedlacek (2016), we set $\lambda = 0.27$ and $x_0 = 1$, and the remaining parameters C and z are pinned down by targeting an unemployment rate of 5.8% and labor share of 68%. The implied cost of entry is $C = 0.18$, or 10.6% of average match output. The implied value of non-market activity is $z = 0.51$, or 44.2% of average wages.

Table 1 presents the values of the key endogenous variables under the benchmark calibration and highlights the comparative statics results. The last column summarizes the effects of a mean-preserving spread in G.[32]

		Value	λ	x_0	C	z	MPS
u^*	unemployment rate	5.8%	-	-	+	+	-
p^*	output per match	1.71	+	+	-	-	+
y^*	output per capita	1.61	+	+	-	-	+
s_L^*	labor share	0.68	-	+	-	+	-
β^*	workers' bargaining power	0.54	+	+	-	-	-
w^*	average wages	1.16	+	+	-	+	+

Both the labor share and workers' effective bargaining power are increasing the minimum firm productivity x_0. The labor share is decreasing in the tail index λ but workers' effective bargaining power is increasing in the tail index λ. The direction of the effect of a mean-preserving spread in G is generally the same as that for an increase in λ. Workers' effective bargaining power β^* is the only variable for which the direction of the effect differs: a mean-preserving spread in G leads to a *decrease* in both the labor share and workers' effective bargaining power. Since successful firms receive the difference between the highest and second-highest productivities among competing firms, it is intuitive that both the labor share and workers' effective bargaining power are lower when there is greater firm heterogeneity or productivity dispersion.

[32] To consider the effect of a mean-preserving spread, we set $x_0 = \bar{x}(1-\lambda)$ where $\bar{x}$ is a constant. The mean of G equals $x_0/(1-\lambda)$, which is always equal to $\bar{x}$, but the variance of G is increasing in λ. We then consider the effect of an increase in λ, which is now a genuine mean-preserving spread. We choose $\bar{x}$ to ensure that $x_0 = 1$ when $\lambda = 0.27$ so that the initial calibration for this exercise is identical to the benchmark calibration.

Average wages are increasing in the minimum productivity x_0 since $w(\theta) = s_L(\theta)p(\theta)$ and both the labor share s_L^* and output per match p^* are increasing in x_0. Also, average wages are decreasing in the cost of entry C since both s_L^* and p^* are decreasing in C. When there is an increase in the value of non-market activity z, the labor share increases but output per match falls. The former effect dominates and average wages are increasing in z. When there is an increase in the tail index λ or a mean-preserving spread in the distribution G, the labor share falls but output per match rises. The increase in expected match output is sufficiently large that this effect dominates and average wages increase.

Notice that if wages were instead determined by Nash bargaining, the *efficient* value of workers' bargaining parameter would be $\beta = 0.54$ under this calibration. However, since the matching elasticity with respect to vacancies is $\eta_m(\theta^*) = 0.18$, applying the standard Hosios condition would result in $\beta = 0.82$. The fact that firms' entry decisions affect both the matching probability for workers *and* the expected match output is thus quantitatively important: efficiency requires that firms are compensated through both productivity and matching rents.

8 Conclusion

This paper offers a *unified* approach to production, matching, and distribution. A process of direct competition between firms to hire workers simultaneously endogenizes both the average match output and the distribution of output between workers and firms – as measured by either the labor share or workers' effective bargaining power. As a result, the curvature of the endogenous production technology and the distribution of output are both influenced by properties of the underlying firm productivity distribution. For example, if this distribution is Pareto, the labor share is decreasing in the *tail index* (a measure of tail fatness) provided that the value of matching rents is not too high.

Mangin and Sedlacek (2016) extends the present model to a dynamic environment with aggregate shocks and shows that a calibrated version of it can account for the dynamics of the labor share, and other variables, over the business cycle. Possible directions for future research include: allowing for ex ante heterogeneity of workers and/or firms; considering alternative meeting technologies; and studying the model's predictions regarding the wage distribution.

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Appendix: Proofs

A.1 Existence and uniqueness of equilibrium. Suppose a firm draws x from the distribution G and n firms compete to hire the same worker. If $n = 1$, the firm's expected net payoff is just $\pi_1 = \int_{x_0}^{\infty}(x - w_R)\ dG(x) - C$. If $n \geq 2$, the expected net payoff is

$$\pi_2(x, n) = \beta(x, n)(x - w(x, n)) - C, \tag{25}$$

where $\beta(x, n)$ is the probability the firm is successful in hiring the worker and $w(x, n) = E(Y_2^n | Y_1^n = x)$ where Y_2^n is the second highest from n draws, and Y_1^n is the highest. Let $H(y, n)$ be the distribution of Y_1^n, i.e. $H(y, n) = G(y)^n$. Now $E(Y_2^n | Y_1^n = x)$ $= E(Y_1^{n-1} | Y_1^{n-1} < x)$, so expected wages as a function of the highest productivity x and the number of firms n is $w(x, n) = \frac{1}{H(x,n-1)} \int_{x_0}^{x} y \ \ dH(y, n-1)$ and thus

$$\pi_2(x, n) = \beta(x, n)\left(x - \frac{1}{H(x,n-1)} \int_{x_0}^{x} y \ \ dH(y, n-1)\right) - C. \tag{26}$$

Now $\beta(x, n)$ is just $G(x)^{n-1} = H(x, n-1)$. Substituting into (26) and using integration by parts, we obtain $\pi_2(x, n) = \int_{x_0}^{x} H(y, n-1)dy - C$. When $n \geq 2$, the expected payoff for a firm is $\pi_2(n) = \int_{x_0}^{\infty} \pi_2(x, n)g(x)dx$. Integrating by parts,

$$\pi_2(n) = [\pi_2(x, n)G(x)]_{x_0}^{\infty} - \int_{x_0}^{\infty} \frac{d}{dx}[\pi_2(x, n)]G(x)dx - C. \tag{27}$$

Now, $\frac{d}{dx}[\pi_2(x, n)] = \frac{d}{dx}\left(\int_{x_0}^{x} H(y, n-1)dy - C\right) = H(x, n-1)$. Also, $[\pi_2(x, n)G(x)]_{x_0}^{\infty} =$ $\lim_{x\to\infty} \pi_2(x, n)$, which is $\int_{x_0}^{\infty} H(y, n-1)dy$, since $G(x) \to 1$ as $x \to \infty$ and $G(1) = 0$. Rearranging, we have

$$\pi_2(n) = \int_{x_0}^{\infty} \beta(x, n)(1 - G(x))dx - C. \tag{28}$$

The number of firms n approaching a given worker is a Poisson random variable with parameter θ, so the expected net payoff given $n \geq 2$ is

$$\pi_2(\theta) = \int_{x_0}^{\infty} \beta(x)(1 - G(x))dx - C, \tag{29}$$

where $\beta(x)$ is the probability of being successful given that $n \geq 2$, namely

$$\beta(x) = \frac{1}{1 - e^{-\theta}} \sum_{n=2}^{\infty} \frac{e^{-\theta}\theta^{n-1}}{(n-1)!} G(x)^{n-1} = \frac{e^{-\theta(1-G(x))} - e^{-\theta}}{1 - e^{-\theta}}. \tag{30}$$

Using $\Pi(\theta) = e^{-\theta}\pi_1 + (1-e^{-\theta})\pi_2(\theta)$, the expected net payoff for a firm is

$$\begin{aligned}\Pi(\theta) &= \int_{x_0}^{\infty} e^{-\theta(1-G(x))}(1-G(x))dx \\ &\quad + e^{-\theta}\left(\left(\int_{x_0}^{\infty} x\ g(x)dx - \int_{x_0}^{\infty}(1-G(x))dx\right) - w_R\right) - C.\end{aligned}$$

By integration by parts and the fact that $\lim_{x\to\infty} x(1-G(x)) = 0$, the zero profit condition $\Pi(\theta) = C$ is equivalent to

$$F(\theta) = \int_{x_0}^{\infty} e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}\left(x_0 - w_R\right) - C = 0. \tag{31}$$

Now $F(\theta)$ is continuous in θ on $[0,\infty)$ and $F(\theta) \to -C$ as $\theta \to \infty$. If $F(0) > 0$, the intermediate value theorem implies that there exists a $\theta > 0$ such that $F(\theta) = 0$. Using integration by parts, $F(0) = \displaystyle\int_{x_0}^{\infty}(1-G(x))dx + (x_0 - w_R) - C = E_G(x) - w_R - C$. So there exists a $\theta > 0$ such that $F(\theta) = 0$ if $E_G(x) > w_R + C$. Otherwise, no firms enter and $\theta = 0$. To prove uniqueness of the equilibrium θ^*, it suffices to show that $F'(\theta) < 0$. Applying Leibniz' integral rule,

$$F'(\theta) = -\left(\int_{x_0}^{\infty} e^{-\theta(1-G(x))}\left(1-G(x)\right)^2 dx + e^{-\theta}(x_0 - w_R)\right) < 0. \tag{32}$$

Therefore, given workers' reservation wage w_R, there exists a unique market tightness $\theta^*(w_R)$ that satisfies the zero profit condition (31) and therefore also (2).

Workers' reservation wage w_R^* maximizes their expected payoff, i.e.

$$w_R^* = \arg\max_{w_R\in[0,\infty)}\ m(\theta^*(w_R))w_G(\theta^*(w_R)) + (1-m(\theta^*(w_R)))z \tag{33}$$

where $\theta^*(w_R)$ satisfies (2). Substituting in $w_G(\theta) = p_G(\theta) - C/q(\theta)$ from (2) and using $m(\theta) = 1 - e^{-\theta}$, this is equivalent to

$$w_R^* = \arg\max\ f(\theta^*(w_R)) - C\theta^*(w_R) + ze^{-\theta^*(w_R)} \tag{34}$$

where $f_G(\theta) \equiv m(\theta)p_G(\theta)$. The first order condition for (34) is $\frac{d\theta^*}{dw_R}(f'(\theta) - C - ze^{-\theta}) = 0$. Since $\frac{d\theta^*}{dw_R} = \frac{-\partial F/\partial w_R}{\partial F/\partial\theta} < 0$, this holds if and only if $f'(\theta) - C = ze^{-\theta}$. Using (39) and (31), $\theta^*(w_R)$ satisfies $f'(\theta) - C = w_R e^{-\theta}$ and hence $w_R^* = z$. Substituting $w_R^* = z$ into (31), the unique equilibrium market tightness $\theta^* \equiv \theta^*(w_R^*)$ satisfies

$$\int_{x_0}^{\infty} e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}\left(x_0 - z\right) = C \tag{35}$$

and $\theta^* > 0$ provided that $E_G(x) > z + C$, which is true if Assumption 2 holds.

A.2 Proof of Proposition 1. Using the definition of $f_G(.)$ and expression (4) for the distribution $H_G(x;\theta)$, we obtain

$$f_G(\theta) = \int_{x_0}^{\infty} \theta e^{-\theta(1-G(x))} x g(x) dx. \tag{36}$$

Applying Leibniz's rule, we have

$$f'_G(\theta) = \int_{x_0}^{\infty} x g(x) e^{-\theta(1-G(x))} dx - \int_{x_0}^{\infty} \theta x g(x) e^{-\theta(1-G(x))}(1-G(x)) dx. \tag{37}$$

By integration by parts on the right integral, and using the fact that $\lim_{x\to\infty} x(1-G(x)) = 0$ (which follows from the finite mean assumption), we have

$$\int_{x_0}^{\infty} \theta x g(x) e^{-\theta(1-G(x))}(1-G(x)) dx = -x_0 e^{-\theta} - \int_{x_0}^{\infty} e^{-\theta(1-G(x))}((1-G(x)) - x g(x)) dx, \tag{38}$$

Substituting (38) into (37),

$$f'_G(\theta) = \int_{x_0}^{\infty} e^{-\theta(1-G(x))}(1-G(x)) dx + x_0 e^{-\theta} > 0, \tag{39}$$

and part (i) is proved. Next, we use Leibniz' rule again to prove part (ii),

$$f''_G(\theta) = -\left(\int_{x_0}^{\infty} e^{-\theta(1-G(x))}(1-G(x))^2 dx + x_0 e^{-\theta} \right) < 0. \tag{40}$$

Clearly, $f(0) = 0$ and $\lim_{\theta\to\infty} f'_G(\theta) = 0$, so parts (iii) and (v) hold. Now consider $\lim_{\theta\to\infty} f_G(\theta)$. Changing variables by setting $t = 1 - G(x)$, we have $f_G(\theta) = \theta \int_0^1 e^{-\theta t} G^{-1}(1-t) dt$. Defining $G^{-1}(y) = 0$ for $y < 0$, we have $G^{-1}(1-t) = 0$ for $t > 1$ so $f_G(\theta) = \theta \int_0^{\infty} e^{-\theta t} G^{-1}(1-t) dt$ and we can apply the initial value theorem for Laplace transforms, which states that for any piecewise continuous function $\phi(t)$, $\lim_{\theta\to\infty} \theta \int_0^{\infty} e^{-\theta t} \phi(t) dt = \lim_{t_0\to 0} \phi(t_0)$. So we have $\lim_{\theta\to\infty} f_G(\theta) = \lim_{t_0\to 0} G^{-1}(1-t_0) = G^{-1}(1) = +\infty$, and part (iv) holds. Using (37), $\lim_{\theta\to 0} f'_G(\theta) = \lim_{\theta\to 0} \int_{x_0}^{\infty} x g(x) e^{-\theta(1-G(x))} dx = \int_{x_0}^{\infty} x g(x) dx = E_G(x)$, so (vi) holds.

Using Lemma 1 below, we first prove that if G is well-behaved then $\sigma_G(\theta) \leq 1$. Starting with the definition found in Arrow, Chenery, Minhas, and Solow (1961),

$$\sigma_G(\theta) = \frac{-f'_G(\theta)(f_G(\theta) - \theta f'_G(\theta))}{\theta f_G(\theta) f''_G(\theta)}. \tag{41}$$

Let $\bar{G}(x) = 1 - G(x)$. Inserting $f'_G(\theta)$ from (39) and $f''_G(\theta)$ from (40) into (41),

$$\sigma_G(\theta) = \frac{\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)dx + x_0 e^{-\theta}\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)dx - \left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)dx + x_0 e^{-\theta}\right)\right)}{\left(\int_{x_0}^{\infty} \theta e^{-\theta\bar{G}(x)}xg(x)dx\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)^2dx + x_0 e^{-\theta}\right)}.$$

Using (38) and simplifying further, we have

$$\sigma_G(\theta) = \frac{\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)dx + x_0 e^{-\theta}\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)\bar{G}(x)dx\right)}{\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)dx\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)^2dx + x_0 e^{-\theta}\right)}. \tag{42}$$

Multiplying out (42) yields

$$\sigma_G(\theta) = \frac{\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)dx\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)\bar{G}(x)dx\right) + x_0 e^{-\theta}\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)\bar{G}(x)dx\right)}{\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)dx\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)^2dx\right) + x_0 e^{-\theta}\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)dx\right)}.$$

Now since $\bar{G}(x) \leq 1$ and both integrands are positive, $\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)\bar{G}(x)dx \leq \int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)dx$. In order to show that $\sigma_G(\theta) \leq 1$, it is sufficient to show that

$$\frac{\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)dx \int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)\bar{G}(x)dx}{\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)dx \int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)^2dx} \leq 1. \tag{43}$$

Rearranging, and using the definition of $\varepsilon_G(x)$, this inequality is equivalent to

$$\frac{\int_{x_0}^{\infty}(1/\varepsilon_G(x))e^{-\theta\bar{G}(x)}xg(x)dx}{\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)dx} \leq \frac{\int_{x_0}^{\infty}(1/\varepsilon_G(x))e^{-\theta\bar{G}(x)}xg(x)\bar{G}(x)dx}{\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)\bar{G}(x)dx}. \tag{44}$$

We can now apply Lemma 1, where $\alpha(x) = 1/\varepsilon_G(x)$, $\varphi(x) = e^{-\theta\bar{G}(x)}xg(x)$, and $\beta(x) = \bar{G}(x)$. We have $\alpha(x) \geq 0$, $\varphi(x) \geq 0$ and $\beta(x) \geq 0$. Since G is well-behaved, $\varepsilon'_G(x) \geq 0$, so $\alpha'(x) \leq 0$ and $\beta'(x) = -g(x) < 0$. Using Lemma 1,

$$\frac{\int_{x_0}^{\infty}(1/\varepsilon_G(x))\varphi(x)dx}{\int_{x_0}^{\infty}\varphi(x)dx} \leq \frac{\int_{x_0}^{\infty}(1/\varepsilon_G(x))\varphi(x)\beta(x)dx}{\int_{x_0}^{\infty}\varphi(x)\beta(x)dx}.$$

Lemma 1. *Let $\alpha(.), \beta(.)$ and $\varphi(.)$ be positive functions defined on $[x_0, \infty)$. Suppose that $\alpha'(x) \leq 0$ and $\beta'(x) < 0$. Then $\int_{x_0}^{\infty}\alpha(x)h(x)dx \leq \int_{x_0}^{\infty}\alpha(x)\hat{h}(x)dx$, where $h(x) \equiv \frac{\varphi(x)}{\int_{x_0}^{\infty}\varphi(x)dx}$ and $\hat{h}(x) \equiv \frac{\varphi(x)\beta(x)}{\int_{x_0}^{\infty}\varphi(x)\beta(x)dx}$.*

Since $\beta'(x) < 0$, β^{-1} exists and $\hat{h}(x) - h(x) \geq 0$ holds if and only if

$$x \leq x_c \equiv \beta^{-1}\left(\frac{\int_{x_0}^{\infty} \varphi(x)\beta(x)dx}{\int_{x_0}^{\infty} \varphi(x)dx}\right),$$

for some critical value $x_c \in [x_0, \infty)$. Now for any $x \in [x_0, x_c]$, $\hat{h}(x) - h(x) \geq 0$ and $\alpha(x) \geq \alpha(x_c)$ since $\alpha'(x) \leq 0$, so

$$\int_{x_0}^{x_c} \alpha(x)(\hat{h}(x) - h(x))dx \geq \int_{x_0}^{x_c} \alpha(x_c)(\hat{h}(x) - h(x))dx. \tag{45}$$

For any $x \in [x_c, \infty)$, $\alpha(x) \leq \alpha(x_c)$ since $\alpha'(x) \leq 0$, but here $\hat{h}(x) - h(x) \leq 0$, so

$$\int_{x_c}^{\infty} \alpha(x)(\hat{h}(x) - h(x))dx \geq \int_{x_c}^{\infty} \alpha(x_c)(\hat{h}(x) - h(x))dx. \tag{46}$$

Using inequalities (45) and (46), we have

$$\begin{aligned} & \int_{x_0}^{\infty} \alpha(x)(\hat{h}(x) - h(x))dx \\ = & \int_{x_0}^{x_c} \alpha(x)(\hat{h}(x) - h(x))dx + \int_{x_c}^{\infty} \alpha(x)(\hat{h}(x) - h(x))dx \\ \geq & \int_{x_0}^{x_c} \alpha(x_c)(\hat{h}(x) - h(x))dx + \int_{x_c}^{\infty} \alpha(x_c)(\hat{h}(x) - h(x))dx \\ = & \alpha(x_c)\left(\int_{x_0}^{\infty} \hat{h}(x)dx - \int_{x_0}^{\infty} h(x)dx\right) = 0. \end{aligned}$$

We now prove that for any distribution G, the elasticity of substitution $\sigma_G(\theta)$ converges to one in the limit as $\theta \to \infty$. Starting with (42) and letting $t = 1 - G(x)$, so $x = G^{-1}(1-t)$ for $t \in (0, 1]$, we have

$$\sigma_G(\theta) = \frac{\left(\int_0^1 e^{-\theta t}\left(\frac{t}{g(G^{-1}(1-t))}\right)dt + x_0 e^{-\theta}\right)\left(\int_0^1 e^{-\theta t} t G^{-1}(1-t)dt\right)}{\left(\int_0^1 e^{-\theta t} G^{-1}(1-t)dt\right)\left(\int_0^1 e^{-\theta t}\left(\frac{t^2}{g(G^{-1}(1-t))}\right)dt + x_0 e^{-\theta}\right)}. \tag{47}$$

Rearranging (47), we have

$$\sigma_G(\theta) = \frac{\left(\int_0^1 e^{-\theta t}\left(G^{-1}(1-t)\frac{t}{g(G^{-1}(1-t))G^{-1}(1-t)}\right)dt + x_0 e^{-\theta}\right)\left(\int_0^1 e^{-\theta t} t G^{-1}(1-t)dt\right)}{\left(\int_0^1 e^{-\theta t} G^{-1}(1-t)dt\right)\left(\int_0^1 e^{-\theta t}\left(tG^{-1}(1-t)\frac{t}{g(G^{-1}(1-t))G^{-1}(1-t)}\right)dt + x_0 e^{-\theta}\right)}.$$

Now define $f_1(t) = G^{-1}(1-t)$ and $f_2(t) = \frac{t}{g(G^{-1}(1-t))G^{-1}(1-t)}$ for $t \in (0,1]$, and let $f_1(t) = f_2(t) = 0$ for $t > 1$. Then $\lim_{\theta\to\infty} \sigma_G(\theta)$ is given by

$$\lim_{\theta\to\infty} \sigma_G(\theta) = \lim_{\theta\to\infty} \frac{\left(\int_0^\infty e^{-\theta t} f_1(t) f_2(t) dt + x_0 e^{-\theta}\right)\left(\int_0^\infty e^{-\theta t} t f_1(t))dt\right)}{\left(\int_0^\infty e^{-\theta t} f_1(t) dt\right)\left(\int_0^\infty e^{-\theta t} t f_1(t) f_2(t) dt + x_0 e^{-\theta}\right)}. \tag{48}$$

Let $t_0 \in (0,1]$. Multiplying each integral in (48) by θ and dividing both the numerator and denominator by $f_1(t_0)f_2(t_0)$ and $t_0 f_1(t_0)$, we have

$$\lim_{\theta\to\infty} \sigma_G(\theta) = \lim_{\theta\to\infty} \frac{\left(\theta \int_0^\infty e^{-\theta t} \frac{f_1(t)f_2(t)}{f_1(t_0)f_2(t_0)} dt + \frac{\theta e^{-\theta}}{f_1(t_0)f_2(t_0)}\right)\left(\theta \int_0^\infty e^{-\theta t} \frac{t f_1(t)}{t_0 f_1(t_0)} dt\right)}{\left(\theta \int_0^\infty e^{-\theta t} \frac{f_1(t)}{f_1(t_0)} dt\right)\left(\theta \int_0^\infty e^{-\theta t} \frac{t f_1(t) f_2(t)}{t_0 f_1(t_0) f_2(t_0)} dt + \frac{\theta e^{-\theta}}{t_0 f_1(t_0) f_2(t_0)}\right)}.$$

Using limit operations and applying the initial value theorem for Laplace transforms,

$$\lim_{\theta\to\infty} \sigma_G(\theta) = \frac{\left(\lim_{t_0\to 0} \frac{f_1(t_0)f_2(t_0)}{f_1(t_0)f_2(t_0)} + 0\right)\left(\lim_{t_0\to 0} \frac{t_0 f_1(t_0)}{t_0 f_1(t_0)}\right)}{\left(\lim_{t_0\to 0} \frac{f_1(t_0)}{f_1(t_0)}\right)\left(\lim_{t_0\to 0} \frac{t_0 f_1(t_0) f_2(t_0)}{t_0 f_1(t_0) f_2(t_0)} + 0\right)} = 1.$$

We also prove that $\sigma_G(\theta) \leq 1$ is equivalent to the result that $\eta_f(\theta)$ is decreasing in θ where $\eta_f(\theta) \equiv f_G'(\theta)\theta / f_G(\theta)$. Differentiating $\eta_f(\theta)$ with respect to θ,

$$\frac{d}{d\theta}\eta_f(\theta) = \frac{f_G''(\theta)\theta f_G(\theta) + f_G'(\theta) f_G(\theta) - (f_G'(\theta))^2\theta}{f_G(\theta)^2} \tag{49}$$

and hence $\frac{d}{d\theta}\eta_f(\theta) \leq 0$ if and only if

$$f_G''(\theta)\theta f_G(\theta) + f_G'(\theta) f_G(\theta) \leq (f_G'(\theta))^2\theta. \tag{50}$$

Comparing the above with (41), inequality (50) is equivalent to $\sigma_G(\theta) \leq 1$ provided that $f_G''(\theta) < 0$, which is true.

Finally, we prove that in the limit as $\theta \to \infty$, the output elasticity $\eta_f(\theta) \to \lambda_G$, the extreme value tail index of G. By definition,

$$\lim_{\theta\to\infty} \eta_f(\theta) = \lim_{\theta\to\infty} \frac{f_G'(\theta)\theta}{f_G(\theta)}, \tag{51}$$

and, using (36) and (39) and simplifying, we have

$$\eta_f(\theta) = \frac{\int_{x_0}^\infty e^{-\theta(1-G(x))}(1-G(x))dx + x_0 e^{-\theta}}{\displaystyle\int_{x_0}^\infty e^{-\theta(1-G(x))} x g(x) dx}. \tag{52}$$

Changing variables by letting $t = 1 - G(x)$, and rearranging, we have

$$\lim_{\theta\to\infty} \eta_f(\theta) = \lim_{\theta\to\infty} \frac{\int_0^1 e^{-\theta t}\left(G^{-1}(1-t)\frac{t}{g(G^{-1}(1-t))G^{-1}(1-t)}\right)dt + x_0 e^{-\theta}}{\int_0^1 e^{-\theta t} G^{-1}(1-t)dt}. \tag{53}$$

Defining $f_1(t) = G^{-1}(1-t)$ and $f_2(t) = \frac{t}{g(G^{-1}(1-t))G^{-1}(1-t)}$ for $t \in (0,1]$, where $f_1(t) = f_2(t) = 0$ for $t > 1$, we obtain

$$\lim_{\theta\to\infty} \eta_f(\theta) = \lim_{\theta\to\infty} \frac{\int_0^1 e^{-\theta t} f_1(t) f_2(t)dt + x_0 e^{-\theta}}{\int_0^1 e^{-\theta t} f_1(t)dt}. \tag{54}$$

Let $t_0 \in (0,1]$. Multiplying each integral in (54) by θ and dividing both the numerator and denominator by $f_1(t_0)$, we have

$$\lim_{\theta\to\infty} \eta_f(\theta) = \lim_{\theta\to\infty} \frac{\theta\int_0^\infty e^{-\theta t}\frac{f_1(t)f_2(t)}{f_1(t_0)}dt + \frac{x_0\theta e^{-\theta}}{f_1(t_0)}}{\theta\int_0^\infty e^{-\theta t}\frac{f_1(t)}{f_1(t_0)}dt}. \tag{55}$$

Using limit operations and applying the initial value theorem for Laplace transforms,

$$\lim_{\theta\to\infty} \eta_f(\theta) = \frac{\lim_{t_0\to 0}\frac{f_1(t_0)f_2(t_0)}{f_1(t_0)} + 0}{\lim_{t_0\to 0}\frac{f_1(t_0)}{f_1(t_0)}} = \lim_{t_0\to 0} f_2(t_0). \tag{56}$$

Changing variables again using $t = 1 - G(x)$ and $f_2(t) = \frac{t}{g(G^{-1}(1-t))G^{-1}(1-t)}$, we have

$$\lim_{\theta\to\infty} \eta_f(\theta) = \lim_{x\to\infty} \frac{1-G(x)}{xg(x)} = \lim_{x\to\infty} \varepsilon_G(x)^{-1}. \tag{57}$$

If G is well-behaved, then $\varepsilon_G'(x) \geq 0$ so $\varepsilon_G(x)^{-1}$ is weakly decreasing in x. Also, for all $x \geq 0$ we have $\varepsilon_G(x)^{-1} \geq 0$. Hence $\lim_{\theta\to\infty} \eta_f(\theta) = \lim_{x\to\infty} \varepsilon_G(x)^{-1} = \alpha$ for some $\alpha \geq 0$. It is straightforward to show that $\lim_{x\to\infty} \frac{d}{dx}\left(\varepsilon_G(x)^{-1}x\right) = \alpha$ and thus $\alpha = \lambda_G$, the tail index of G, using Definition 4. Note that if Assumption 1 holds and thus G has a finite mean, we have $\lambda_G < 1$ since well-behaved distributions have a finite $n - th$ moment if and only if $\lambda_G < 1/n$. Also, $\lambda_G \geq 0$ since G has unbounded upper support. Therefore $\lambda_G \in [0,1)$.

A.3 Derivation of expression (9). Since the zero profit condition can also be expressed as $q(\theta)J = C$, using (35) delivers

$$J = \frac{1}{q(\theta)}\left(\int_{x_0}^\infty e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}(x_0 - z)\right), \tag{58}$$

or, equivalently, using $q(\theta) = m(\theta)/\theta$ and rearranging,

$$J = \frac{1}{m(\theta)} \int_{x_0}^{\infty} \theta e^{-\theta(1-G(x))} \left(\frac{1-G(x)}{xg(x)} \right) xg(x)dx + \frac{\theta e^{-\theta}}{m(\theta)} (x_0 - z)\,. \tag{59}$$

Using definition (1) for $\varepsilon_G(x)$ and the definition of $H^e_G(x;\theta)$, and defining $\mu(\theta) \equiv \theta e^{-\theta}/m(\theta)$, we obtain:

$$J = \pi_G(\theta) + \mu(\theta)(x_0 - z) \tag{60}$$

where $\pi_G(\theta) = \int_{x_0}^{\infty} \varepsilon_G(x)^{-1}\ x\ dH^e_G(x;\theta)$, as defined in (10).

A.4 Proof of Proposition 2.

First of all, $\pi_G(\theta)/p_G(\theta)$ is given by

$$\frac{\pi_G(\theta)}{p_G(\theta)} = \frac{\int_{x_0}^{\infty} \varepsilon_G(x)^{-1}\ x\ dH^e_G(x;\theta)}{\int_{x_0}^{\infty} x\ dH^e_G(x;\theta)}. \tag{61}$$

Using the definition of $\varepsilon_G(x)$ and simplifying, we have

$$\frac{\pi_G(\theta)}{p_G(\theta)} = \frac{\int_{x_0}^{\infty} e^{-\theta(1-G(x))}(1-G(x))dx}{\int_{x_0}^{\infty} e^{-\theta(1-G(x))}xg(x)dx}. \tag{62}$$

We first prove that $\pi_G(\theta)/p_G(\theta)$ is decreasing in θ. Letting $\bar{G}(x) = 1 - G(x)$ and differentiating (62), we have

$$\frac{d}{d\theta}\left(\frac{\pi_G(\theta)}{p_G(\theta)}\right) = \frac{-\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}(\bar{G}(x))^2 dx}{\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)dx} + \frac{\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)dx \int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)\bar{G}(x)dx}{\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)dx\right)^2}. \tag{63}$$

Rearranging, $\frac{d}{d\theta}\left(\frac{\pi_G(\theta)}{p_G(\theta)}\right) \leq 0$ if and only if

$$\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)dx \int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)\bar{G}(x)dx \leq \int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)dx \int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}(\bar{G}(x))^2 dx, \tag{64}$$

which is equivalent to inequality (43) established in the proof of Proposition 1. Therefore $\frac{d}{d\theta}\left(\frac{\pi_G(\theta)}{p_G(\theta)}\right) \leq 0$ for all θ. Next, to determine $\lim_{\theta\to\infty} \frac{\pi_G(\theta)}{p_G(\theta)}$, (62) and (52) yield

$$\frac{\pi_G(\theta)}{p_G(\theta)} = \eta_f(\theta) - \frac{x_0 e^{-\theta}}{\displaystyle\int_{x_0}^{\infty} e^{-\theta(1-G(x))}xg(x)dx}. \tag{65}$$

Since the second term on the right-hand side of (65) goes to zero as $\theta \to \infty$, we have $\lim_{\theta\to\infty} \frac{\pi_G(\theta)}{p_G(\theta)} = \lambda_G$ using the fact that $\lim_{\theta\to\infty} \eta_f(\theta) = \lambda_G$ by Proposition 1.

Expressing (12) using (62) and $\mu(\theta) = \theta e^{-\theta}/m(\theta)$, labor share is given by

$$s_L(\theta; G) = 1 - \left(\frac{\int_{x_0}^{\infty} e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}(x_0 - z)}{\int_{x_0}^{\infty} e^{-\theta(1-G(x))} x g(x) dx} \right). \tag{66}$$

Letting $\bar{G}(x) = 1 - G(x)$, and differentiating with respect to θ, we obtain

(67)
$$\frac{d}{d\theta} s_L(\theta; G) = \frac{\left(\begin{array}{c} \left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)^2 dx + e^{-\theta}(x_0 - z)\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)} x g(x) dx\right) \\ -\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x) x g(x) dx\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x) dx + e^{-\theta}(x_0 - z)\right) \end{array} \right)}{\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)} x g(x) dx\right)^2}.$$

Now $\frac{d}{d\theta} s_L(\theta; G) > 0$ if and only if

$$\begin{array}{rl} & \left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x) x g(x) dx\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x) dx\right) + e^{-\theta}(x_0 - z)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x) x g(x) dx\right) \\ < & \left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)^2 dx\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)} x g(x) dx\right) + e^{-\theta}(x_0 - z)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)} x g(x) dx\right). \end{array}$$

Since $\bar{G}(x) < 1$ for all $x > x_0$, it suffices to show that

$$\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x) x g(x) dx\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x) dx\right) \le \left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)^2 dx\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)} x g(x) dx\right)$$

which is identical to inequality (64) that is established above and applies to well-behaved distributions. Next, we have $\lim_{\theta\to\infty} \frac{\pi_G(\theta)}{p_G(\theta)} = \lambda_G$ from above and therefore, using (12), we have

$$\lim_{\theta\to\infty} s_L(\theta; G) = 1 - \lambda_G - (x_0 - z) \lim_{\theta\to\infty} \frac{\mu(\theta)}{p_G(\theta)} \tag{68}$$

and

$$\lim_{\theta\to\infty} \frac{\mu(\theta)}{p_G(\theta)} = \lim_{\theta\to\infty} \frac{\theta e^{-\theta}}{f_G(\theta)} = 0 \tag{69}$$

since $e^{-\theta} \to 0$ and $f_G(\theta) \to \infty$ as $\theta \to \infty$. Hence $\lim_{\theta\to\infty} s_L(\theta; G) = 1 - \lambda_G$.

Workers' effective bargaining power $\beta_G(\theta)$ can be expressed as

$$1 - \beta_G(\theta) = (1 - s_L(\theta; G)) \left(\frac{p_G(\theta)}{p_G(\theta) - z} \right). \tag{70}$$

Therefore, since $\frac{d}{d\theta}s_L(\theta;G) > 0$, to show that $\frac{d}{d\theta}\beta_G(\theta) > 0$ it suffices to show that $\frac{d}{d\theta}\left(\frac{p_G(\theta)}{p_G(\theta)-z}\right) < 0$, which is true since $p'_G(\theta) > 0$ by Lemma 2 below. Using (70), we also have $\lim_{\theta\to\infty}\beta_G(\theta) = 1-\lambda_G$ since $\lim_{\theta\to\infty}\frac{p_G(\theta)}{p_G(\theta)-z} = 1$.

A.5 Proof of Proposition 3.

Before proving Proposition 3, we establish the following lemma.

Lemma 2. *If G is well-behaved, output per match $p_G(\theta)$ is increasing in the market tightness θ.*

Suppose that G is well-behaved. Using the fact that $p_G(\theta) = f_G(\theta)/m(\theta)$ and differentiating, we have $p'_G(\theta) > 0$ if and only if

$$h(\theta) \equiv \frac{f'_G(\theta)\theta}{f_G(\theta)} - \frac{m'(\theta)\theta}{m(\theta)} > 0. \tag{71}$$

Since $h(0) = 0$, it suffices to prove $h'(\theta) > 0$. Differentiating and simplifying (71) using (49), we have $h'(\theta) > 0$ iff

$$\frac{f''_G(\theta)\theta f_G(\theta) + f'_G(\theta)f_G(\theta) - (f'_G(\theta))^2\theta}{f_G(\theta)^2} > \frac{e^{-\theta}(1-e^{-\theta}-\theta)}{(1-e^{-\theta})^2}. \tag{72}$$

Using (41) and the result that $\sigma_G(\theta) \leq 1$ for any well-behaved distribution G, the left-hand side of (72) is greater than or equal to zero. So it suffices to show that $1-e^{-\theta}-\theta < 0$, which is easily verified for all $\theta > 0$.

By the implicit function theorem, $\frac{d\theta^*}{dz} = -\frac{\partial F/\partial z}{\partial F/\partial\theta}$ where $F(\theta) = \int_{x_0}^{\infty} e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}(x_0-z) - C$. We have

$$F'(\theta) = -\left(\int_{x_0}^{\infty} e^{-\theta(1-G(x))}\left(1-G(x)\right)^2 dx + e^{-\theta}(x_0-z)\right) < 0. \tag{73}$$

Since $z \leq x_0$, we have

$$\frac{d\theta^*}{dz} = \frac{-e^{-\theta}}{\left(\int_{x_0}^{\infty} e^{-\theta(1-G(x))}(1-G(x))^2dx + e^{-\theta}\left(x_0-z\right)\right)} < 0. \tag{74}$$

Also, $\frac{d\theta^*}{dC} = -\frac{\partial F/\partial C}{\partial F/\partial\theta}$ where $\frac{\partial F}{\partial C} = -1$. Using $\frac{\partial F}{\partial\theta}$ from above, we have

$$\frac{d\theta^*}{dC} = \frac{-1}{\left(\int_{x_0}^{\infty} e^{-\theta(1-G(x))}(1-G(x))^2dx + e^{-\theta}\left(x_0-z\right)\right)} < 0. \tag{75}$$

Now, since $f_G'(\theta) > 0$ and $u'(\theta) < 0$, the fact that $\frac{d\theta^*}{dz} < 0$ and $\frac{d\theta^*}{dC} < 0$ implies that $\frac{dy^*}{dz} < 0$, $\frac{dy^*}{dC} < 0$, $\frac{du^*}{dz} > 0$, and $\frac{du^*}{dC} > 0$. Finally, since $p_G'(\theta) > 0$ by Lemma 2, the fact that $\frac{d\theta^*}{dz} < 0$ and $\frac{d\theta^*}{dC} < 0$ implies that $\frac{dp^*}{dz} < 0$ and $\frac{dp^*}{dC} < 0$.

Starting with (66), and using (67) and (74),

$$
\begin{aligned}
\frac{ds_L^*}{dz} &= \frac{\partial s_L}{\partial \theta}\frac{d\theta^*}{dz} + \frac{\partial s_L}{\partial z} \\
&= \frac{-e^{-\theta}\left(\begin{array}{c}\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)^2dx + e^{-\theta}(x_0 - z)\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)dx\right) \\ -\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)xg(x)dx\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)dx + e^{-\theta}(x_0 - z)\right)\end{array}\right)}{\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)^2dx + e^{-\theta}(x_0 - z)\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)dx\right)^2} \\
&\quad + \frac{e^{-\theta}}{\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)dx}.
\end{aligned}
$$

where $\bar{G}(x) = 1 - G(x)$. Simplifying and rearranging, $\frac{ds_L^*}{dz} > 0$ if and only if

$$
\frac{\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)xg(x)dx\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)dx + e^{-\theta}(x_0 - z)\right)}{\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}\bar{G}(x)^2dx + e^{-\theta}(x_0 - z)\right)\left(\int_{x_0}^{\infty} e^{-\theta\bar{G}(x)}xg(x)dx\right)} > 0,
$$

which is clearly true.

The effect of a change in z on workers' effective bargaining power is given by $\frac{d\beta^*}{dz} = \frac{\partial\beta}{\partial\theta}\frac{d\theta^*}{dz} + \frac{\partial\beta}{\partial z}$. Rearranging (70), we have

$$
\beta_G(\theta) = 1 - (1 - s_L(\theta; G))\left(\frac{p_G(\theta)}{p_G(\theta) - z}\right). \tag{76}
$$

It is clear that the direct effect of z on $\beta_G(\theta)$ is negative, i.e. $\frac{\partial\beta}{\partial z} < 0$. Since $\frac{\partial\beta}{\partial\theta} > 0$ and $\frac{d\theta^*}{dz} < 0$, the indirect effect is also negative and therefore $\frac{d\beta^*}{dz} < 0$.

A.6 Proof of constrained efficiency.

The social planner's solution θ_P satisfies the first-order condition $f_G'(\theta) - ze^{-\theta} = C$, and the equilibrium θ^* satisfies (35). Substituting $f_G'(\theta)$ from (39) into the first-order condition, it is clear that $\theta^* = \theta_P$. Using (40), the second-order condition, $f_G''(\theta) + ze^{-\theta} < 0$, also holds.

A.7 Properties of the Lower Incomplete Gamma Function.

Fact 1. *The function $\gamma(s,x)$ satisfies: (i) the recurrence relation: $\gamma(s,x) = (s-1)\gamma(s-1,x) - x^{s-1}e^{-x}$; (ii) $\frac{\partial}{\partial x}\gamma(s,x) = x^{s-1}e^{-x} > 0$; (iii) $\frac{\partial}{\partial s}\gamma(s,x) = \int_0^x t^{s-1}e^{-t}(\ln t)dt$; (iv) $\lim_{x\to\infty}\gamma(s,x) = \Gamma(s)$; and (v) $\gamma(1,x) = 1 - e^{-x}$.*

For standard properties such as Fact 1, see Andrews, Askey, and Roy (2000).

Definition 6. *For any* s, $x \in \mathbb{R}^+$, $\varepsilon(s,x)$ *is the elasticity of* $\gamma(s,x)$ *wrt* x,

$$\varepsilon(s,x) \equiv \frac{x^s e^{-x}}{\gamma(s,x)}. \tag{77}$$

Lemma 3. *The elasticity* $\varepsilon(s,x)$ *has the following properties: (i)* $\frac{\partial}{\partial s}\varepsilon(s,x) > 0$*; (ii)* $\frac{\partial}{\partial x}\varepsilon(s,x) < 0$; *(iii)* $\lim_{x\to 0}\varepsilon(s,x) = s$*; and (iv)* $\lim_{x\to\infty}\varepsilon(s,x) = 0$.

Proof of Lemma 3. Differentiating (77) with respect to s, we obtain

$$\frac{\partial}{\partial s}\varepsilon(s,x) = x^s e^{-x}\left(\frac{\int_0^x (\ln x - \ln t)t^{s-1}e^{-t}dt}{\gamma(s,x)^2}\right) > 0.$$

Differentiating (77) with respect to x, we have

$$\frac{\partial}{\partial x}\varepsilon(s,x) = \frac{x^{s-1}e^{-x}}{\gamma(s,x)}\left(s - x - \frac{x^s e^{-x}}{\gamma(s,x)}\right) < 0. \tag{78}$$

To see this, observe that $\frac{\partial}{\partial x}\varepsilon(s,x) < 0$ if and only if $s - x < \varepsilon(s,x)$. Applying Fact 1 (i), this is true provided that $x > \gamma(s+1,x)/\gamma(s,x)$. Multiplying both sides by $x^s e^{-x}$ and rearranging, this is true if and only if $\varepsilon(s+1,x) > \varepsilon(s,x)$, which follows from $\frac{\partial}{\partial s}\varepsilon(s,x) > 0$. Parts (iii) and (iv) follow from L'Hôpital's rule.

A.8 "Frictionless" benchmark economy. Suppose that we relax two key frictions: *(i)* the fact that firms can approach only one worker; and *(ii)* the fact that firms can hire only one worker. We consider only the limit economy where each firm can approach a large number of workers and there is no restriction on the number of workers a firm can hire.

Suppose that entering firms pay the cost C to enter but they can send out M "offers" simultaneously. The number of "offers" received by each worker is a Poisson random variable with parameter θM. Each worker produces output at the level of the highest productivity x among the firms who approach that worker. For finite M, the distribution of output per worker, $H_G(x;\theta,M)$, is given by

$$H_G(x;\theta,M) = \begin{cases} e^{-\theta M\left(\frac{x}{x_0}\right)^{-1/\lambda}} & \text{if } x \in [x_0,\infty) \\ e^{-\theta M} & \text{if } x \in [0,x_0) \end{cases} \tag{79}$$

Now consider the limit G_0 of the distribution G as $x_0 \to 0$ and define the corresponding endogenous distribution $H_0(x;\theta) \equiv H_{G_0}(x;\theta,M)$.[33] To ensure that H_0 is indeed a cdf, we normalize $M = x_0^{-1/\lambda}$. Clearly, in the limit as $x_0 \to 0$, we have $M \to \infty$. The endogenous distribution is given by $H_0(x;\theta) = e^{-\theta x^{-1/\lambda}}$ for all

[33]This means that we also require $z = 0$ to satisfy the assumption that $0 \leq z \leq x_0$.

$x \in [0, \infty)$, the Fréchet distribution.[34] There is no mass point because there is no unemployment in this limit. Output per capita is given by $f(\theta) = \Gamma(1-\lambda)\theta^{\lambda}$ and we obtain an exact Cobb-Douglas aggregate production function:

$$Y = \Gamma(1-\lambda)K^{\lambda}L^{1-\lambda}. \tag{80}$$

A.9 Proof of Proposition 4. By the implicit function theorem, we have $\frac{d\theta^*}{d\lambda} = -\frac{\partial F/\partial\lambda}{\partial F/\partial\theta}$. Using (31), we have

$$F(\theta) = x_0\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}(x_0 - z), \tag{81}$$

and differentiating with respect to θ and then using Fact 1 (i) yields

$$\frac{\partial F}{\partial\theta} = -\left(x_0\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) + e^{-\theta}(x_0 - z)\right). \tag{82}$$

Applying Fact 1 (iii) and simplifying yields

$$\frac{\partial F}{\partial\lambda} = x_0\theta^{\lambda-1}\left(\gamma(1-\lambda,\theta) + \lambda\int_0^{\theta} t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt\right).$$

Again using (82), plus the fact that $\int_0^{\theta} t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt > 0$, we have

$$\frac{d\theta^*}{d\lambda} = \frac{x_0\theta^{\lambda-1}\left(\gamma(1-\lambda,\theta) + \lambda\int_0^{\theta} t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt\right)}{x_0\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) + e^{-\theta}(x_0 - z)} > 0. \tag{83}$$

Also, the fact that $\frac{du^*}{d\lambda} < 0$ follows from $\frac{d\theta^*}{d\lambda} > 0$ and $u'(\theta) < 0$. Next, $\frac{dy^*}{d\lambda} = \frac{\partial f}{\partial\theta}\frac{d\theta^*}{d\lambda} + \frac{\partial f}{\partial\lambda}$. Now $f'(\theta) > 0$ and $\frac{d\theta^*}{d\lambda} > 0$ so it suffices to show that $\frac{\partial f}{\partial\lambda} > 0$. Using Fact 1 (iii), we obtain

$$\frac{\partial f}{\partial\lambda} = x_0\theta^{\lambda}\left(\int_0^{\theta} t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt\right) > 0. \tag{84}$$

Similarly, $\frac{dp^*}{d\lambda} = \frac{\partial p}{\partial\theta}\frac{d\theta^*}{d\lambda} + \frac{\partial p}{\partial\lambda}$. Since $p'(\theta) > 0$ and $\frac{d\theta^*}{d\lambda} > 0$, it suffices to show that $\frac{\partial p}{\partial\lambda} > 0$, which follows from (84).

We have $\frac{d\theta^*}{dx_0} = -\frac{\partial F/\partial x_0}{\partial F/\partial\theta} > 0$ since $\frac{\partial F}{\partial x_0} = \lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta} > 0$, and therefore $\frac{du^*}{dx_0} < 0$ also. The effect of x_0 on output per capita is given by $\frac{dy^*}{dx_0} = \frac{\partial f}{\partial\theta}\frac{d\theta^*}{dx_0} + \frac{\partial f}{\partial x_0}$. Since $f'(\theta) > 0$ and $\frac{d\theta^*}{dx_0} > 0$, it suffices to show that $\frac{\partial f}{\partial x_0} > 0$, which is clearly true since $f(\theta) = x_0\theta^{\lambda}\gamma(1-\lambda,\theta)$. Finally, $\frac{dp^*}{dx_0} = \frac{\partial p}{\partial\theta}\frac{d\theta^*}{dx_0} + \frac{\partial p}{\partial x_0}$, where $p'(\theta) > 0$ and $\frac{d\theta^*}{dx_0} > 0$, and $\frac{\partial p}{\partial x_0} > 0$ is clear, hence $\frac{dp^*}{dx_0} > 0$.

[34] This approach is mathematically similar to that used in Eaton, Kortum, and Sotelo (2012) and Oberfield (2013) to obtain exact Fréchet distributions.

Next, since $\mu(\theta)/p(\theta) = \varepsilon(1-\lambda,\theta)/x_0$ using Definition 6, we have

$$s_K^* = \lambda + \left(\frac{x_0 - z}{x_0}\right)\varepsilon(1-\lambda,\theta) \tag{85}$$

where $\theta^*(\lambda)$ solves the zero profit condition

$$x_0\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + (x_0 - z)e^{-\theta} = C. \tag{86}$$

Rearranging (86) and substituting into the expression for capital share using Definition 6, we obtain $s_K^* = \frac{C\theta^{1-\lambda}}{x_0\gamma(1-\lambda,\theta)}$. Differentiating s_K^* with respect to λ,

$$\frac{ds_K^*}{d\lambda} = \frac{C}{x_0}\left(\frac{\partial}{\partial\theta}\left(\frac{\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}\right)\frac{d\theta^*}{d\lambda} + \frac{\partial}{\partial\lambda}\left(\frac{\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}\right)\right). \tag{87}$$

Using Fact 1 (ii), we have

$$\frac{\partial}{\partial\theta}\left(\frac{\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}\right) = \frac{(1-\lambda)\theta^{-\lambda}}{\gamma(1-\lambda,\theta)} - \frac{\theta^{1-2\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)^2}. \tag{88}$$

Applying Fact 1 (iii) and simplifying,

$$\frac{\partial}{\partial\lambda}\left(\frac{\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}\right) = \frac{-\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)^2}\left(\int_0^\theta t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt\right). \tag{89}$$

Letting $B = \int_0^\theta t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt$ and then substituting (88) and (89) into (87),

$$\frac{ds_K^*}{d\lambda} = \frac{C\theta^{-\lambda}}{x_0\gamma(1-\lambda,\theta)}\left((1-\lambda-\varepsilon(1-\lambda,\theta))\frac{d\theta^*}{d\lambda} - \frac{\theta B}{\gamma(1-\lambda,\theta)}\right) \tag{90}$$

Applying Fact 1 (i) and Definition 6, and simplifying, we have $\frac{ds_K^*}{d\lambda} > 0$ if and only if $\frac{d\theta^*}{d\lambda} > \frac{\theta B}{\gamma(2-\lambda,\theta)}$. Substituting in $\frac{d\theta^*}{d\lambda}$ from (83) and simplifying, $\frac{ds_K^*}{d\lambda} > 0$ if and only if

$$\gamma(2-\lambda,\theta)\gamma(1-\lambda,\theta) > B\left(\frac{x_0 - z}{x_0}\right)\theta^{2-\lambda}e^{-\theta}. \tag{91}$$

Now suppose that $\frac{x_0 - z}{x_0} < \frac{1-\lambda}{2-\lambda}$. To prove (91), it suffices to show

$$\gamma(2-\lambda,\theta)\gamma(1-\lambda,\theta) > B\left(\frac{1-\lambda}{2-\lambda}\right)\theta^{2-\lambda}e^{-\theta}. \tag{92}$$

To prove this, we introduce a generalized hypergeometric function defined by

$$F_{2,2}(a_1, a_2; b_1, b_2; z) \equiv \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n}{(b_1)_n (b_2)_n} \frac{z^n}{n!},$$

where $(a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)}$, the Pochhammer symbol or ascending factorial function. Calculating the integral B, we have:

$$B = (\ln \theta)\gamma(1-\lambda, \theta) - \left[(\ln x)\gamma(1-\lambda, x) - \frac{x^{1-\lambda}}{(1-\lambda)^2} F_{2,2}(1-\lambda, 1-\lambda; 2-\lambda, 2-\lambda; -x) \right]_0^\theta .$$

As $\lim_{x\to 0} \frac{x^{1-\lambda}}{(1-\lambda)^2} F_{2,2}(1-\lambda, 1-\lambda; 2-\lambda, 2-\lambda; -x) = \lim_{x\to 0} (\ln x)\gamma(1-\lambda, x) = 0$,

$$B = \frac{\theta^{1-\lambda}}{(1-\lambda)^2} F_{2,2}(1-\lambda, 1-\lambda; 2-\lambda, 2-\lambda; -\theta). \tag{93}$$

Inequality (92) can now be stated purely in terms of generalized hypergeometric functions using $\gamma(x, z) = z^x x^{-1} F_{1,1}(x; x+1; -z)$, a standard identity (See, for example, Andrews et al. (2000)). Rearranging, (92) is equivalent to

$$\frac{e^{-\theta} F_{2,2}(1-\lambda, 1-\lambda; 2-\lambda, 2-\lambda; -\theta)}{F_{1,1}(1-\lambda; 2-\lambda; -\theta) F_{1,1}(2-\lambda; 3-\lambda; -\theta)} < 1. \tag{94}$$

To establish (94) and hence prove that $ds_K^*/d\lambda > 0$, it suffices to prove Lemma 4. Inequality (94) is the special case where $a = 1-\lambda$ and $x = \theta$.

Lemma 4. *For any $a \geq 0$ and any $x > 0$, we have*

$$e^{-x} F_{2,2}(a, a; a+1, a+1; -x) < F_{1,1}(a; a+1; -x) F_{1,1}(a+1; a+2; -x). \tag{95}$$

First, we use the following result found in Miller and Paris (2012) just after Eq. (5.3), obtained by specialization of 9.1 (34) in Luke (1969),

$$F_{2,2}(a, f; b, c; -x) = \sum_{k=0}^{\infty} \frac{(a)_k (c-f)_k}{(b)_k (c)_k} \frac{x^k}{k!} F_{1,1}(a+k; b+k; -x). \tag{96}$$

Setting $f = a$ and $b = c = a+1$ in (96), and using the fact that $(1)_k = k!$,

$$F_{2,2}(a, a; a+1, a+1; -x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(a+1)_k^2} x^k F_{1,1}(a+k; a+1+k; -x).$$

Next, we apply Kummer's first transformation, $F_{1,1}(y; z; -x) = e^{-x} F_{1,1}(z-y; z; x)$ to all $F_{1,1}$ terms. (See, for example, Andrews et al. (2000), [Eq. 4.1.11]). Replacing $F_{1,1}(1; a+2; x)$ with its definition and cancelling the term e^{-2x} on both sides, inequality

(95) is equivalent to

$$\sum_{k=0}^{\infty} \frac{(a)_k \; x^k}{(a+1)_k^2} F_{1,1}(1; a+1+k; x) < F_{1,1}(1; a+1; x) \sum_{k=0}^{\infty} \frac{x^k}{(a+2)_k}. \tag{97}$$

Since all terms in both series are positive now, we can simply compare coefficients of like powers of x. Inequality (97) holds provided that for all $k \in \mathbb{N}$,

$$\frac{(a)_k}{(a+1)_k^2} F_{1,1}(1; a+1+k; x) < F_{1,1}(1; a+1; x) \frac{1}{(a+2)_k}. \tag{98}$$

First, it is straightforward to verify that the following holds:

$$\frac{(a)_k (a+2)_k}{(a+1)_k^2} = \frac{a(a+k+1)}{(a+1)(a+k)} \leq 1.$$

Also, $F_{1,1}(1; a+1+k; x) < F_{1,1}(1; a+1; x)$ for all $k \in \mathbb{N}$ since $\frac{\partial F_{1,1}(a_1; b_1; x)}{\partial b_1} < 0$. (See Erdelyi, Magnus, Oberhettinger, and Tricomi (1953) for this derivative.)

A.10 Sufficient conditions. We first provide a sufficient condition under which workers' effective bargaining power β^* is decreasing in λ.

Proposition 5. *If G is Pareto, workers' effective bargaining power β^* is decreasing in λ if labor share s_L^* is decreasing in λ and, in addition, we have:*

$$\frac{x_0 - z}{z} > \frac{\eta_{p^*}(\lambda)}{\eta_{s_K^*}(\lambda)} \tag{99}$$

where $\eta_{p^*}(\lambda) \equiv \frac{dp^*}{d\lambda} \frac{\lambda}{p^*}$ and $\eta_{s_K^*}(\lambda) \equiv \frac{ds_K^*}{d\lambda} \frac{\lambda}{s_K^*}$.

Using (70) and differentiating, we obtain

$$\frac{d}{d\lambda}(1 - \beta^*) = \frac{ds_K^*}{d\lambda} \left(\frac{p^*}{p^* - z} \right) - \frac{z s_K^*}{(p^* - z)^2} \left(\frac{dp^*}{d\lambda} \right).$$

Rearranging, we have $\frac{d}{d\lambda}(1 - \beta^*) > 0$ if and only if

$$\left(\frac{p^* - z}{z} \right) \frac{ds_K^*}{d\lambda} \frac{\lambda}{s_K^*} > \frac{dp^*}{d\lambda} \frac{\lambda}{p^*}.$$

If labor share s_L^* is decreasing in λ then $\frac{d}{d\lambda}(1 - \beta^*) > 0$ if and only if

$$\frac{p^* - z}{z} > \frac{\eta_{p^*}(\lambda)}{\eta_{s_K^*}(\lambda)}.$$

Finally, since $p^* \geq x_0$, a sufficient condition for $\frac{d}{d\lambda}(1-\beta^*) > 0$ is (99).

Next, we provide sufficient conditions for the effects of x_0 on the labor share s_L^* and workers' effective bargaining power β^*.

Proposition 6. *If G is Pareto,*

(i) *labor share s_L^* is increasing in the minimum productivity x_0 if and only if the value of matching rents is not too high:*

$$\frac{x_0 - z}{z} < \frac{-1}{\eta_\varepsilon(\theta^*)\varepsilon_{\theta^*}(x_0)} \tag{100}$$

where $\eta_\varepsilon(\theta) \equiv \frac{d}{d\theta}\left(\frac{\mu(\theta)}{p(\theta)}\right)\frac{\theta}{\mu(\theta)/p(\theta)}$ and $\eta_{\theta^*}(x_0) \equiv \frac{d\theta^*}{dx_0}\frac{x_0}{\theta^*}$; and

(ii) *workers' effective bargaining power β^* is increasing in the minimum productivity x_0 if condition (100) holds.*

Differentiating s_L^* with respect to x_0, we obtain

$$\frac{ds_L^*}{dx_0} = -\frac{z}{x_0^2}\varepsilon(1-\lambda,\theta^*) - \left(\frac{x_0 - z}{x_0}\right)\left(\frac{\partial\varepsilon(1-\lambda,\theta)}{\partial\theta}\frac{d\theta^*}{dx_0}\right)$$

and therefore $\frac{ds_L^*}{dx_0} > 0$ if and only if

$$\frac{\varepsilon(1-\lambda,\theta^*)}{x_0}\left(\frac{z}{x_0} + \left(\frac{x_0 - z}{x_0}\right)\frac{\partial\varepsilon(1-\lambda,\theta)}{\partial\theta}\frac{d\theta^*}{dx_0}\frac{x_0}{\varepsilon(1-\lambda,\theta^*)}\right) < 0.$$

Simplifying, using $\eta_\varepsilon(\theta) \equiv \frac{d}{d\theta}\left(\frac{\mu(\theta)}{p(\theta)}\right)\frac{\theta}{\mu(\theta)/p(\theta)}$ and $\eta_{\theta^*}(x_0) \equiv \frac{d\theta^*}{dx_0}\frac{x_0}{\theta^*}$, we have $\frac{ds_L^*}{dx_0} > 0$ if and only if the following holds:

$$\frac{z}{x_0} + \left(\frac{x_0 - z}{x_0}\right)\eta_\varepsilon(\theta^*)\eta_{\theta^*}(x_0) < 0,$$

using the fact that $x_0\mu(\theta)/p(\theta) = \varepsilon(1-\lambda,\theta)$. Rearranging, $\frac{ds_L^*}{dx_0} > 0$ if and only if (100) holds. Finally, using (70) and differentiating, we obtain

$$\frac{d(1-\beta^*)}{dx_0} = \frac{ds_K^*}{dx_0}\left(\frac{p^*}{p^* - z}\right) + s_K^*\left(\frac{\partial}{\partial\theta}\left(\frac{p(\theta)}{p(\theta) - z}\right)\frac{d\theta^*}{dx_0} + \frac{\partial}{\partial x_0}\left(\frac{p^*}{p^* - z}\right)\right).$$

We know that $\frac{d}{d\theta}\left(\frac{p(\theta)}{p(\theta)-z}\right) < 0$ and $\frac{d\theta^*}{dx_0} > 0$ from above. Also, $\frac{\partial p}{\partial x_0} > 0$ so we have $\frac{\partial}{\partial x_0}\left(\frac{p^*}{p^*-z}\right) < 0$ and therefore $\frac{d}{dx_0}(1-\beta^*) < 0$ provided that $\frac{ds_K^*}{dx_0} < 0$. Therefore, if condition (100) holds, we obtain $\frac{d\beta^*}{dx_0} > 0$.