# Consumer Choice and Private Information in Monetary Exchange*

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**Abstract**

We introduce consumer choice into a competitive search model of monetary exchange. In contrast to standard search models featuring bilateral meetings, there is a general meeting technology which allows consumers to meet multiple sellers and *choose* a seller with whom to trade. Consumer choice is influenced by random utility shocks. When buyers' utility shocks are private information, there is inefficiency of both entry and quantities traded – even at the Friedman rule. Without consumer choice, this inefficiency cannot be eliminated. With consumer choice, however, greater seller entry can alleviate this inefficiency by reducing the informational rents available to buyers. In fact, in the competitive limit where the seller-buyer ratio becomes large, consumer choice eliminates the effects of private information and delivers efficiency at the Friedman rule.

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# 1 Introduction

Consumer choice is an important feature of monetary exchange. When consumers purchase goods, they typically choose from a number of goods that are available simultaneously from a range of competing sellers. Choice is often idiosyncratic: different consumers might make different choices when faced with the same range of goods, and the same consumer might make different choices at different points in time. Discrete choice models with random utility shocks have been used extensively to study these kinds of choices in the large literature following Anderson, De Palma, and Thisse (1992), but these models do not feature monetary exchange.

Search-theoretic models have become the standard way of modelling the microfoundations of monetary exchange, as surveyed in Lagos, Rocheteau, and Wright (2017). However, meetings are typically one-on-one (or bilateral): each buyer meets at most one seller during a single period of time and chooses to either trade or wait. While many papers have incorporated random utility shocks or match-specific preference shocks into the process of monetary exchange, such as Lagos and Rocheteau (2005), these shocks typically influence only the quantities traded and the payments – not the choice of seller – because meetings are bilateral. In this way, there is no genuine role for what we call *consumer choice*, i.e. buyers' choice of seller.

This paper develops a new model that features both consumer choice and monetary exchange. To do so, we introduce the possibility of consumer choice into the monetary framework of Rocheteau and Wright (2005), hereafter denoted RW. This framework shares the convenience of the Lagos and Wright (2005) alternating structure, but it also features endogenous seller entry, which is important for our results.

We study *competitive search equilibrium.* Buyers and sellers choose to enter submarkets in which terms of trade, or contracts, are posted by market makers. After entering a submarket, agents commit to trading at the terms posted in that submarket. Within each submarket, there are search frictions that govern how agents meet. Directed or competitive search is a natural alternative to bargaining in our environment because buyers can meet many sellers in a single meeting. At the same time, it is a natural benchmark for welfare analysis since directed or competitive search is often used to decentralize the constrained efficient allocation in search-theoretic environments, as discussed in Wright, Kircher, Julien, and Guerrieri (2021).

Our model has two main features that are necessary for consumer choice.

First, search frictions within submarkets are modelled using a meeting technology that features *many-on-one meetings* (sometimes called *multilateral*). We consider the general class of invariant meeting technologies introduced in Lester, Visschers, and Wolthoff (2015). In the model, each seller meets exactly *one* buyer during any given period of time, but a buyer may meet *many* sellers. In particular, a buyer can meet either no sellers, one seller, or more than one sellers, but they can trade with only one seller per period. A *meeting* is an opportunity for buyers to choose a seller.

Second, after a meeting takes place, the buyer draws an i.i.d. preference or utility shock specific to each seller in the meeting. The buyer then chooses to purchase from the seller that maximizes their net utility. The pair consisting of a buyer and their chosen seller is called a *match*. Sellers cannot observe buyers' utility shocks; they are private information for the buyer. We interpret buyers' utility shocks as idiosyncratic preference or "taste" shocks.

With consumer choice, the distribution of utilities of *chosen* goods is endogenous and depends on the seller-buyer ratio. In particular, the seller-buyer ratio affects both the expected value of this endogenous distribution. More sellers per buyer means that each buyer can choose from a greater number of sellers (on average), which increases the average utility of the goods that are actually chosen by buyers in equilibrium. As a result, both the *average utility* of a chosen good and the *average surplus* depends directly on the seller-buyer ratio.

After choosing a seller with whom to trade, buyers choose the quantity of the good to purchase and make the corresponding payment. We focus on incentive-compatible direct revelation mechanisms that induce buyers to reveal their private information to their chosen seller. We derive a sufficient condition on the meeting technology under which existence and uniqueness of competitive search equilibrium is guaranteed.

In equilibrium, there is only one active submarket and all sellers offer the same non-linear price schedule that specifies both the quantity traded and the payment in real dollars for any given realization of the buyer's utility shock. Within any meeting, trades may or may not be liquidity constrained. Buyers may spend all of their money, some of their money, or none of it.

In terms of efficiency, there are two margins: an *extensive margin* (seller entry) and an *intensive margin* (quantity traded). With consumer choice, the extensive margin has two components since seller entry directly affects both the number of trades *and* the expected trade surplus. There may be inefficiencies on both the intensive and extensive margins. In particular, outside the Friedman rule, there are various possibilities for ranges of underconsumption and overconsumption relative to the efficient quantity, and there may be either under-entry or over-entry of sellers.

In general, the Friedman rule does not deliver efficiency along *either* the extensive or the intensive margin. First, there is underconsumption of all goods. Second, there may be either under-entry, over-entry, or efficient entry of sellers at the Friedman rule. These inefficiencies are due to the presence of private information.

In our model, there is an important and novel interaction between private information and consumer choice. In the presence of consumer choice, a higher seller-buyer ratio induces a first-order stochastic dominance shift in the distribution of chosen goods. Specifically, the distribution of chosen goods for a higher seller-buyer ratio first-order stochastically dominates the distribution for a lower seller-buyer ratio. Importantly, because buyers' utility shocks are private information, this shift in the distribution affects the extent of buyers' private information, which is endogenous in the presence of consumer choice. To illustrate the interaction between consumer choice and private information, we show that the typical value of buyers' informational rents – in a sense we define precisely – is decreasing in the seller-buyer ratio.

In this way, greater seller entry can alleviate the private information distortion by impacting the distribution of utility shocks for chosen goods, which is the source of the private information. We show that, in the competitive limit where the seller-buyer ratio becomes large, the dispersion of chosen goods goes to zero, thus eliminating the private information available to buyers. We obtain a single price equilibrium in which all buyers trade. In the competitive limit, the Friedman rule can deliver the efficient allocation despite the presence of private information. More generally, outside the competitive limit, greater seller entry can alleviate the private information distortion.

In the absence of consumer choice, this inefficiency at the Friedman rule – which is due to the private information distortion – cannot be eliminated or reduced because the distribution of chosen goods is exogenous and is not affected by seller entry.

## 2 Related literature

As discussed, our model builds on the environment in RW, which shares the alternating centralized and decentralized markets of Lagos and Wright (2005) but features endogenous seller entry. In RW, the focus is on comparing different market structures (e.g. bargaining and competitive search) that feature bilateral meetings, while our paper examines the effect of consumer choice on monetary exchange in an environment featuring private information and competitive search.

Our paper is related to the wide literature on directed and competitive search surveyed in Wright et al. (2021). In particular, we contribute to the literature on directed or competitive search and private information, including Faig and Jerez (2005), Menzio (2007), Guerrieri (2008), Guerrieri, Shimer, and Wright (2010), Moen and Rosen (2011), and Davoodalhosseini (2019). In our environment, both buyers and sellers are ex ante identical and buyers' private utility shocks are realized *after* meetings take place. Importantly, meetings are many-on-one in our environment, allowing buyers to *choose* sellers within meetings. The sequential nature of search in our model, in which buyers first choose a submarket using directed or competitive search and then face a "noisy" process of choosing or matching among the random subset of sellers they meet, shares some similarities with the model of sequentially mixed search developed in Shi (2020). However, in our model buyers' choice of seller within meetings is driven by private utility shocks rather than prices.

Related papers that feature many-on-one or multilateral meetings in monetary environments include Julien, Kennes, and King (2008) and Galenianos and Kircher (2008). Julien et al. (2008) introduces multilateral meetings and directed search into the framework of Shi (1995) and Trejos and Wright (1995) with divisible goods and indivisible money. Galenianos and Kircher (2008) develops a model featuring ex ante heterogeneity, private information, and multilateral meetings in which indivisible goods are allocated according to auctions in money holdings. In both Julien et al. (2008) and Galenianos and Kircher (2008), sellers can meet multiple buyers and either money or goods are indivisible. In our paper, by contrast, buyers can meet multiple sellers and both money and goods are divisible.[1]

While we study the effects of consumer choice, some related papers consider monetary environments featuring buyer preference shocks that are private information. Ennis (2008) incorporates private, match-specific buyer preference shocks into the monetary framework of Lagos and Wright (2005). Faig and Jerez (2006) and Dong and Jiang (2014) examine the effect of inflation on the extent of quantity discounts

[1] An alternative approach is Head and Kumar (2005), which combines the monetary search framework of Shi (1997, 1999) with the price-posting mechanism of Burdett and Judd (1983), which allows buyers to observe a random sample of prices posted by sellers and choose the lowest price. See also Herrenbrueck (2017), which extends the framework of Head and Kumar (2005).

when buyers' valuations are private information, thus extending the theory of non-linear pricing developed in Mussa and Rosen (1978) and Maskin and Riley (1984). Faig and Jerez (2006), which builds on Faig and Jerez (2005), is effectively a special case of our model in which there is no seller entry, no consumer choice, no individual rationality (IR) constraint, and the distribution of utility shocks is uniform. Dong and Jiang (2014) considers a similar environment that features an IR constraint and price posting with undirected search. More recently, Choi and Rocheteau (2021) develops a search model of retail banking in which consumers' liquidity needs are private information. All of these papers feature bilateral meetings without consumer choice.[2]

Bajaj and Mangin (2023) generalizes the current framework to allow for the possibility of meetings in which consumers are uninformed, in addition to meetings in which consumers make an informed choice. We focus on a Poisson meeting technology rather than the general meeting technology used in this paper. We use the model to study how the degree of informed choice affects the welfare cost of inflation. When we calibrate the model to U.S. money demand data, we find that a greater degree of consumer choice can significantly increase the welfare cost of inflation.

# 3 Model

Time is discrete and continues forever. Each period $t \in \{0, 1, 2, ....\}$ is divided into two subperiods as in Lagos and Wright (2005). During the day, there is a frictionless, centralized market and at night there is a frictional, decentralized market. As in RW, there is a continuum of agents divided into two types: *buyers* and *sellers*. Buyers are ex ante identical and sellers are ex ante identical. The sets of buyers and sellers are denoted $B$ and $S$ respectively. While all agents both produce and consume during the day, buyers and sellers differ at night: buyers wish to consume (but cannot produce) and sellers do not wish to consume (but can produce).

There is a fixed measure of buyers and we normalize $|B| = 1$. All buyers participate in the night market at zero cost, but there is an entry decision by sellers. Only a subset $\bar{S}_t \subseteq S$ of sellers of measure $n_t$ enter the night market. Sellers may or may not choose to enter the night market at cost $K > 0$ and thus $n_t \in \mathbb{R}_+$ is endogenous.[3]

[2] Dong (2010) considers the effect of product variety in a monetary search model where firms can invest to expand product variety. In a related paper, Silva (2017) incorporates endogenous product variety into a monetary search model featuring monopolistic competition.

[3] We assume the set $S$ is sufficiently large that $n_t \leq |S|$ always.

Since $|B| = 1$, the measure of sellers who enter, $n_t$, is also the seller-buyer ratio.

Money is perfectly divisible. The aggregate money supply at date $t$ is $M_t \in \mathbb{R}_+$, which grows at a constant rate $\gamma \in \mathbb{R}_+$, i.e. $M_{t+1} = \gamma M_t$. Money is either injected into the economy ($\gamma > 1$) or withdrawn ($\gamma < 1$) by lump sum transfers during the day. We assume these transfers are to buyers only, and we restrict attention to policies where $\gamma \geq \beta$, where $\beta$ is the discount factor. When $\gamma = \beta$ (the Friedman rule), we only consider equilibria obtained by taking the limit as $\gamma \to \beta$ from above.

In the day market, the price of goods is normalized to one and the relative price of money is denoted by $\phi_t$. The real value of a quantity of money $m_t$ held by an agent at date $t$ is defined as $z_t \equiv \phi_t m_t$ and the aggregate real money supply is $Z_t \equiv \phi_t M_t$. We will focus on steady-state equilibria where all of the aggregate real variables are constant. Since $M_{t+1}/M_t = \gamma$, this implies that in steady state $\phi_{t+1}/\phi_t = 1/\gamma$.

In the night market, prices are determined in competitive search equilibrium, which we discuss in Section 6. The night market has some novel features that enable the possibility of consumer choice.

**Many-on-one meetings.** A *meeting* is an opportunity for a buyer to choose from among a subset of sellers. While all sellers meet exactly one buyer, a buyer can meet possibly *many* sellers. In particular, each buyer can meet either no sellers, one seller, or more than one seller.

Specifically, the number of sellers in a meeting with buyer $i$ is a random variable $N_i$. For any given seller-buyer ratio $n$, the probability that a buyer meets $k \in \{0, 1, 2, ...\}$ sellers is given by a *meeting technology* $\mathbb{P}_k : \mathbb{N} \to [0, 1]$ where $\mathbb{P}_k(n) = \Pr(N_i = k) \in [0, 1]$, the probability a buyer meets $k$ sellers. The meeting technology is a discrete probability distribution with expected value $\sum_{k=0}^{\infty} n\mathbb{P}_k(n) = n$.

Let $\alpha : \mathbb{R}^+ \to [0, 1]$ be a function that represents the endogenous probability $\alpha(n)$ that a buyer meets at least one seller, i.e. $\alpha(n) \equiv 1 - \mathbb{P}_0(n)$. Since all sellers meet exactly one buyer, the probability that a seller has the opportunity to trade equals $\alpha(n)/n$, the probability that the seller's good is *chosen*.

**Buyer's choice of seller.** After a meeting takes place, the buyer draws a seller-specific random utility shock $a$ for each seller the buyer meets. The buyer then chooses a single seller with whom to trade in that subperiod.[4] The pair consisting of a buyer

[4] Similarly to standard discrete choice models, we assume that consumers choose to purchase from a single firm in each meeting.

and their chosen seller is called a *match*.

**Distribution of utility shocks.** The random utility shocks $a$ are drawn from a bounded, continuous distribution with cdf $G$ and pdf $g = G'$. This distribution is known to all agents. Importantly, the realizations of the utility shocks are not observed by sellers; they are private information for the buyer.

We assume that the distribution $G$ is not degenerate, and Assumptions 1 and 2 are both maintained throughout the paper.

**Assumption 1.** *The distribution of utility shocks has twice-differentiable cdf $G$, where $G' > 0$ and $G'' \leq 0$, and $G$ has bounded support $A = [a_0, \bar{a}] \subseteq \mathbb{R}_+$.*

**Assumption 2.** *The distribution of utility shocks has an increasing hazard rate, i.e. $h'_G(a) \geq 0$ where $h_G : A \to \mathbb{R}$ is defined by*

$$h_G(a) \equiv \frac{g(a)}{1 - G(a)}. \tag{1}$$

This assumption implies that the virtual valuation function is strictly increasing, where the virtual valuation function $\psi_G : A \to \mathbb{R}$ is defined by $\psi_G(a) \equiv a - \frac{1}{h_G(a)}$.

**Buyer and seller utility.** Sellers can produce on demand any quantity $q \in \mathbb{R}_+$ of a divisible good and the cost of production is $c(q)$, where $c : \mathbb{R}_+ \to \mathbb{R}_+$ and we assume that $c(0) = 0$, $c'(q) > 0$, and $c''(q) \geq 0$ for all $q > 0$. A buyer who consumes quantity $q$ of a good with utility $a$ receives utility $au(q)$, where $u : \mathbb{R}_+ \to \mathbb{R}_+$ and we assume that $u(0) = 0$, $u'(0) = \infty$, $u'(q) > 0$, and $u''(q) < 0$ for all $q > 0$.

The instantaneous utility of a buyer who meets a seller at night at date $t$ is

$$U_t^b = \nu(x_t) - y_t + \beta E_{\tilde{G}_t}(au(q_{a,t})), \tag{2}$$

and the instantaneous utility of a seller who is chosen by a buyer at night at date $t$ is

$$U_t^s = \nu(x_t) - y_t - \beta E_{\tilde{G}_t}(c(q_{a,t})), \tag{3}$$

where $x_t$ is the quantity consumed and $y_t$ is the quantity produced during the day, $q_{a,t}$ is the quantity consumed at night, $a$ is the *utility* of the good consumed, and $\tilde{G}_t$ is the *distribution of chosen goods* at time $t$, which we introduce in the next section.

We assume $\nu'(x) > 0$ and $\nu''(x) < 0$ for all $x$, and that there exists $x^*$ such that $\nu'(x^*) = 1$. We normalize $\nu(x^*) - x^* = 0$.

## 4 Distribution of chosen goods

The distribution of utility shocks of the goods actually *chosen* by buyers is endogenous. For any $n \in \mathbb{R}_+$ this distribution depends on both the equilibrium seller-buyer ratio $n$ and the equilibrium choices made by buyers. For brevity, we refer to $G$ simply as the *distribution of available goods* and $\tilde{G}$ as the *distribution of chosen goods*. In this section, we derive the distribution of chosen goods $\tilde{G}$ and some of its properties.

The distribution of chosen goods depends on $n$, the ratio of sellers to buyers, and on the equilibrium choices by buyers regarding which seller to purchase from. We will later prove that, in any equilibrium we consider, buyers always choose the highest utility seller they meet.[5] Therefore, the distribution of chosen goods equals the distribution across buyers of the highest utility shock $a$ among the sellers a buyer meets, conditional on meeting a seller. It is this distribution we will discuss here.

Throughout the remainder of this paper, it is convenient to assume that the meeting technology $\mathbb{P}_k$ is *invariant*, as defined in Lester et al. (2015).

**Assumption 3.** *The meeting technology $\mathbb{P}_k$ is* invariant*, i.e. for all $y \in [0, 1]$,*

$$\sum_{k=0}^{\infty} \mathbb{P}_k(n) y^k = \mathbb{P}_0(n(1-y)) \tag{4}$$

*where $\mathbb{E}_{\mathbb{P}}(N_i) = n$ and $\mathbb{P}_0 : \mathbb{R}^+ \to [0, 1]$ is continuous and infinitely differentiable.*

The assumption of invariance is useful because the function $\mathbb{P}_0$ captures everything we need to know about the meeting technology. Examples of invariant search technologies include the Poisson distribution, the Geometric distribution, and the entire family of negative binomial distributions. For further details, see Lester et al. (2015).

Lemma 1 presents some useful properties of the matching elasticity that follow from the invariance assumption. The matching elasticity will be important when we consider efficiency of entry and it is defined by:

$$\eta_\alpha(n) \equiv \frac{\alpha'(n)n}{\alpha(n)}. \tag{5}$$

[5] We will also show that this distribution is the one that reflects the social planner's choices.

**Lemma 1.** *If $\mathbb{P}_k$ is invariant and $n > 0$, then*

1. *The matching elasticity $\eta_\alpha$ is decreasing in $n$, i.e. $\eta'_\alpha(n) < 0$.*
2. *We have $\lim_{n\to 0} \eta_\alpha(n) = 1$, $\lim_{n\to\infty} \eta_\alpha(n) = 0$, and $\eta_\alpha(n) \in (0, 1)$.*

For invariant meeting technologies, Lemma 2 presents an expression for the distribution of chosen goods.

**Lemma 2.** *Suppose that $\mathbb{P}_k$ is invariant and $n > 0$. For all $a \in A$, the cdf of the distribution of chosen goods is given by*

$$\tilde{G}(a; n) = \frac{\mathbb{P}_0(n(1 - G(a)) - \mathbb{P}_0(n)}{\alpha(n)}. \tag{6}$$

Lemma 3 states that the distribution of chosen goods first-order stochastically dominates the distributions of available goods, and the average utility of a *chosen* good $\tilde{a}(n) \equiv E_{\tilde{G}}(a)$ is greater than the average utility of an *available* good, $E_G(a)$. In the limit as $n \to 0$, the distribution of chosen goods converges to the distribution of available goods. In the limit as $n \to \infty$, the distribution of chosen goods converges to a degenerate distribution with support $A = \{\bar{a}\}$.

**Lemma 3.** *Suppose that $\mathbb{P}_k$ is invariant and $n > 0$.*

1. *In the limit as $n \to 0$, we have $\tilde{G}(a; n) \to G(a)$ and $\tilde{a}(n) \to E_G(a)$.*
2. *In the limit as $n \to \infty$, we have $\tilde{G}(a; n) \to 0$ for all $a \in [a_0, \bar{a})$, and $\tilde{a}(n) \to \bar{a}$.*
3. *The distribution of chosen goods $\tilde{G}(a; n)$ first-order stochastically dominates the distribution of available goods $G(a)$ and $\tilde{a}(n) > E_G(a)$.*

We now state an assumption that will prove useful. This assumption can be shown to hold for all distributions in the negative binomial family (e.g. the Poisson and the Geometric distributions).

**Assumption 4.** *The meeting technology $\mathbb{P}_k$ is invariant and*

$$\frac{d}{dx}\left(\frac{-\mathbb{P}''_0(x)x}{\mathbb{P}'_0(x)}\right) > 0. \tag{7}$$

Lemma 4 implies, among other things, that the average utility of a chosen good $\tilde{a}(n)$ is strictly increasing in $n$, i.e. $\tilde{a}'(n) > 0$. Intuitively, average utility is increasing in the seller-buyer ratio because more sellers per buyer means greater choice of seller.

**Lemma 4.** *Suppose that $\mathbb{P}_k$ satisfies Assumption 4 and $n > 0$. For any $f : A \to \mathbb{R}_+$ such that $f' > 0$, we have $\tilde{f}'(n) > 0$ where $\tilde{f}(n) \equiv \int_{a_0}^{\bar{a}} f(a) d\tilde{G}(a; n)$.*

Lemma 4 is useful in its own right, but it also implies Corollary 1.

**Corollary 1.** *Suppose that $\mathbb{P}_k$ satisfies Assumption 4. If $n > n' > 0$, the distribution $\tilde{G}(a; n)$ first-order stochastically dominates the distribution $\tilde{G}(a; n')$.*

The distribution of chosen goods $\tilde{G}(a; n)$ has hazard rate $h_{\tilde{G}}(a; n)$ defined by

$$h_{\tilde{G}}(a; n) \equiv \frac{\tilde{g}(a; n)}{1 - \tilde{G}(a; n)}. \tag{8}$$

Lemma 5 provides an expression for the hazard rate of $\tilde{G}$ and states that this hazard rate is increasing for any invariant meeting technology $\mathbb{P}_k$. Therefore, $\tilde{G}$ has an increasing virtual valuation function. This result follows immediately from Lemma 1, which implies that $\eta_\alpha$ is decreasing, plus our assumption that $G$ has an increasing hazard rate, $h_G(a)$.

**Lemma 5.** *Suppose that $\mathbb{P}_k$ is invariant and $n > 0$.*

1. *The hazard rate of the distribution of chosen goods $\tilde{G}(a; n)$ is given by*

$$h_{\tilde{G}}(a; n) = \eta_\alpha(n(1 - G(a))) h_G(a). \tag{9}$$

2. *The distribution of chosen goods $\tilde{G}(a; n)$ has an increasing hazard rate, i.e.*

$$h'_{\tilde{G}}(a; n) \equiv \frac{\partial h_{\tilde{G}}(a; n)}{\partial a} \geq 0. \tag{10}$$

Lemma 6 states that, for any given seller-buyer ratio $n > 0$, there is a unique utility shock $a_f(n)$ such that the two densities cross, i.e. $\tilde{g}(a; n) = g(a)$. The value $a_f(n)$ is a fixed point of the transformation from $g(a)$ to $\tilde{g}(a; n)$ that is induced by consumer choice. Roughly speaking, this utility shock is "equally likely" as a draw from both the distribution of available goods and the distribution of chosen goods.

For all goods with higher utility shocks, i.e. $a > a_f(n)$, the density of chosen goods $\tilde{g}(a;n)$ is higher than the density of available goods $g(a)$. So, the range of utilities $[a_f(n), \bar{a}]$ is *over-represented* with choice. For all goods with lower utility shocks, i.e. $a < a_f(n)$, the density of chosen goods $\tilde{g}(a;n)$ is higher than the density of available goods $g(a)$. So, the range of utilities $[a_0, a_f(n)]$ is *under-represented* with choice.

**Lemma 6.** *Suppose that $\mathbb{P}_k$ is invariant and $n > 0$. There exists a unique $a_f(n)$ such that $\tilde{g}(a;n) = g(a)$. We have $\tilde{g}(a;n) > g(a)$ if and only if $a > a_f(n)$. Moreover, $a_f'(n) > 0$ and $\lim_{n\to\infty} a_f(n) = \bar{a}$.*

Figure 1 compares the density $\tilde{g}(a;n)$ of the distribution of chosen goods (for $n = 3$ and $n = 10$) and the density of the distribution of available goods $g(a)$. In this example, $G$ is uniform on $[0, 1]$ and $\mathbb{P}_k$ is Poisson.

For any given seller-buyer ratio $n$, the density $\tilde{g}(a;n)$ is lower than the density $g(a)$ for low values, $a < a_f(n)$, and higher for high values, $a > a_f(n)$. The unique intersection of these two densities is at the value $a_f(n)$. Intuitively, we observe this shift in density from $g(a)$ to $\tilde{g}(a;n)$ because consumer choice shifts the distribution towards the higher utility goods that are chosen by buyers. This effect is stronger when the seller-buyer ratio $n$ is higher because buyers have greater choice.

Figure 1 also suggests that the dispersion of chosen goods is lower than that of available goods. The utility shocks of chosen goods are closer together because they are closer to the upper bound $\bar{a}$ as a result of choice. Again, this effect is stronger when the seller-buyer ratio $n$ is higher. This suggests that the effect of private information, which is endogenous here, may be decreasing in the level of seller entry when there is consumer choice. Later, in Section 11, we will discuss exactly how seller entry and consumer choice affect the informational rents accruing to buyers in equilibrium.

# 5 Planner's problem

Before we consider competitive search equilibrium, we solve the planner's problem. We assume the planner is constrained by the same search frictions and meeting technology as the decentralized market. We say that the planner's solution achieves the constrained efficient allocation, which we refer to simply as the *efficient allocation*.

Given the cost of seller entry $K > 0$, the planner chooses a seller-buyer ratio $n^*$, a function $q^* : A \to \mathbb{R}_+$, and a distribution of chosen goods $\tilde{G} : A \to [0, 1]$, to maximize

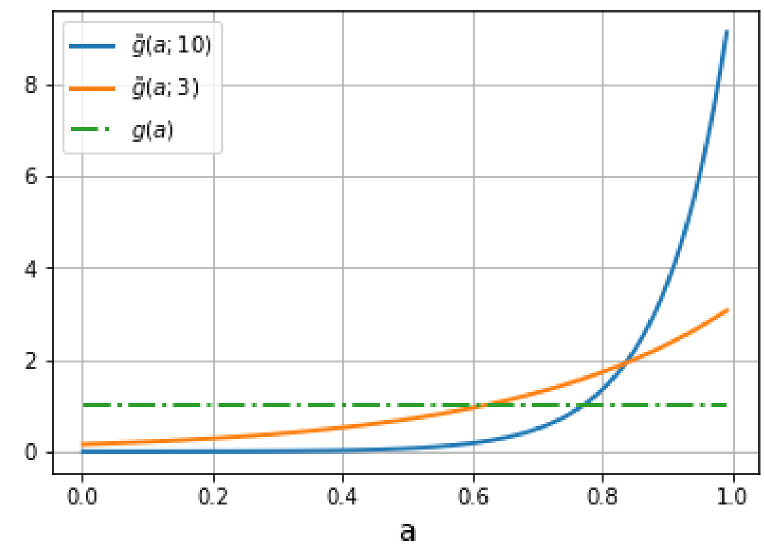


Figure 1: Example of densities of distributions of chosen vs available goods for $n = 3$ and $n = 10$. *Note:* $G$ is uniform on $[0, 1]$ and meeting technology is Poisson.

the total surplus created minus the total cost of seller entry, subject to the constraints he faces. That is, the planner solves the following problem:

$$\max_{n \in \mathbb{R}_+, \{q_a\}_{a \in A}} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[ au(q_a) - c(q_a) \right] d\tilde{G}(a; n) - nk \right\} \tag{11}$$

where $\tilde{G}$ represents the planner's optimal choice of seller for each buyer.[6] The planner must take into account the fact that buyers' expected utility from consumption in the night market depends not only on the meeting probability and the quantity of goods traded, but also on the expected utility of the good purchased.

Define $s_a \equiv au(q_a) - c(q_a)$, the trade surplus (or match surplus) for a good of utility $a$. Let $q_a^*$ denote the efficient quantity of good $a$ and define $s_a^* \equiv au(q_a^*) - c(q_a^*)$. Assume that $s_0^* \geq 0$ where $s_0^* \equiv a_0 u(q_0) - c(q_0)$ and $q_0 = q(a_0)$, so there is (weakly) positive trade surplus for all goods. Define the *expected trade surplus* by

$$\tilde{s}(n; \{q_a\}_{a \in A}) \equiv \int_{a_0}^{\bar{a}} [au(q_a) - c(q_a)] d\tilde{G}(a; n). \tag{12}$$

[6] The planner's distribution of chosen goods will turn out to be equal to the buyers' distribution of chosen goods, so we use the same notation, $\tilde{G}$, for simplicity.

For simplicity of notation, throughout the paper we sometimes suppress the dependence of the expected trade surplus $\tilde{s}(n;\{q_a\}_{a\in A})$ on the function $q : A \to \mathbb{R}_+$ and let $\tilde{s}(n)$ denote $\tilde{s}(n;\{q_a\}_{a\in A})$ and $\tilde{s}'(n)$ denote $\partial\tilde{s}(n)/\partial n$.

The following assumption ensures the existence of a social optimum where $n^* > 0$. Intuitively, this condition says that the expected trade surplus in the limit as $n \to 0$, i.e. $\lim_{n\to 0} \tilde{s}(n)$, must be greater than $K$.[7] It follows from our assumptions that, for all $a \in A$, there exists a unique $q_a^* \in \mathbb{R}_+$ such that $au'(q_a^*) = c'(q_a^*)$.

**Assumption 5.** *The cost of entry is not too high:* $E_G[au(q_a^*) - c(q_a^*)] > K$.

We are now ready to describe the planner's solution. Proposition 1 states that there exists a unique social optimum $(n^*, \{q_a^*\}_{a\in A})$ with $n^* > 0$ and provides the necessary conditions for an efficient allocation.

**Proposition 1.** *If* $\mathbb{P}_k$ *is invariant, there exists a unique social optimum* $(n^*, \{q_a^*\}_{a\in A})$ *and it satisfies:*

1. *For any* $a \in A$, *the quantity* $q_a^* > 0$ *solves*

$$au'(q_a^*) = c'(q_a^*). \tag{13}$$

2. *The seller-buyer ratio* $n^* > 0$ *satisfies*

$$\alpha'(n^*)\tilde{s}(n^*;\{q_a^*\}_{a\in A}) + \alpha(n^*)\tilde{s}'(n^*;\{q_a^*\}_{a\in A}) = K. \tag{14}$$

3. *The distribution of chosen goods is given by (6).*

Equation (14) can be rewritten as a version of the *generalized Hosios condition* derived in Mangin and Julien (2021). This result generalizes the well-known Hosios (1990) condition states that entry is constrained efficient only if sellers' surplus share equals the matching elasticity. To see the connection, defining the matching elasticity by (5) and the surplus elasticity by $\eta_s(n) \equiv \tilde{s}'(n)n/\tilde{s}(n)$, condition (14) says

$$\underbrace{\eta_\alpha(n)}_{\text{matching elasticity}} + \underbrace{\eta_s(n;\{q_a\}_{a\in A})}_{\text{surplus elasticity}} = \underbrace{\frac{nk}{\alpha(n)\tilde{s}(n;\{q_a\}_{a\in A})}}_{\text{sellers' surplus share}}. \tag{15}$$

[7] Since $\tilde{G} \to G$ as $n \to 0$, as verified in Lemma 3, we have $\lim_{n\to 0} \tilde{s}(n) = E_G[au(q_a^*) - c(q_a^*)]$

We have not yet discussed equilibrium, but it is useful to refer to the term on the right as the sellers' surplus share. Given that our equilibrium features free entry of sellers at cost $K$, sellers' total expected payoff will be equal to the total cost of seller entry, $nk$, and the total surplus created is $\alpha(n)\tilde{s}(n)$. Therefore, the term on the right will be sellers' surplus share in equilibrium. The generalized Hosios condition (15) says that constrained efficiency requires sellers' surplus share to be equal to the matching elasticity plus the surplus elasticity.

Since $s_a^*$ is increasing in $a$, Lemma 3 implies that the expected trade surplus $\tilde{s}(n)$ is increasing in the seller-buyer ratio, i.e. $\tilde{s}'(n) > 0$.[8] Therefore, the surplus elasticity $\eta_s(n)$ is positive. Intuitively, more sellers per buyer means greater choice for buyers, which increases both the average utility of chosen goods and the quantities traded (since $q_a^*$ is increasing in $a$), thus increasing the average trade surplus. Equivalently, there is a positive externality arising from the effect of seller entry on the average surplus when there is consumer choice. When the generalized Hosios condition (15) holds, both the search externalities and this "choice externality" are internalized.

**No choice.** When there is no consumer choice, (14) is the standard Hosios condition. In this case, we have $\alpha'(n^*)\tilde{s}(\{q_a^*\}_{a\in A}) = K$ where $\tilde{s}(\{q_a\}_{a\in A}) \equiv \int_{a_0}^{\bar{a}} [au(q_a) - c(q_a)]dG(a)$. This is because $\tilde{G}(a;n) = G(a)$ and the expected surplus no longer depends on the seller-buyer ratio $n$ because there is no "choice externality".

# 6 Competitive search equilibrium

Competitive search is an equilibrium concept developed in Moen (1997) and Shimer (1996). A large literature on directed or competitive search has followed. The basic idea is that either buyers or sellers, or market makers, can post prices or contracts that specify the terms of trade offered. Search is directed in the sense that buyers and sellers choose which *submarket* to enter, where each submarket corresponds to a particular specification of the terms of trade. Commitment is key: buyers and sellers who enter a submarket *commit* to trade at the terms specified within that submarket. Within each submarket, there are search frictions.

As in Rocheteau and Wright (2005), we assume there are agents called "market

[8] It is established in the proof of Proposition 1 that both $q_a^*$ and $s_a^*$ are increasing in $a$.

makers" who can open submarkets by posting terms of trade or contracts.[9] Market makers take into account the expected relationship between the posted terms of trade or contracts and the seller-buyer ratio $n$. In our environment, market makers post contracts $\{(q_a, d_a)\}_{a\in A}$ which specify the quantity of the good $q_a$ and the payment in real dollars $d_a$ *contingent on the buyer's utility shock for their chosen seller.*

Within each submarket, meetings take place, buyers choose sellers, and trade occurs as described in Section 3.

Within meetings, buyers' utility shocks are private information and they cannot be observed directly by any seller (including their chosen seller). However, buyers may choose to reveal their private information within matches through their choice of contract $(q_a, d_a)$ offered by the chosen seller. By the revelation principle, it is without loss of generality to focus on incentive-compatible direct mechanisms $\{(q_a, d_a)\}_{a\in A}$ that induce buyers to truthfully reveal their private information to their chosen sellers.

Within each period, the timing is as follows. At the start of each day, the market makers announce the submarkets $\{(q_a, d_a)\}_{a\in A}$ that will be open that night, implying an expected $n$ for each submarket. During the day, agents trade in the centralized market and readjust their real balances, and then choose a submarket in which to trade at night, in a manner consistent with expectations. During the night, agents trade goods and money in the decentralized market in their chosen submarket, where they are bound by the posted contracts $\{(q_a, d_a)\}_{a\in A}$ in that submarket.

Let $\Omega$ denote the set of open submarkets, where each submarket $\omega \in \Omega$ is characterized by $(\{(q_a, d_a)\}_{a\in A}, n)_\omega$. Let $W^b$ and $W^s$ denote the value functions for buyers and sellers respectively in the day market, and let $V^b$ and $V^s$ denote the value functions for buyers and sellers respectively in the night market.

**Centralized market.** In the CM, a buyer with real balance $z$ solves:

$$W^b(z) = \max_{\hat{z},x,y\in\mathbb{R}_+} \{\nu(x) - y + \beta V^b(\hat{z})\}, \tag{16}$$

[9] While these "market makers" are not able to clear the market, we use this term in order to be consistent with the terminology in Rocheteau and Wright (2005).

subject to $\hat{z} + x = z + T + y$, where $T$ is her real transfer and $\hat{z}$ is the real balances carried forward into that period's decentralized market. Substituting into (16) yields

$$W^b(z) = z + T + \max_{\hat{z},x\in\mathbb{R}_+} \{\nu(x) - x - \hat{z} + \beta V^b(\hat{z})\}. \tag{17}$$

Thus, the buyer's $\hat{z}$ is independent of $z$, and $W^b(z) = z + W^b(0)$, which is linear.

Similarly, a seller with real balance $z_s$ in the centralized market solves:

$$W^s(z_s) = \max_{\hat{z},x,y\in\mathbb{R}_+} \left\{\nu(x) - y + \beta \max\left[V^s(\hat{z}), W^s\left(\frac{\hat{z}}{\gamma}\right)\right]\right\}, \tag{18}$$

subject to $\hat{z} + x = z_s + y$. Substituting into (18), we obtain

$$W^s(z_s) = z_s + \max_{\hat{z},x\in\mathbb{R}_+} \left\{\nu(x) - x - \hat{z} + \beta \max\left[V^s(\hat{z}), W^s\left(\frac{\hat{z}}{\gamma}\right)\right]\right\}. \tag{19}$$

Thus, the seller's $\hat{z}$ is independent of $z_s$, and $W^s(z_s) = z_s + W^s(0)$.

**Decentralized market.** The equilibrium distribution of chosen goods $\tilde{G}$ is given by buyers' optimal choices of sellers. In any meeting, the buyer chooses the seller that maximizes $v_a \equiv au(q_a) - d_a/\gamma$, the buyer's ex post trade surplus.

For a seller in the decentralized night market,

$$V^s(z_s) = \max_{\omega\in\Omega} \left\{ \begin{array}{c} \frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[-c(q_a) + W^s\left(\frac{z_s+d_a}{\gamma}\right)\right] d\tilde{G}(a;n) \\ + \left[1 - \frac{\alpha(n)}{n}\right] W^s\left(\frac{z_s}{\gamma}\right) \end{array} \right\} - K \tag{20}$$

where each submarket $\omega \in \Omega$ is characterized by $(\{(q_a, d_a)\}_{a\in A}, n)$. A seller chooses $\omega$ among the set of open submarkets and has the opportunity to trade only if chosen. While all sellers meet exactly one buyer, the probability a seller is *chosen* is $\alpha(n)/n$. It is straightforward to verify that the seller's choice of real balances is $\hat{z} = 0$.[10]

For a buyer in the decentralized night market,

$$V^b(z) = \max_{\omega\in\Omega} \left\{ \begin{array}{c} \alpha(n) \int_{a_0}^{\bar{a}} \mathbf{1}_a \left[au(q_a) + W^b\left(\frac{z-d_a}{\gamma}\right)\right] d\tilde{G}(a;n) \\ + \left[1 - \alpha(n) \int_{a_0}^{\bar{a}} \mathbf{1}_a \, d\tilde{G}(a;n)\right] W^b\left(\frac{z}{\gamma}\right) \end{array} \right\} \tag{21}$$

[10]Using $W^s(z_s) = z_s + W^s(0)$, (20) simplifies to $V^s(z_s) = z_s/\gamma + V^s(0)$. Substituting into (19), the choice of $\hat{z}$ is given by the first order condition $-1 + \beta/\gamma \leq 0$, where $-1 + \beta/\gamma = 0$ if $\hat{z} > 0$. Since we only consider the case $\gamma = \beta$ by taking the limit as $\gamma \to \beta$ from above, $\hat{z} = 0$.

where $\mathbf{1}_a$ is an indicator function that is equal to one if $z \geq d_a$ and zero otherwise. A buyer chooses $\omega$ among the set of open submarkets and gets the opportunity to trade if she meets at least one seller and has sufficient money $z$ to pay the posted $d_a$ for her chosen good. If she either fails to meet a seller or does not have sufficient money, she does not trade. Using $W^b(z) = z + W^b(0)$ we obtain

$$V^b(z) = \max_{\omega \in \Omega} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \mathbf{1}_a \left[ au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a; n) + \frac{z}{\gamma} + W^b(0) \right\}. \tag{22}$$

Thus, the buyer's choice of $z$ from (17) is given by

$$\max_{z \in \mathbb{R}_+} \left\{ -z + \beta \max_{\omega \in \Omega} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[ au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a; n) + \frac{z}{\gamma} \right\} \right\} \tag{23}$$

subject to the liquidity constraint, $d_a \leq z$ for all $a \in A$.

Defining $i \equiv \frac{\gamma - \beta}{\beta}$, the nominal interest rate, the above problem is equivalent to

$$\max_{z \in \mathbb{R}_+,\, \omega \in \Omega} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[ au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a; n) - i \frac{z}{\gamma} \right\}, \tag{24}$$

subject to $d_a \leq z$ for all $a \in A$ plus the constraint that a submarket with posted contracts $\{(q_a, d_a)\}_{a \in A}$ will attract measure $n$ of sellers per buyer, where $n$ satisfies

$$\frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[ -c(q_a) + \frac{d_a}{\gamma} \right] d\tilde{G}(a; n) \leq K \tag{25}$$

and $n \geq 0$ with complementary slackness.

Before presenting the results with private information, we first describe the full information benchmark. Later, when we define equilibrium with private information, there will be additional constraints imposed on problem (24).

# 7 Full information benchmark

We first briefly present the *full information* equilibrium as a benchmark. By "full information," we refer to a version of the model in which, prior to trade occurring, the *chosen* seller can directly observe the buyer's utility shock for that seller.

For simplicity, since we present the full information case only as a benchmark, we do not consider buyers' individual rationality (IR) constraint. That is, we assume that buyers commit to trading when they enter a submarket. For the full model with private information, we introduce the IR constraint for buyers in addition to an incentive compatibility (IC) constraint.

We define competitive search equilibrium for the full information case as follows. We restrict attention to steady-state monetary equilibria where $z > 0$ and $n > 0$. It will turn out that there is a unique solution to the market makers' problem and there is only one active submarket in equilibrium. Anticipating this result, we simply denote equilibrium by $(\{(q_a, d_a)\}_{a \in A}, z, n)$ and define it as follows.

**Definition 1.** *A* full-info competitive search equilibrium *is a list* $(\{(q_a, d_a)\}_{a \in A}, z, n)$ *and a distribution of chosen goods* $\{\tilde{G}(a; n)\}_{a \in A}$ *where* $(q_a, d_a) \in \mathbb{R}^2_+$ *for all* $a \in A$*,* $\tilde{G}(a; n) \in [0, 1]$ *for all* $a \in A$*, and* $z, n \in \mathbb{R}_+ \backslash \{0\}$*, such that* $(\{(q_a, d_a)\}_{a \in A}, z, n)$ *maximizes (24) subject to constraint (25) and the liquidity constraint* $d_a \leq z$ *for all* $a \in A$*, and* $\{\tilde{G}(a; n)\}_{a \in A}$ *represents buyers' optimal choices of sellers.*

## 7.1 No choice

As a benchmark, we first characterize equilibrium in our model *without consumer choice*, which is very closely related to Rocheteau and Wright (2005). To do this, we assume that buyers cannot choose sellers but instead are randomly matched with sellers in meetings. This is equivalent to an environment with bilateral (one-on-one) meetings where the probability a buyer meets a seller is $\alpha(n)$.

**Proposition 2.** *Without consumer choice, for any* $i > 0$ *there exists a unique full-information competitive search equilibrium and it satisfies:*

1. *For any* $a \in A$*, the quantity* $q_a > 0$ *solves*

$$au'(q_a) = \left(1 + \frac{i}{\alpha(n)}\right) c'(q_a), \tag{26}$$

   *or equivalently,*

$$(1 - \phi^{RW}) au'(q_a) = c'(q_a) \tag{27}$$

*where*

$$\phi^{RW} = \frac{i}{i + \alpha(n)}. \tag{28}$$

2. *The seller-buyer ratio $n > 0$ is strictly decreasing in $K$ and satisfies*

$$\frac{\alpha'(n)\tilde{s}(\{q_a\}_{a \in A})}{1 + (1 - \eta_\alpha(n))\frac{i}{\alpha(n)}} = K. \tag{29}$$

3. *We have $d_a = z$ for all $a \in A$ where $z > 0$ is given by*

$$\frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[ -c(q_a) + \frac{z}{\gamma} \right] dG(a) = K. \tag{30}$$

4. *The distribution of chosen goods is $G$.*

With full information, buyers spend all of their money holdings, $d_a = z$. However, the quantities traded $q_a$ differ across meetings and depend on the buyer's utility shock $a$. Both money holdings $z$ and the equilibrium seller-buyer ratio $n$ depend on the quantities traded at each utility shock $a$ as well as on the distribution of utility shocks $G$. The term $\phi^{RW}$ represents the degree of underconsumption relative to the efficient quantity $q_a^*$. The term $\phi^{RW}$, and thus the degree of underconsumption, is increasing in the nominal interest rate.

As in RW, the equilibrium condition (29) is equal to the standard Hosios condition, $\alpha'(n)s(q) = K$, which is required for efficiency of entry, only at the Friedman rule, i.e. $i \to 0$. We can recover the results in RW as a limiting case of Proposition 2 where $A \to \{1\}$. In RW, there are no utility shocks, a single quantity $q$ is traded, a single payment $d$ is made, and all trades generate the same surplus.

## 7.2 Consumer choice

We now consider what happens when we introduce consumer choice. In this case, the distribution of chosen goods $\tilde{G}$ is endogenous and depends on the equilibrium seller-buyer ratio.[11] With consumer choice, a version of the generalized Hosios condition holds in competitive search equilibrium.

[11] We omit the proof of Proposition 3 because it is lengthy and can be derived from the proof of Proposition 6 by eliminating the IR and IC constraints.

**Proposition 3.** *Suppose that $\mathbb{P}_k$ is invariant. For any $i > 0$, there exists a unique full-information competitive search equilibrium and it satisfies:*

1. *For any $a \in A$, the quantity $q_a > 0$ solves*

$$au'(q_a) = \left(1 + \frac{i}{\alpha(n)}\right) c'(q_a), \tag{31}$$

   *or equivalently,*

$$(1 - \phi^{RW})au'(q_a) = c'(q_a). \tag{32}$$

2. *The seller-buyer ratio $n > 0$ is strictly decreasing in $K$ and satisfies*

$$\alpha'(n)\tilde{s}(n; \{q_a\}_{a \in A}) + \alpha(n)\tilde{s}'(n; \{q_a\}_{a \in A}) = K. \tag{33}$$

3. *We have $d_a = z$ for all $a \in A$ where $z > 0$ is given by*

$$\frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[ -c(q_a) + \frac{z}{\gamma} \right] d\tilde{G}(a; n) = K. \tag{34}$$

4. *The distribution of chosen goods is given by (6).*

The only difference between the equilibrium condition (33) and the planner's condition (14) is that the quantities $q_a$ traded in equilibrium may be different than the efficient quantities $q_a^*$. Since the expected match surplus $\tilde{s}(n; \{q_a\}_{a \in A})$ depends not only on the seller-buyer ratio $n$ but also on the quantities $q_a$, seller entry is not necessarily efficient. Instead, seller entry is efficient *provided that the quantity traded is efficient*, i.e. $q_a = q_a^*$ for all $a \in A$.

By contrast, in the environment without choice (including in RW), the Hosios-like equilibrium condition (29) depends directly on the nominal interest rate $i$. This condition is equivalent to the standard Hosios condition because the average trade surplus is $\tilde{s}(\{q_a\}_{a \in A}) \equiv \int_{a_0}^{\bar{a}} [au(q_a) - c(q_a)] dG(a)$, which no longer depends on the seller-buyer ratio $n$. In contrast to the model with choice, this condition is equivalent (conditional on quantities $q_a$) to the standard Hosios condition $\alpha'(n)\tilde{s}(\{q_a\}_{a \in A})$ *only in the case where $i \to 0$.*

### 7.3 Consumption and entry

There are two margins for efficiency: the *intensive margin* (related to quantity traded or consumption) and the *extensive margin* (related to seller entry).

We say that there is *underconsumption* of any good of utility $a$ whenever the quantity traded in equilibrium is less than the efficient quantity, i.e. $q_a < q_a^*$, and there is *overconsumption* whenever $q_a > q_a^*$. We say that there is *under-entry* of sellers when the equilibrium seller-buyer ratio is less than the efficient ratio, i.e. $n < n^*$, and there is *over-entry* whenever $n > n^*$.

Proposition 4 states that there are inefficiencies on both the intensive and extensive margin when there is full information. In particular, when $i > 0$, there is underconsumption and there may be either under-entry, over-entry, or efficient entry of sellers. The Friedman rule ($i \to 0$) delivers efficiency on both margins.

**Proposition 4.** *In any full information competitive search equilibrium,*

1. *If $i > 0$, there is underconsumption for all $a \in (a_0, \bar{a}]$.*

2. *If $i > 0$, there may be either under-entry, over-entry, or efficient entry of sellers.*

3. *The Friedman rule delivers efficiency, i.e. $n = n^*$ and $q_a = q_a^*$ for all $a \in A$.*

With full information, when the Friedman rule holds, i.e. $i \to 0$, the equilibrium conditions (31) and (33) reduce to the planner's first-order conditions, (13) and (14). In particular, (33) is equivalent to generalized Hosios condition when $i \to 0$. This means that both the search externalities and the "choice externality" (which arises due to consumer choice) are internalized. With competitive search, the Friedman rule thus delivers efficiency along both the intensive margin (i.e. $q_a = q_a^*$ for all $a \in A$) and the extensive margin (i.e. $n = n^*$). Importantly, with consumer choice, the extensive margin has two components because seller entry affects both the trading probability for buyers *and* the size of the expected surplus. Because the generalized Hosios condition holds endogenously at the Friedman rule, both components are internalized because seller entry follows automatically from efficiency of trading quantities.

As a benchmark, we also describe these results without consumer choice.

**Proposition 5.** *Without consumer choice, in any full-information equilibrium,*

1. *If $i > 0$, there is underconsumption for all $a \in (a_0, \bar{a}]$.*

2. *If $i > 0$, there is under-entry of sellers.*

3. *The Friedman rule delivers efficiency, i.e. $n = n^*$ and $q_a = q_a^*$ for all $a \in A$.*

Without consumer choice, including in the special case of RW, there is underconsumption outside the Friedman rule, i.e. $q_a < q_a^*$, and there is always under-entry, i.e. $n < n^*$. At the Friedman rule ($i \to 0$), the efficient quantity is traded and the equilibrium condition (29) reduces to the standard Hosios condition, so seller entry is efficient too.

# 8 Private information equilibrium

We now turn to our main case where buyers have private information about their utility shocks. That is, buyers' utility shocks cannot be observed by any sellers, including their chosen seller. We focus on incentive-compatible direct mechanisms that induce buyers to reveal their private information to their chosen sellers.

Given this, we need to impose on problem (24) two additional constraints: an incentive compatibility (IC) constraint and an individual rationality (IR) constraint. The IR constraint for buyers is

$$au(q_a) - \frac{d_a}{\gamma} \geq 0 \tag{35}$$

for all $a \in A$. This condition states that buyers must receive a (weakly) positive ex post trade surplus, otherwise they will not trade. The IC constraint is given by

$$au(q_a) - \frac{d_a}{\gamma} \geq au(q_{a'}) - \frac{d_{a'}}{\gamma} \tag{36}$$

for all $a, a' \in A$. Intuitively, this condition states that a buyer with utility shock $a$ cannot do better by choosing a contract $(q_{a'}, d_{a'})$ instead of $(q_a, d_a)$.

We restrict attention to steady-state monetary equilibria where $z > 0$ and $n > 0$. We will later prove that there is a unique solution to the market makers' problem and thus there is only one active submarket in equilibrium. Anticipating this result, we simply denote equilibrium by $(\{(q_a, d_a)\}_{a \in A}, z, n)$ and define it as follows.

**Definition 2.** *A* competitive search equilibrium *is a list $(\{(q_a, d_a)\}_{a \in A}, z, n)$ and a distribution of chosen goods $\{\tilde{G}(a; n)\}_{a \in A}$ where $(q_a, d_a) \in \mathbb{R}_+^2$ for all $a \in A$, $\tilde{G}(a; n) \in$*

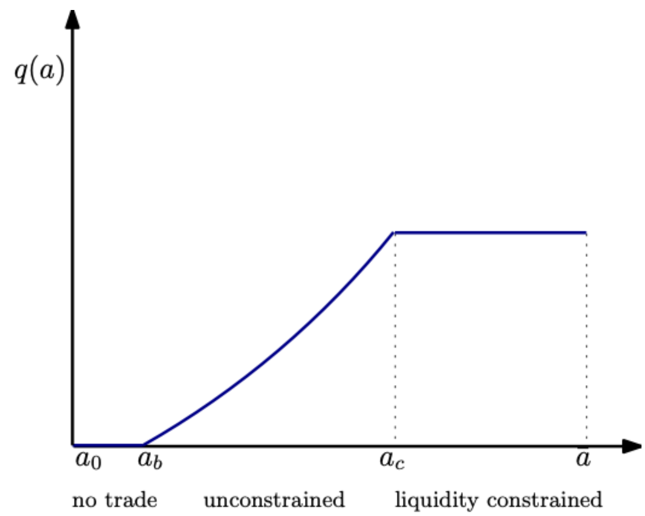


Figure 2: Example of no-trade, unconstrained, and liquidity constrained ranges

$[0,1]$ *for all* $a \in A$*, and* $z, n \in \mathbb{R}_+ \backslash \{0\}$*, such that* $(\{(q_a, d_a)\}_{a\in A}, z, n)$ *maximizes (24) subject to constraint (25), the liquidity constraint* $d_a \leq z$ *for all* $a \in A$*, plus the IR constraint (35) and the IC constraint (36), and* $\{\tilde{G}(a; n)\}_{a\in A}$ *represents buyers' optimal choices of sellers.*

Lemma 7 tells us that there exists a non-empty range of utility shocks $a$ such that trade does not occur in equilibrium, i.e. $q_a = 0$. When the good chosen by a buyer within a meeting falls within this range, we call such meetings *no-trade meetings*. There may also exist a non-empty range of utility shocks such that buyers' purchases are constrained by their money holdings, i.e. $d_a = z$. When the good chosen by a buyer within a meeting falls within this range, we call such meetings *liquidity constrained*.

**Lemma 7.** *In any equilibrium where* $i > 0$*, there exist* $a_b$*,* $a_c \in (a_0, \bar{a}]$ *such that*

1. *No-trade range:* $q_a = 0$ *and* $d_a = 0$ *for all* $a \in [a_0, a_b]$.
2. *Unconstrained range:* $q_a > 0$ *and* $d_a < z$ *for all* $a \in (a_b, a_c)$.
3. *Liquidity constrained range:* $q_a = q_{a_c} > 0$ *and* $d_a = z$ *for all* $a \in [a_c, \bar{a}]$.

Before presenting Proposition 6, it will be useful to define $\rho(a; n) \equiv 1 - \tilde{G}(a; n)$, the probability that a chosen good has utility greater than $a$. We also define $\varepsilon_\rho(a; n) \equiv$

$-a\rho'(a;n)/\rho(a;n)$, the elasticity of $\rho(a;n)$ with respect to $a$, where $\rho'(a;n) \equiv \frac{\partial \rho(a;n)}{\partial a}$. This elasticity can be calculated as follows:

$$\varepsilon_{\rho}(a;n) \ = \frac{a}{I_{\tilde{G}}(a;n)}. \tag{37}$$

For simplicity, we assume $a_0 = 0$ throughout the rest of the paper. We also make the following assumption, which is necessary to ensure the existence of equilibrium, throughout the remainder of the paper.

**Assumption 6.** *The cost of entry is not too high:* $E_G[au(q_a^0) - c(q_a^0)] > K$.

Assumption 6 says the expected trade surplus in the limit as $n \to 0$ must be greater than $K$, otherwise no sellers enter. Since $\tilde{G} \to G$ as $n \to 0$ by Lemma 3, $\lim_{n\to 0} \tilde{s}(n) = E_G[au(q_a^0) - c(q_a^0)]$ where $q_a^0 \equiv \lim_{n\to 0} q_a(n)$ is given by Lemma 8.[12]

**Lemma 8.** *For all* $a \in [a_0, a_b]$, $q_a^0 = 0$ *and, for all* $a \in (a_b, \bar{a}]$, $q_a^0$ *satisfies*

$$(a - I_G(a))\, u'(q_a) = c'(q_a) \tag{38}$$

*where* $a_b^0 \in [a_0, \bar{a})$ *is the unique solution to* $\psi_G(a) = 0$.

Assumption 7 is a sufficient but not a necessary condition for the existence of equilibrium. It is used to prove that the function $q(a)$ is weakly increasing, which is required for ensuring that the IC constraint holds for any $a \in A$. This condition says that the function $\mathbb{P}_0$ is not too convex. This assumption will be true, for example, if the meeting technology $\mathbb{P}_k$ is negative binomial with parameter $r \geq 1$ (which includes both the Poisson and the Geometric meeting technologies).

**Assumption 7.** *The meeting technology* $\mathbb{P}_k$ *is invariant and, for all* $x \in \mathbb{R}^+$,

$$\frac{\mathbb{P}_0''(x)\mathbb{P}_0(x)}{\mathbb{P}_0'(x)^2} \leq 2. \tag{39}$$

We can now present our main result, which establishes the existence and uniqueness of equilibrium and provides a characterization.

**Proposition 6.** *Suppose that* $\mathbb{P}_k$ *satisfies Assumption 7. For any* $i > 0$*, there exists a unique competitive search equilibrium and it satisfies:*

[12] Assumption 6 is more complicated than Assumption 5 because $q_a$ depends on $n$ in equilibrium, but the planner's solution $q_a^*$ is independent of $n$.

1. *No-trade range. For any* $a \in [a_0, a_b]$, $q_a = 0$ *and* $d_a = 0$.

2. *Unconstrained range. For any* $a \in (a_b, a_c]$, *the quantity* $q_a > 0$ *solves:*

$$(a - \phi(a;n))u'(q_a) = c'(q_a) \tag{40}$$

*where*

$$\phi(a;n) = \left(1 - \frac{1}{\delta}\right) I_{\tilde{G}}(a;n) - \left(\frac{1}{\delta}\right) \frac{i}{\alpha(n)\tilde{g}(a;n)} \tag{41}$$

*and*

$$\delta = \frac{1}{1 - \varepsilon_\rho(a_b;n)} \left(1 + \frac{i}{\alpha(n)\rho(a_b;n)}\right). \tag{42}$$

*Also,* $d_a/\gamma = au(q_a) - \int_{a_0}^{a} u(q_x)dx$.

3. *Liquidity constrained range. For any* $a \in [a_c, \bar{a}]$, $q_a = q_{a_c}$ *and* $d_a = d_{a_c}$.

4. *The value of* $a_c$ *satisfies*

$$\frac{i\bar{a}}{\alpha(n)} = \int_{a_c}^{\bar{a}} (a - a_c)\tilde{g}(a;n)dx + (\delta - 1)(\bar{a} - a_c)(1 - \tilde{G}(a_c;n)). \tag{43}$$

5. *Real money holdings* $z > 0$ *is given by* $z = d_{a_c}$.

6. *The seller-buyer ratio* $n > 0$ *is strictly decreasing in* $K$ *and satisfies*

$$\alpha'(n)\tilde{s}(n;\{q_a\}_{a\in A}) + \alpha(n)\tilde{s}'(n;\{q_a\}_{a\in A}) = K. \tag{44}$$

7. *The zero profit condition is satisfied:*

$$\frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[-c(q_a) + \frac{d_a}{\gamma}\right] d\tilde{G}(a;n) = K. \tag{45}$$

8. *The distribution of chosen goods is given by (6).*

The equilibrium distribution of chosen goods $\tilde{G}$ is the same as the planner's because buyers always choose the highest utility seller they meet. The distribution of chosen goods therefore equals the distribution across buyers of the highest utility $a$ among the sellers a buyer meets, conditional on meeting at least one seller.

As before, a version of the generalized Hosios condition holds *endogenously* in our environment featuring competitive search since the equilibrium condition (44) is equivalent in form to the planner's condition (14). The only difference between the equilibrium condition (44) and the planner's condition (14) is that the quantities $q_a$ traded in equilibrium may be different than the efficient quantities $q_a^*$. Since the expected trade surplus $\tilde{s}(n; \{q_a\}_{a \in A})$ depends not only on the seller-buyer ratio $n$ but also on the quantities $q_a$, seller entry is not necessarily efficient. However, seller entry is efficient *provided that the quantity traded is efficient*, i.e. $q_a = q_a^*$ for all $a \in A$.

## 8.1 No choice

As a benchmark, we also describe equilibrium without consumer choice. To do this, we assume that buyers cannot choose sellers but instead are randomly matched with sellers in meetings. This is equivalent to an environment with bilateral (one-on-one) meetings where the probability a buyer meets a seller is $\alpha(n)$.

**Corollary 2.** *Without consumer choice, for any $i > 0$ there exists a unique competitive search equilibrium and it satisfies:*

1. *No-trade range. For any $a \in [a_0, a_b]$, $q_a = 0$ and $d_a = 0$.*
2. *Unconstrained range. For any $a \in (a_b, a_c]$, the quantity $q_a > 0$ solves:*

$$(a - \phi(a; n))u'(q_a) = c'(q_a) \tag{46}$$

*where*

$$\phi(a; n) = \left(1 - \frac{1}{\delta}\right) I_G(a) - \left(\frac{1}{\delta}\right) \frac{i}{\alpha(n)g(a)} \tag{47}$$

*and*

$$\delta = \frac{1}{1 - \varepsilon_\rho(a_b)} \left(1 + \frac{i}{\alpha(n)\rho(a_b)}\right). \tag{48}$$

*Also, $d_a/\gamma = au(q_a) - \int_{a_0}^{a} u(q_x)dx$.*

3. *Liquidity constrained range. For any* $a \in [a_c, \bar{a}]$*,* $q_a = q_{a_c}$ *and* $d_a = d_{a_c}$*.*

4. *The value of* $a_c$ *satisfies*

$$\frac{i\bar{a}}{\alpha(n)} = \int_{a_c}^{\bar{a}} (a - a_c) g(a) dx + (\delta - 1)(\bar{a} - a_c)(1 - G(a_c)). \tag{49}$$

5. *Real money holdings* $z > 0$ *is given by* $z = d_{a_c}$*.*

6. *The seller-buyer ratio* $n > 0$ *is strictly decreasing in* $K$ *and satisfies*

$$\frac{\alpha'(n)\tilde{s}(\{q_a\}_{a\in A})}{\delta + (1-\delta)\eta_\alpha(n)} = K. \tag{50}$$

7. *The zero profit condition is satisfied:*

$$\frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[ -c(q_a) + \frac{d_a}{\gamma} \right] dG(a) = K. \tag{51}$$

8. *The distribution of chosen goods is* $G$*.*

The differences between the equilibria with and without choice are the same as for the full information benchmark. One difference is that the distribution of chosen goods is equal to the exogenous distribution of available goods, $G(a)$, and it no longer depends on the equilibrium seller-buyer ratio $n$. As a result, $\rho(a_b)$ and $\varepsilon_\rho(a_b)$ no longer depend directly on $n$.

Another difference is that the Hosios-like equilibrium condition (50) depends directly on the nominal interest rate $i$, as in RW. This condition is closely related to the standard Hosios condition, but this condition is not equivalent (conditional on quantities $q_a$) to the standard Hosios condition $\alpha'(n)\tilde{s}(\{q_a\}_{a\in A}) = K$ even as $i \to 0$.

# 9 Consumption and entry

Consider expression (41), which gives us the equilibrium quantities for the trading range that is unconstrained, $a \in (a_b, a_c]$. Given that the efficient quantity $q_a^*$ satisfies $au'(q_a^*) = c'(q_a^*)$, it is clear that we have *underconsumption* if $\phi(a; n) > 0$, *overconsumption* if $\phi(a; n) < 0$, and *efficient* consumption if $\phi(a; n) = 0$.

To better understand expression (41), we can interpret it as a weighted average of two terms, where the endogenous weights are $1/\delta \in (0,1]$ and $1 - 1/\delta \in [0,1)$.

$$\phi(a;n) = \left(1 - \frac{1}{\delta}\right) \underbrace{I_{\tilde{G}}(a;n)}_{\text{weakly positive, } \geq 0} + \left(\frac{1}{\delta}\right) \underbrace{\frac{-i}{\alpha(n)\tilde{g}(a;n)}}_{\text{negative, } <0} \tag{52}$$

Whether or not we have equilibrium overconsumption or underconsumption for a good of utility $a$ depends on the relative weights given to each of these two terms, as well as their values at $a$. If the positive term dominates, we have underconsumption, while if the negative term dominates we have overconsumption. If the two terms exactly offset each other, we have efficient consumption at utility $a$.

Proposition 7 describes the three possible equilibrium outcomes in terms of underconsumption or overconsumption ranges for $i > 0$ (as depicted in Figure 3). With private information, we always have a no-trading range, i.e. $a_b > a_0$, but we include the limit as $a_b \to a_0$ as a case here.

**Proposition 7.** *Suppose that $\mathbb{P}_k$ satisfies Assumption 7. Let $a_u \equiv \max\{a_c, a_d\}$ where $a_d \equiv a_c - \phi(a_c)$, and let $a_p$ solve $\tilde{G}(a_p;n) = 1 + \frac{i}{\alpha(n)(1-\delta)}$. For any $i > 0$, there are three possible outcomes in competitive search equilibrium:*

1. *If $a_p \leq a_c$, there is underconsumption on $(a_0, a_p)$, overconsumption on $(a_p, a_u)$, and underconsumption on $(a_u, \bar{a}]$.*
2. *If $a_p \geq a_c$, there is underconsumption on $(a_0, \bar{a}]$.*
3. *If $a_b \to a_0$, there is overconsumption on $(a_0, a_d)$ and underconsumption on $(a_d, \bar{a}]$.*

Given that the generalized Hosios condition holds endogenously under competitive search, we know that the equilibrium seller-buyer ratio $n$ is efficient *provided that the quantities traded $q_a$ are efficient*. However, the quantities traded are not efficient whenever $i > 0$ and therefore seller entry is not necessarily efficient. Proposition 8 states that there can be over-entry, under-entry, or efficient entry of sellers outside the Friedman rule. We can find examples of each possibility.

**Proposition 8.** *In any competitive search equilibrium where $i > 0$, there may be either under-entry, over-entry, or efficient entry of sellers.*

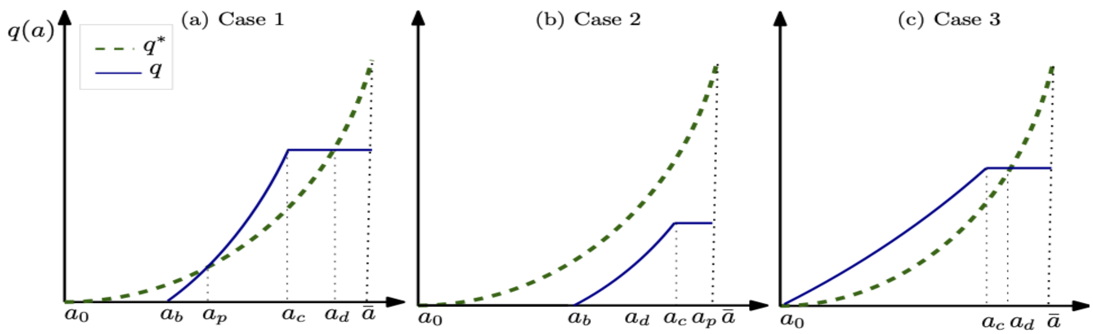


Figure 3: Examples of the three cases of under/over consumption in Proposition 7

While we know that entry must be efficient if the quantity traded is efficient, the converse is not true. There are examples where entry is efficient but the quantities traded are not. When this occurs, the efficiency of entry is really just "coincidental".

## 10 Friedman rule

With full information, there is efficiency along both the intensive and extensive margins when the Friedman rule is imposed. That is, both the quantity traded and the level of entry of sellers are efficient. In our environment, there can be inefficiencies along *both* margins at the Friedman rule. These inefficiencies are both due to buyers' private information, not the presence of consumer choice.

**Corollary 3.** *Suppose that $\mathbb{P}_k$ satisfies Assumption 7. At the Friedman rule, competitive search equilibrium satisfies:*

1. *No-trade range. For any $a \in [a_0, a_b]$, $q_a = 0$, and $d_a = 0$.*
2. *Unconstrained range. For all $a \in (a_b, \bar{a}]$, the quantity $q_a$ satisfies*

   (53) $$\left(a - \varepsilon_\rho(a_b; n) I_{\tilde{G}}(a; n)\right) u'(q_a) = c'(q_a).$$

   *Also, $d_a/\gamma = au(q_a) - \int_{a_0}^{a} u(q_x)dx$.*
3. *No meetings are liquidity constrained: $a_c = \bar{a}$.*
4. *Parts 5-8 from Proposition 6 hold.*

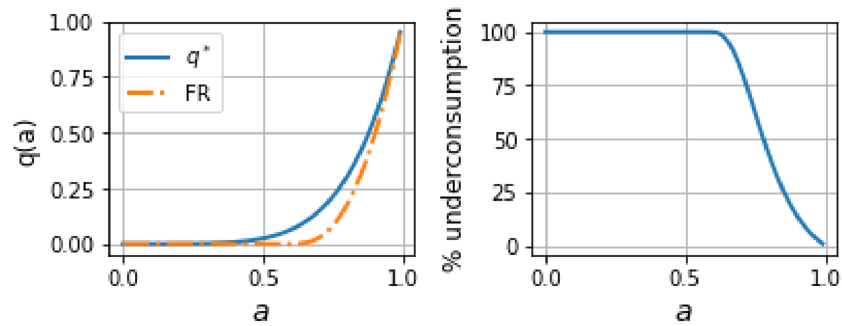


Figure 4: Equilibrium underconsumption at the Friedman rule

Proposition 9 tells us that the Friedman rule results in *underconsumption,* i.e. $q_a < q_a^*$ for all $a \in (a_0, \bar{a})$. Given that there is underconsumption at the Friedman rule, seller entry is not necessarily efficient. Notice that there is *no distortion at the top*, i.e. $q_a = q_a^*$ at $a = \bar{a}$, but there is *downwards distortion below*, both standard results in mechanism design.

Figure 4 illustrates an example. It is clear that the extent of underconsumption is decrasing in the value of the utility shock $a$ and goes to zero as $a \to \bar{a}$.

**Proposition 9.** *Suppose that $\mathbb{P}_k$ satisfies Assumption 7. At the Friedman rule, there is underconsumption for all $a \in (a_0, \bar{a})$ and there can be either under-entry, over-entry, or efficient entry of sellers.*

The reason why the Friedman rule does not yield efficiency along the intensive margin is not only because there is underconsumption in no-trade meetings. Even if we consider meetings that *do* result in trade, there is underconsumption. Intuitively, sellers need to compensate for the fact that there is a range of meetings in which no trade occurs. Sellers compensate for the no-trade meetings by charging higher prices over the trading range, which implies that less than the efficient quantity is consumed even within the trading range. The extent of underconsumption at the Friedman rule is governed by the *informational rents*, $I_{\tilde{G}}(a; n)$, which we discuss in Section 11.

In addition to underconsumption, Proposition 9 says that there may be either under-entry or over-entry of sellers. Therefore, the Friedman rule does not generally deliver efficiency along either the intensive or extensive margin.

### 10.1 No choice

Without choice, there can also be inefficiencies along both the intensive and extensive margins when the Friedman rule is imposed. As discussed, these inefficiencies are due to buyers' private information.

**Corollary 4.** *Without choice, at the Friedman rule, equilibrium satisfies:*

1. *No-trade range. For any $a \in [a_0, a_b]$, $q_a = 0$, and $d_a = 0$.*
2. *Unconstrained range. For all $a \in (a_b, \bar{a}]$, the quantity $q_a$ satisfies*

   $$(a - \varepsilon_\rho(a_b) I_G(a))\, u'(q_a) = c'(q_a). \tag{54}$$

   *Also, $d_a/\gamma = au(q_a) - \int_{a_0}^{a} u(q_x)dx$.*
3. *No meetings are liquidity constrained: $a_c = \bar{a}$.*
4. *Parts 5-8 from Corollary 2 hold.*

It is clear that, without consumer choice, the Friedman rule still results in *under-consumption,* i.e. $q_a < q_a^*$ for all $a \in (a_0, \bar{a})$. We again have *no distortion at the top*, i.e. $q_a = q_a^*$ at $a = \bar{a}$, but *downwards distortion below.* However, over-entry is not possible without choice: there is always *under-entry* of sellers at the Friedman rule.

**Proposition 10.** *Without choice, at the Friedman rule, there is underconsumption for all $a \in (a_0, \bar{a})$ and under-entry of sellers.*

## 11 Informational rents and choice

At the Friedman rule, the only source of inefficiency is buyers' private information. In our model, there is an important interaction between consumer choice and private information. In the presence of consumer choice, a higher seller-buyer ratio shifts the distribution of chosen goods towards the highest utility shock $\bar{a}$, thus increasing the mean of the distribution and reducing its dispersion. This shift in the distribution alleviates the distortion due to private information by reducing informational rents – in a sense we will now make precise.

Without choice, informational rents are defined by $I_G(a;n) = 1/h_G(a)$, or

$$I_G(a) = \frac{1 - G(a)}{g(a)}. \tag{55}$$

The virtuation valuation function can be written as $\psi_G(a) \equiv a - I_G(a)$. We know that $I_G(a)$ is decreasing in $a$ because $G$ has an increasing hazard rate. Therefore, the virtuation valuation function is increasing in $a$.

With consumer choice, informational rents are defined by $I_{\tilde{G}}(a;n) = 1/h_{\tilde{G}}(a;n)$,

$$I_{\tilde{G}}(a;n) = \frac{1 - \tilde{G}(a;n)}{\tilde{g}(a;n)}. \tag{56}$$

where $\tilde{G}$ is given by (6). We know this term is decreasing in $a$ because $\tilde{G}$ has an increasing hazard rate by Lemma 5. As the seller-buyer ratio $n$ increases, there is FOSD shift in the distribution $\tilde{G}$ which shifts greater density towards higher values of $a$. As a result, we expect that the "typical" informational rents, $I_{\tilde{G}}(a;n)$, might decrease due to this indirect effect of $n$ because for higher utility shocks $a$, the informational rents $I_{\tilde{G}}(a;n)$ are lower.

However, there is also a direct effect of $n$. For any given $a$, the informational rents $I_{\tilde{G}}(a;n)$ are increasing in $n$. To see this, we know from Lemma 5 that

$$I_{\tilde{G}}(a;n) = \frac{I_G(a)}{\eta_\alpha(n(1 - G(a)))}.$$

Given that $\eta'_\alpha < 0$, informational rents $I_{\tilde{G}}(a;n)$ are increasing in $n$ for any given $a$.

So, the overall effect of an increase in $n$ on the "typical" informational rents is unclear. To answer this question, we need to be more precise about what we mean by "typical" informational rents.

If we were to define "typical" informational rents simply as the average across all chosen goods, i.e. $\int I_{\tilde{G}}(a;n)\tilde{g}(a;n)da$, this expression would always be *higher* for chosen goods than available goods, i.e. $\int I_{\tilde{G}}(a;n)\tilde{g}(a;n)da > \int I_G(a)g(a)da$. This follows immediately from the FOSD relation because $\tilde{G}(a;n) < G(a)$ for all $a \in A$ implies $\int \tilde{G}(a;n)da < \int G(a)da$.

In order for this question to be interesting, we need to compare informational rents for the two distributions at a "typical" value of $a$ – instead of simply averaging across all chosen goods. Consider any function $a : \mathbb{R}_+ \to A$ where $a(n)$ represents a "typical" chosen good. Proposition 11 provides a necessary and sufficient condition for informational rents at $a(n)$ to be *strictly decreasing* in the seller-buyer ratio $n$. If this condition holds then, as $n$ increases, the effect of private information decreases.

In particular, because $\tilde{G} \to G$ as $n \to 0$ by Lemma 3, condition (57) in Proposition 11 implies that informational rents at the value $a(n)$ are *lower* with consumer choice than without choice. That is, $I_{\tilde{G}}(a(n); n) < I_G(a(n))$, i.e. consumer choice reduces the "typical" value of informational rents. Note that condition (57) implies that $a'(n) > 0$ is necessary.

**Proposition 11.** *Suppose that $\mathbb{P}_k$ is invariant and $n > 0$. Let $a : \mathbb{R}_+ \to A$. The informational rents at $a(n)$ are decreasing in $n$, i.e. $\frac{d}{dn} I_{\tilde{G}}(a(n); n) < 0$, if and only if*

$$\frac{a'(n)n}{a(n)} \left( \frac{1}{\eta_{\eta_\alpha}(x)} \frac{h'_G(a)a}{h_G(a)} + \frac{g(a)a}{1 - G(a)} \right) > 1 \tag{57}$$

*at $a = a(n)$, where $x = n(1 - G(a))$ and $\eta_{\eta_\alpha}(x) \equiv \frac{-\eta'_\alpha(x)x}{\eta_\alpha(x)}$.*

Lemma 9 can be used to derive a simpler sufficient condition.

**Lemma 9.** *If $\mathbb{P}_k$ satisfies Assumption 7 then $\eta_{\eta_\alpha}(x) \in (0, 1)$.*

Lemma 9, together with $h'_G(a) \geq 0$, implies that a sufficient condition for (57) is

$$\frac{a'(n)n}{a(n)} \left( \frac{h'_G(a)a}{h_G(a)} + \frac{g(a)a}{1 - G(a)} \right) > 1. \tag{58}$$

For any given $n > 0$, this condition holds if the function $a(n)$ increases sufficiently sharply with $n$, the hazard rate $h_G(a)$ increases sufficiently sharply with $a$, and $1 - G(a)$ decreases sufficiently sharply with $a$.

Simplifying further, this sufficient condition is equivalent to

$$\frac{a'(n)n}{a(n)} \left( \frac{g'(a)(1 - G(a))}{g(a)^2} + 2 \right) > \frac{1 - G(a)}{g(a)a}. \tag{59}$$

In order to determine whether or not this holds, we first need to define a function $a(n)$ which is intended to represent "typical" informational rents. A natural choice is the value $\hat{a}(n)$,which we define as the unique value of the utility shock at which $\frac{\partial}{\partial n} \tilde{g}(a; n) = 0$. This is because, like the average $\tilde{a}(n)$, it is increasing in $n$ and it converges to $\bar{a}$ in the limit as $n \to \infty$. Moreover, unlike the average $\tilde{a}(n)$, we can prove that it is always an over-represented good (i.e. its density increases when the seller-buyer ratio increases).

We define "typical" *informational rents* as informational rents as follows:

$$I_{\tilde{G}}(\hat{a}(n);n) = \frac{1-\tilde{G}(\hat{a}(n);n)}{\tilde{g}(\hat{a}(n);n)}. \tag{60}$$

**Lemma 10.** *Suppose $\mathbb{P}_k$ satisfies Assumption 4 and $n > 0$. There exists a unique $\hat{a}(n)$ such that $\frac{\partial}{\partial n}\tilde{g}(a;n) = 0$ and $\frac{\partial}{\partial n}\tilde{g}(a;n) > 0$ if and only if $a > \hat{a}(n)$. We have $\hat{a}'(n) > 0$ and $\lim_{n\to\infty} \hat{a}(n) = \bar{a}$, and $\hat{a}(n) > a_f(n)$, i.e. it is an over-represented good.*

The curvature of the function $\hat{a}(n)$ is crucial for determining whether or not "typical" informational rents are increasing in $n$. This curvature depends on properties of both the distribution $G$ and the meeting technology $\mathbb{P}_k$.

**Lemma 11.** *Suppose that $\mathbb{P}_k$ satisfies Assumption 4 and $n > 0$. We have*

$$\frac{\hat{a}'(n)n}{\hat{a}(n)} = \frac{1-G(\hat{a})}{g(\hat{a})\hat{a}}\left(1 + \frac{\eta'_\alpha(n)n}{\eta_A(\hat{x})(1-\eta_\alpha(n))}\right) \tag{61}$$

*where $\hat{a} = \hat{a}(n)$, $\eta_A(x) \equiv \frac{A'(x)x}{A(x)}$ and $A(x) \equiv \frac{-\mathbb{P}''_0(x)x}{\mathbb{P}'_0(x)}$ and $\hat{x} = n(1-G(\hat{a}))$.*

Combining Proposition 11 and Lemma 11, we obtain the following result.

**Proposition 12.** *Suppose that $\mathbb{P}_k$ satisfies Assumption 4 and $n > 0$. The informational rents at $\hat{a}(n)$ are decreasing in $n$, i.e. $\frac{d}{dn}I_{\tilde{G}}(\hat{a}(n);n) < 0$, if and only if*

$$\left(1 + \frac{\eta'_\alpha(n)n}{\eta_A(\hat{x})(1-\eta_\alpha(n))}\right)\left(\frac{1}{\eta_{\eta_\alpha}(x)}\frac{h'_G(a)a}{h_G(a)} + \frac{g(a)a}{1-G(a)}\right) > 1. \tag{62}$$

*Moreover, in the limit as $n \to \infty$, we have $I_{\tilde{G}}(\hat{a}(n);n) \to 0$.*

Given that $\eta_{\eta_\alpha}(x) < 1$ by Lemma 9, a sufficient condition is

$$\left(1 + \frac{\eta'_\alpha(n)n}{\eta_A(\hat{x})(1-\eta_\alpha(n))}\right)\left(\frac{g'(\hat{a})(1-G(\hat{a}))}{g(\hat{a})^2} + 2\right) > 1. \tag{63}$$

Observe that if we had $\eta'_\alpha(n) = 0$, the increasing hazard condition on the distribution $G$, which is equivalent to $\frac{g'(a)(1-G(a))}{g(a)^2} > -1$ for all $a \in A$, would be sufficient. In general, however, $\eta'_\alpha(n) < 0$ by Lemma 1, so this condition is more restrictive. Whether or not condition (63) holds depends not only on the properties of the distribution $G$, but also on the seller-buyer ratio $n$ and the properties of the meeting technology $\mathbb{P}_k$.

**Example.** For example, if the distribution $G$ is uniform on $[0,1]$ and the meeting technology $\mathbb{P}_k$ is Poisson, a sufficient condition for (62) is $n \geq 2$. To see this, if $G$ is uniform then $g'(a) = 0$ and it suffices to show

$$\frac{1}{\eta_A(\hat{x})} \frac{-\eta_\alpha'(n)n}{\eta_\alpha(n)} \frac{\eta_\alpha(n)}{1-\eta_\alpha(n)} < \frac{1}{2}.$$

Given that $\eta_{\eta_\alpha}(x) < 1$ by Lemma 9, it suffices to show that

$$\frac{1}{\eta_A(\hat{x})} \frac{\eta_\alpha(n)}{1-\eta_\alpha(n)} < \frac{1}{2}.$$

If the meeting technology $\mathbb{P}_k$ is Poisson, then $A(x) = x$ and $\eta_A(x) = 1$, so

$$\frac{\eta_\alpha(n)}{1-\eta_\alpha(n)} < \frac{1}{2}$$

suffices. This is true provided that $\eta_\alpha(n) < 1/3$ or $n \geq 2$.

# 12 Competitive limit

With consumer choice, greater seller entry can reduce the informational rents available to buyers, thus diminishing the distortion due to buyers' private information. Given this is the *only* source of inefficiency at the Friedman rule, this suggests that, if the market is sufficiently competitive, i.e. the seller-buyer ratio is sufficiently high, consumer choice may be able to eliminate this inefficiency altogether.

Consider the limit as the entry cost $K \to 0$. We know from Proposition 6 that $n$ is strictly decreasing in $K$. In the limiting case where $K \to 0$, the equilibrium seller-buyer ratio $n \to \infty$. We refer to this as the *competitive limit* because all buyer meet a large number of sellers in each meeting.

First, we describe the full-information benchmark. As $K \to 0$, we have $n \to \infty$ and the distribution of chosen goods $\tilde{G}(a;n)$ converges to a degenerate distribution with support $A = \{\bar{a}\}$. In all meetings, the quantity traded $q_a = q_{\bar{a}}$ where $q_{\bar{a}}$ solves

$$\bar{a}u'(q_{\bar{a}}) = (1+i)\,c'(q_{\bar{a}}) \tag{64}$$

by Proposition 3, using the fact that $\alpha(n) \to 1$ as $n \to \infty$. With full information, the Friedman rule delivers efficiency – both in general and in the competitive limit.

Now consider competitive search equilibrium under private information. As $n \to \infty$, the distribution of chosen goods $\tilde{G}(a; n)$ converges to a degenerate distribution with support $A = \{\bar{a}\}$. Proposition 13 says that, in this limiting case, we obtain a single price equilibrium in which all buyers trade. All meetings are liquidity constrained and the quantity traded is $q_{\bar{a}}$, which is below the efficient quantity. For all trades, the payment is $d_{\bar{a}} = z$, i.e. all money is used in trade.

**Proposition 13.** *Suppose that $\mathbb{P}_k$ satisfies Assumption 7. For any $i > 0$, in the competitive limit as $K \to 0$, equilibrium satisfies:*

1. *The seller-buyer ratio $n \to \infty$.*
2. *The distribution of chosen goods has support $A \to \{\bar{a}\}$.*
3. *The quantity traded $q_{\bar{a}}$ satisfies*

$$\bar{a}u'(q_{\bar{a}}) = (1+i)c'(q_{\bar{a}}). \tag{65}$$

4. *We have $d_{\bar{a}}/\gamma = z/\gamma$ and $d_{\bar{a}}/\gamma = a_c u(q^*_{a_c}) - \int_{a_0}^{a_c} u(q^*_x)dx$ where*

$$a_c = \frac{\bar{a}}{1+i}. \tag{66}$$

With consumer choice, as the seller-buyer ratio $n$ increases, the endogenous distribution of chosen goods $\tilde{G}$ shifts towards a distribution with greater mass concentrated at higher values that are closer to the upper bound $\bar{a}$. In the limit as $a \to \bar{a}$, we know that $I_{\tilde{G}}(\bar{a}; n) \to 0$, i.e. there is *no distortion at the top.* In the limit as $K \to 0$, the seller-buyer ratio goes to infinity and all chosen goods are at the "top", i.e. the upper bound $\bar{a}$. This eliminates the "information friction" altogether. In this limiting case, the equilibrium is similar to the full information benchmark.

While the "monetary friction" is not eliminated in the competitive limit as $K \to 0$, this friction is eliminated at the Friedman rule as $i \to 0$. Corollary 5 follows immediately from Proposition 13 by letting $i \to 0$, which delivers $a_c = \bar{a}$.

**Corollary 5.** *Suppose that $\mathbb{P}_k$ satisfies Assumption 7. In the competitive limit as $K \to 0$, at the Friedman rule, equilibrium satisfies:*

1. *The seller-buyer ratio $n \to \infty$.*

2. *The distribution of chosen goods has support $A \to \{\bar{a}\}$.*

3. *The quantity traded $q_{\bar{a}}$ satisfies*

$$\bar{a}u'(q_{\bar{a}}) = c'(q_{\bar{a}}). \tag{67}$$

4. *We have $d_{\bar{a}}/\gamma = z/\gamma$ and $d_{\bar{a}}/\gamma = \bar{a}u(q^*_{\bar{a}}) - \int_{a_0}^{\bar{a}} u(q^*_x)dx$.*

At the Friedman rule, consumer choice delivers efficiency in the competitive limit. This result contrasts with our benchmark case with private information but without choice, which we discuss next.

**No choice.** Consider the benchmark without choice. If we take the limit as $K \to 0$, we have $n \to \infty$ but the distribution of chosen goods does not depend on the seller-buyer ratio, i.e. $\tilde{G}(a; n) = G(a)$. Only the trading probability $\alpha(n)$ is affected by the change in $n$, and we have $\alpha(n) \to 1$ and $\eta_\alpha(n) \to 0$ in the limit as $n \to \infty$. Therefore, equilibrium is characterized by the same equations as in Corollary 2 except that $\alpha(n) \to 1$ and $\eta_\alpha(n) \to 0$. As before, we do not have efficiency in terms of either the quantities traded or seller entry. Moreover, without choice, we do not obtain efficiency *even at the Friedman rule* due to the persistence of the information friction, which cannot be eliminated even in the competitive limit as $n \to \infty$.

# 13 Conclusion

This paper introduces consumer choice into a competitive search model of monetary exchange. There is a general meeting technology which allows consumers to meet multiple sellers and *choose* a seller with whom to trade, and consumer choice is influenced by random utility shocks that are private information.

We find that the Friedman rule does not generally implement the efficient allocation: there may be under-consumption of all goods and either under-entry or over-entry of sellers. In the absence of consumer choice, this inefficiency due to private information cannot be eliminated. With consumer choice, however, there is an important interaction between private information and consumer choice due to the fact that the distribution of chosen goods is endogenous. We find that greater buyer entry can alleviate this inefficiency by reducing the informational rents available to buyers, thus diminishing the extent of private information. In the competitive limit where the seller-buyer ratio becomes large, the effect of private information is eliminated and the Friedman rule can deliver efficiency when there is consumer choice.

Bajaj and Mangin (2023) extends the current framework to incorporate a parameter that represents the degree of informed choice and uses the model to study the effect of consumer choice on the welfare cost of inflation. When we calibrate the model to U.S. money demand data, we find that a greater degree of consumer choice can significantly increase the welfare cost of inflation. In future work, it would be interesting to use the generality of the framework in the present paper to examine whether the welfare cost of inflation is affected by the nature of the meeting technology.

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# Appendix: Proofs

## Proofs for Section 4

### Useful lemmas

Before presenting the proofs for Section 4, we provide some lemmas that will be useful. Lemma 12 summarizes some properties of invariant meeting technologies that follow directly from the definition.

**Lemma 12.** *If $\mathbb{P}_k$ is an invariant meeting technology, then*

1. *We have $\mathbb{P}_0'(x) < 0$ and $\mathbb{P}_0''(x) > 0$ for all $x \in \mathbb{R}^+ \backslash \{0\}$.*
2. *We have $\lim_{x\to 0} \mathbb{P}_0(x) = 1$ and $\lim_{x\to 0} \mathbb{P}_0'(x) = -1$.*
3. *We have $\lim_{x\to\infty} \mathbb{P}_0(x) = 0$, $\lim_{x\to\infty} \mathbb{P}_0'(x) = 0$, and $\lim_{x\to\infty} \mathbb{P}_0''(x) = 0$.*

Corollary 6 states some properties of the function $\alpha$ representing buyers' meeting probability, $\alpha(n) \equiv 1 - \mathbb{P}_0(n)$, that follow immediately from Lemma 12.

**Corollary 6.** *If $\mathbb{P}_k$ is an invariant meeting technology, then*

1. *We have $\alpha'(n) < 0$ and $\alpha''(n) < 0$ for all $n \in \mathbb{R}^+ \backslash \{0\}$.*
2. *We have $\lim_{n\to 0} \alpha(n) = 0$ and $\lim_{n\to 0} \alpha'(n) = 1$.*
3. *We have $\lim_{n\to\infty} \alpha(n) = 1$, $\lim_{n\to\infty} \alpha'(n) = 0$, and $\lim_{n\to\infty} \alpha''(n) = 0$.*

### Proof of Lemma 1

*Part 1.* We can start by writing

(68)
$$\eta_\alpha(n) \equiv \frac{\alpha'(n)n}{\alpha(n)} = \frac{-\mathbb{P}_0'(n)n}{1 - \mathbb{P}_0(n)}.$$

Given that $\mathbb{P}_k$ is invariant, we know that $-\mathbb{P}_0'(n)n = \mathbb{P}_1(n)$ and thus

(69)
$$\eta_\alpha(n) = \frac{\mathbb{P}_1(n)}{1 - \mathbb{P}_0(n)}.$$

That is, if $\mathbb{P}_k$ is invariant then $\eta_\alpha(n)$ equals the probability that $k = 1$ conditional on $k \geq 1$. Clearly, this probability is decreasing in $n$. Therefore, we have $\eta'_\alpha(n) < 0$.

*Part 2.* First, the fact that $\lim_{n\to 0} \eta_\alpha(n) = 1$ follows from Corollary 6 and L'Hopital's rule. Second, we know that $\eta'_\alpha(n) < 0$ from Part 1, so the fact that $\lim_{n\to 0} \eta_\alpha(n) = 1$ implies $\eta_\alpha(n) < 1$ for any $n > 0$. Finally, we have

$$\lim_{n\to\infty} \eta_\alpha(n) = \lim_{n\to\infty} \frac{\mathbb{P}_1(n)}{1 - \mathbb{P}_0(n)} = 0, \tag{70}$$

which completes the proof. ■

**Proof of Lemma 2**

Using the fact that the distribution of the maximum of $k \geq 1$ draws is $(G(a))^k$, and weighting by the probability $\mathbb{P}_k(n)$ that exactly $k$ sellers meet a buyer, conditional on $k \geq 1$, we have

$$\tilde{G}(a;n) = \frac{\sum_{k=1}^{\infty} \mathbb{P}_k(n)(G(a))^k}{\alpha(n)}. \tag{71}$$

Given that we assume the meeting technology $\mathbb{P}_k$ is invariant, we have $\sum_{k=0}^{\infty} \mathbb{P}_k(n) y^k = \mathbb{P}_0(n(1-y))$ and substituting into the above yields (6). ■

**Proof of Lemma 3**

*Part 1.* Taking the limit as $n \to 0$, we have

$$\lim_{n\to 0} \tilde{G}(a;n) = \lim_{n\to 0} \left( \frac{\mathbb{P}_0(n(1-G(a))) - \mathbb{P}_0(n)}{\alpha(n)} \right) = G(a) \tag{72}$$

using L'Hopital's rule and the fact that $\lim_{z\to 0} \mathbb{P}_0(z) = 1$ and $\lim_{z\to 0} \mathbb{P}'_0(z) = -1$ by Lemma 12. Therefore, $\tilde{a}(n) \to E_G(a)$.

*Part 2.* Taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \tilde{G}(a;n) = \lim_{n\to\infty} \left( \frac{\mathbb{P}_0(n(1-G(a))) - \mathbb{P}_0(n)}{\alpha(n)} \right) = 0 \tag{73}$$

for any $a \in [a_0, \bar{a})$ and $\lim_{n\to\infty} \tilde{G}(\bar{a};n) = 1$, using the fact that $\lim_{z\to\infty} \mathbb{P}_0(z) = 0$ by Lemma 12. Therefore, $\tilde{a}(n) \to \bar{a}$.

*Part 3.* For $n > 0$, we have $\tilde{G}(a;n) < G(a)$ for all $a \in A$. To see this, let $w_k(n) = \mathbb{P}_k(n)/\alpha(n)$. Using (71), $\tilde{G}(a;n) = \sum_{k=1}^{\infty} w_k(n)(G(a))^k$. Since $\tilde{G}(a;n)$ is a weighted average of the term $(G(a))^k$ for all $k \geq 1$, and $(G(a))^k < G(a)$ for all $k > 1$ and $a \in (a_0, \bar{a})$, and $G(a)^k = G(a)$ for $k = 1$ and $a = a_0$ or $a = \bar{a}$, we have $\tilde{G}(a;n) < G(a)$. So $\tilde{G}(a;n)$ first order stochastically dominates $G(a)$ and $\tilde{a}(n) > E_G(a)$. ■

**Proof of Lemma 5**

*Part 1.* Starting with (6) and letting $x = n(1 - G(a))$, we have

$$\tilde{G}(a;n) = \frac{\mathbb{P}_0(x) - \mathbb{P}_0(n)}{\alpha(n)} \tag{74}$$

and therefore, using $\alpha(n) = 1 - \mathbb{P}_0(n)$, we obtain

$$1 - \tilde{G}(a;n) = \frac{\alpha(x)}{\alpha(n)}. \tag{75}$$

Next, differentiating with respect to $a$ yields

$$\tilde{g}(a;n) = \frac{ng(a)\alpha'(x)}{\alpha(n)}. \tag{76}$$

Therefore, we obtain

$$\frac{1 - \tilde{G}(a;n)}{\tilde{g}(a;n)} = \frac{\alpha(x)}{ng(a)\alpha'(x)}, \tag{77}$$

which, using the fact that $x = n(1 - G(a))$, is equivalent to

$$\frac{1 - \tilde{G}(a;n)}{\tilde{g}(a;n)} = \frac{\alpha(x)}{\alpha'(x)x}\left(\frac{1 - G(a)}{g(a)}\right). \tag{78}$$

The hazard rate of the distribution $\tilde{G}(a;n)$ is thus given by

$$h_{\tilde{G}}(a;n) = \eta_\alpha(n(1 - G(a))h_G(a) \tag{79}$$

where $\eta_\alpha(x) = \alpha'(x)x/\alpha(x)$ and $h_G(a) = (1 - G(a))/g(a)$.

*Part 2.* This result follows immediately from (79) plus Lemma 1, which implies that $\eta'_\alpha(x) < 0$ and therefore $\eta_\alpha(x)$ is increasing in $a$, plus our assumption that $G$ has an increasing hazard rate, i.e. $h'_G(a) > 0$. ■

**Proof of Lemma 4**

Applying Leibniz' integral rule gives us

$$\tilde{f}'(n) = \int_{a_0}^{\bar{a}} f(a)\frac{\partial \tilde{g}(a;n)}{\partial n}da. \tag{80}$$

First, we show that there exists a unique cutoff $\hat{a} \in A$ such that $\frac{\partial \tilde{g}(a;n)}{\partial n} = 0$, and we have $\frac{\partial \tilde{g}(a;n)}{\partial n} > 0$ for $a > \hat{a}$ and $\frac{\partial \tilde{g}(a;n)}{\partial n} < 0$ for $a < \hat{a}$. To start with, we have

$$\tilde{g}(a;n) = \frac{-ng(a)\mathbb{P}'_0(n(1-G(a)))}{\alpha(n)}. \tag{81}$$

Differentiating (81) with respect to $n$, we obtain

$$\frac{\partial \tilde{g}(a;n)}{\partial n} = \frac{g(a)\mathbb{P}'_0(x)}{\alpha(n)}\left(\frac{-x\mathbb{P}''_0(x)}{\mathbb{P}'_0(x)} - (1-\eta_\alpha(n))\right) \tag{82}$$

where $x = n(1-G(a))$. Since $\mathbb{P}'_0(x) < 0$ by Lemma 12, $\frac{\partial \tilde{g}(a;n)}{\partial n} > 0$ if and only if

$$\frac{-\mathbb{P}''_0(x)x}{\mathbb{P}'_0(x)} < 1 - \eta_\alpha(n). \tag{83}$$

If Assumption 4 holds, then $\frac{d}{dx}\left(\frac{-\mathbb{P}''_0(x)x}{\mathbb{P}'_0(x)}\right) > 0$. So there exists a unique solution $x$, and therefore a unique solution $a$, such that the above holds with equality. Defining $\hat{a}$ as the unique solution to this equality, we have $\frac{\partial \tilde{g}(a;n)}{\partial n} > 0$ if and only if $a > \hat{a}$. We can rewrite $\tilde{f}'(n)$ as follows:

$$\tilde{f}'(n) \equiv \int_{a_0}^{\hat{a}} f(a)\frac{\partial \tilde{g}(a;n)}{\partial n}da + \int_{\hat{a}}^{\bar{a}} f(a)\frac{\partial \tilde{g}(a;n)}{\partial n}da. \tag{84}$$

We therefore have $\tilde{f}'(n) > 0$ if and only if

$$\int_{\hat{a}}^{\bar{a}} f(a)\frac{\partial \tilde{g}(a;n)}{\partial n}da > -\int_{a_0}^{\hat{a}} f(a)\frac{\partial \tilde{g}(a;n)}{\partial n}da > 0. \tag{85}$$

Given that $f' > 0$, and both sides of (85) are positive, by definition of $\hat{a}$, a sufficient condition for $\tilde{f}'(n) > 0$ is

$$\int_{\hat{a}}^{\bar{a}} f(\hat{a}) \frac{\partial \tilde{g}(a;n)}{\partial n} da \geq - \int_{a_0}^{\hat{a}} f(\hat{a}) \frac{\partial \tilde{g}(a;n)}{\partial n} da, \tag{86}$$

which is true if and only if $\int_{\hat{a}}^{\bar{a}} \frac{\partial \tilde{g}(a;n)}{\partial n} da \geq - \int_{a_0}^{\hat{a}} \frac{\partial \tilde{g}(a;n)}{\partial n} da$, or equivalently $\int_{a_0}^{\bar{a}} \frac{\partial \tilde{g}(a;n)}{\partial n} da \geq 0$. Applying Leibniz' integral rule again, $\int_{a_0}^{\bar{a}} \frac{\partial \tilde{g}(a;n)}{\partial n} da = \frac{\partial}{\partial n} \int_{a_0}^{\bar{a}} \tilde{g}(a;n) da = 0$, since $\int_{a_0}^{\bar{a}} \tilde{g}(a;n) da = 1$, and thus $\tilde{f}'(n) > 0$. ■

**Proof of Corollary 1**

Consider any $f : A \to \mathbb{R}_+$ such that $f' > 0$. Suppose that Assumption 4 holds, so that $\frac{-\mathbb{P}_0''(x)x}{\mathbb{P}_0'(x)}$ is strictly increasing in $x$. For any $n_1$ and $n_2$ such that $n_1 > n_2$, Lemma 3 implies $\tilde{f}(n_1) > \tilde{f}(n_2)$, i.e. $\int_{a_0}^{\bar{a}} f(a) d\tilde{G}(a;n_1) > \int_{a_0}^{\bar{a}} f(a) d\tilde{G}(a;n_2)$. Thus $\tilde{G}(a;n_1) \leq \tilde{G}(a;n_2)$ and $\tilde{G}(a;n_1)$ first order stochastically dominates $\tilde{G}(a;n_2)$. ■

### 13.0.1 Proof of Lemma 6

By definition, $a_f(n)$ solves $\tilde{g}(a;n) = g(a)$. Using (76), this implies that

$$\frac{n g(a_f) \alpha'(x_f)}{\alpha(n)} = g(a_f), \tag{87}$$

or, equivalently,

$$\alpha'(x_f) = \frac{\alpha(n)}{n}, \tag{88}$$

where $x_f = n(1 - G(a_f))$. For any given $n > 0$, the LHS is decreasing in $x$ because $\alpha'' < 0$ by Corollary 6 and is therefore increasing in $a$ because $x = n(1 - G(a))$ and $dx/da = -ng(a)$. For $a = a_0$, we have $x = n$ and thus $\alpha'(x) = \alpha'(n) < \frac{\alpha(n)}{n}$ because $\eta_\alpha(n) < 1$ by Lemma 1. For $a = \bar{a}$, we have $x = 0$ and thus $\alpha'(x) = \lim_{x \to 0} \alpha'(x) = 1 > \frac{\alpha(n)}{n}$ by Lemma 6. Therefore, there exists a unique solution $a_f(n)$.

Also, we have $\tilde{g}(a;n) > g(a)$ if and only if $\alpha'(x) > \frac{\alpha(n)}{n} = \alpha'(x_f)$ where $x_f = n(1 - G(a_f))$, which holds if and only if $x < x_f$ because $\alpha'' < 0$ by Corollary 6. So, $\tilde{g}(a;n) > g(a)$ if and only if $n(1 - G(a)) < n(1 - G(a_f))$, which is true if and only if $G(a) > G(a_f)$, i.e. $a > a_f$. Finally, we have $a_f'(n) > 0$ and $\lim_{n \to \infty} a_f(n) = \bar{a}$. ■

## Proofs for Section 5

### Proof of Proposition 1

The first-order condition with respect to $q_a$ is

$$\alpha(n)[au'(q_a) - c'(q_a)]\tilde{g}(a;n) = 0 \tag{89}$$

and the first order-condition with respect to $n$ is

$$\alpha'(n)\tilde{s}(n;\{q_a\}_{a\in A}) + \alpha(n)\tilde{s}'(n;\{q_a\}_{a\in A}) = K. \tag{90}$$

We can verify that $s_a^* = au(q_a^*) - c(q_a^*)$ is strictly increasing in $a$. Differentiating $s_a^*$,

$$\frac{ds_a^*}{da} = u(q_a^*) + [au'(q_a^*) - c'(q_a^*)]\,\frac{dq^*}{da}. \tag{91}$$

Since $au'(q_a^*) - c'(q_a^*) = 0$ by (89) if $n^* > 0$, we have $\frac{ds_a^*}{da} = u(q_a^*) > 0$ for all $a \in (a_0, \bar{a}]$. Given that $s_a^*$ is strictly increasing in $a$ and $s_0^* \geq 0$ where $s_0^* \equiv a_0 u(q_0) - c(q_0)$, we have $s_a^* \geq 0$ for all $a \in A$. Therefore, all chosen goods $a \in A$ are traded if $a_0 > 0$, and $q_a$ satisfies $au'(q_a) = c'(q_a)$. If $a_0 = 0$, we have $q_a = 0$ since $\lim_{q\to 0} c'(q)/u'(q) = 0$.

Since $s_a^*$ is strictly increasing in $a$, the planner chooses the seller with the highest utility shock $a$. The distribution of chosen goods, $\tilde{G}(a;n)$, is therefore equal to (6).

Existence and uniqueness of the solution to the planner's problem follows from Proposition 6, which is proven below. For the planner's problem, we know that $s_a^* \geq 0$ for all $a \in A$ and thus all chosen goods are traded. Setting $i = 0$ in Proposition 6 results in equilibrium conditions that are equivalent to the planner's first-order conditions. It follows that there exists a unique solution to the planner's problem with $n^* > 0$ provided that Assumption 6 holds, except that $q_a^0 = q_a^*$ since $q_a^*$ does not depend directly on $n$. That is, Assumption 5 suffices. ■

## Proofs for Section 7

### Proof of Proposition 2

Without choice, the distribution of chosen goods is equal to $G$. Other than changing $\tilde{G}$ to $G$, the proofs are identical to those for Proposition 3 except for Part 2. We omit the proof of Proposition 3 because it is lengthy and can be derived from the proof of Proposition 6 by eliminating the IR and IC constraints.

*Part 2.* Replacing $\tilde{G}(a;n)$ in (132) with $G(a)$, the first-order condition for $n$ is

$$\alpha'(n)\int_{a_0}^{\bar{a}}(1-\delta)v_a+\delta s_a dG(a)+\alpha(n)\frac{\partial}{\partial n}\int_{a_0}^{\bar{a}}(1-\delta)v_a+\delta s_a dG(a)=\delta K. \tag{92}$$

Since $\tilde{G}(a;n)=G(a)$, which no longer depends on $n$, the second term on the left is zero and we have

$$\alpha'(n)\int_{a_0}^{\bar{a}}(1-\delta)v_a+\delta s_a dG(a)=\delta K. \tag{93}$$

Rearranging, this is equivalent to

$$\alpha'(n)\int_{a_0}^{\bar{a}}\frac{1-\delta}{\delta}\left(au(q_a)-\frac{d_a}{\gamma}\right)+[au(q_a)-c(q_a)]dG(a)=K. \tag{94}$$

Since (104) implies $\int_{a_0}^{\bar{a}}\frac{d_a}{\gamma}dG(a)=\int_{a_0}^{\bar{a}}c(q_a)dG(a)+\frac{nK}{\alpha(n)}$,

$$\alpha'(n)\int_{a_0}^{\bar{a}}\frac{1-\delta}{\delta}\left(au(q_a)-c(q_a)-\frac{nK}{\alpha(n)}\right)+[au(q_a)-c(q_a)]dG(a)=K. \tag{95}$$

Simplifying further yields

$$\alpha'(n)\int_{a_0}^{\bar{a}}[au(q_a)-c(q_a)]dG(a)=K[\delta+(1-\delta)\eta_\alpha(n)] \tag{96}$$

where $\eta_\alpha(n)\equiv\frac{\alpha'(n)n}{\alpha(n)}$. This is equivalent to

$$\frac{\alpha'(n)\tilde{s}(\{q_a\}_{a\in A})}{\delta+(1-\delta)\eta_\alpha(n)}=K \tag{97}$$

where $\tilde{s}(\{q_a\}_{a\in A})\equiv\int_{a_0}^{\bar{a}}s_a dG(a)$. With full information, we have $\delta=1+\frac{i}{\alpha(n)}$ from (139). Substituting into (97), we obtain (29). ■

**Proof of Proposition 4**

*Part 1.* For any $a \in (a_0, \bar{a}]$, we have $(1 - \phi^{RW})au'(q_a) = c'(q_a)$ and $\phi^{RW} > 0$, so there is underconsumption relative to the efficient allocation, i.e. $q_a < q_a^*$.

*Part 2.* The equilibrium $n$ satisfies

$$\alpha'(n)\tilde{s}(n; \{q_a\}_{a\in A}) + \alpha(n)\tilde{s}'(n; \{q_a\}_{a\in A}) = K \tag{98}$$

and the efficient $n^*$ satisfies

$$\alpha'(n^*)\tilde{s}(n^*; \{q_a^*\}_{a\in A}) + \alpha(n^*)\tilde{s}'(n^*; \{q_a^*\}_{a\in A}) = K. \tag{99}$$

We know from above that $q_a^* > q_a$ for any $a \in (a_0, \bar{a})$, but we cannot infer anything about whether there is under-entry ($n < n^*$), over-entry ($n > n^*$), or efficient entry ($n = n^*$). We can find examples of equilibria for each of these three possibilities.

*Part 3.* Letting $i \to 0$, we have $\phi^{RW} = \frac{i}{i+\alpha(n)} \to 0$, so $q_a = q_a^*$ for all $a \in A$. Given that $q_a = q_a^*$, the equilibrium condition (33) is equivalent to the planner's first-order condition (14) and thus $n = n^*$. ■

**Proof of Proposition 5**

*Part 1.* For any $a \in (a_0, \bar{a}]$, we have $(1 - \phi^{RW})au'(q_a) = c'(q_a)$ and $\phi^{RW} > 0$, so there is underconsumption relative to the efficient allocation, i.e. $q_a < q_a^*$.

*Part 2.* The equilibrium $n$ satisfies

$$\frac{\alpha'(n)\tilde{s}(\{q_a\}_{a\in A})}{1 + (1 - \eta_\alpha(n))\frac{i}{\alpha(n)}} = K \tag{100}$$

and the efficient $n^*$ satisfies

$$\alpha'(n^*)\tilde{s}(\{q_a^*\}_{a\in A}) = K. \tag{101}$$

We know that $q_a^* > q_a$ for any $a \in (a_0, \bar{a}]$. Thus $\tilde{s}(\{q_a^*\}_{a\in A}) > \tilde{s}(\{q_a\}_{a\in A})$ and therefore, since $K$ is constant, we have

$$\alpha'(n^*) < \frac{\alpha'(n)}{1 + (1 - \eta_\alpha(n))\frac{i}{\alpha(n)}}. \tag{102}$$

Now, $1 + (1 - \eta_\alpha(n))\frac{i}{\alpha(n)} \geq 1$ since $\eta_\alpha(n) < 1$, so $\alpha'(n^*) < \alpha'(n)$. Since $\alpha''(n) < 0$ by Lemma 12, we have $n^* > n$ and there is always under-entry of sellers.

*Part 3.* Letting $i \to 0$, we have $\phi^{RW} = \frac{i}{i+\alpha(n)} \to 0$, so $q_a = q_a^*$ for all $a \in A$. In addition, as $i \to 0$, condition (29) is equivalent to the planner's first-order condition (14) and thus $n = n^*$. ■

## Proof of Proposition 6 in Section 6

Our strategy is to solve for the equilibrium in two stages. First, we take $z$ and $n$ as given and solve for $\{(q_a, d_a)\}_{a \in A}$ (inner maximization problem). Second, we solve for $z$ and $n$ (outer maximization problem) given the solutions for $\{(q_a, d_a)\}_{a \in A}$.

We first solve the inner and outer maximization problems. Next, we use the results to prove Parts 1 to 8 of Proposition 6. Finally, we prove existence and uniqueness of equilibrium. Proofs for all lemmas used in this section to prove Proposition 6 are found at the end of this section (unless included earlier).

### Stage 1. Inner maximization problem

In the first stage, taking $z > 0$ and $n > 0$ as given (we later prove this), the market makers' problem is to maximize (24) subject to (25) at equality, plus a liquidity constraint $d_a \leq z$ for all $a \in A$, the IC constraint (36), and the IR constraint (35). Ignoring constants, the market maker's inner maximization problem is:

$$\max_{\{(q_a, d_a)\}_{a \in A}} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[ a u(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a; n) - i \frac{z}{\gamma} \right\}, \tag{103}$$

subject to

$$\frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[ -c(q_a) + \frac{d_a}{\gamma} \right] d\tilde{G}(a; n) = K, \tag{104}$$

and, for all $a, a' \in A$,

$$d_a \leq z, \tag{105}$$

$$au(q_a) - \frac{d_a}{\gamma} \geq au(q_{a'}) - \frac{d_{a'}}{\gamma}, \tag{106}$$

$$au(q_a) - \frac{d_a}{\gamma} \geq 0, \tag{107}$$

$$d_a, q_a \geq 0. \tag{108}$$

To solve the inner maximization problem (103), we transform the above problem as follows. Defining $v_a \equiv au(q_a) - d_a/\gamma$, the buyer's ex post trading surplus, and $\dot{v}_a \equiv v'(a)$, the following lemma simplifies the (IC) constraint. This is a standard result and the proof is omitted.

**Lemma 13.** *The incentive compatibility (IC) constraint holds if and only if (i)* $q'(a) \geq 0$*, and (ii)* $\dot{v}_a = u(q_a)$.

We can now use $v_a \equiv au(q_a) - d_a/\gamma$ and Lemma 13 to re-write the problem as an optimal control problem where $q_a$ is the control variable, $v_a$ is the state variable, and $\delta$ is the Lagrange multiplier associated with the seller entry constraint (104). For simplicity, we assume that $a_0 = 0$.

In the first stage, we take $z$, $n$, $\delta$ as given and later solve for these. Given that $a_0 = 0$, we have $v_0 = 0$. Using $v_a \equiv au(q_a) - d_a/\gamma$ to eliminate $d_a$ in the above, and substituting in the constraint (104), the inner maximization problem becomes

$$\max_{\{(q_a, v_a)\}_{a \in A}} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \{(1-\delta)v_a + \delta [au(q_a) - c(q_a)]\} \tilde{g}(a; n) da - \delta n k - i\frac{z}{\gamma} \right\}, \tag{109}$$

subject to $v_0 = 0$ and, for all $a \in A$,

$$au(q_a) - v_a \leq \frac{z}{\gamma}, \tag{110}$$

$$\dot{v}_a = u(q_a), \tag{111}$$

$$q'(a) \geq 0, \tag{112}$$

$$q_a, v_a \geq 0. \tag{113}$$

The inner maximization problem is a standard optimal control problem with $q_a$ as the control variable and $v_a$ as the state variable. We can therefore apply the Maximum

Principle to find the necessary conditions for the optimal path of the control and state variables. To solve the inner maximization problem, we ignore the condition $q'(a) \geq 0$ and later verify that it holds in Lemma 17. Ignoring the constants, the current value Hamiltonian for the optimal control problem is:

$$H = \alpha(n)\{(1-\delta)v_a + \delta\left[au(q_a) - c(q_a)\right]\}\tilde{g}(a;n) + \lambda_a u(q_a) \tag{114}$$

where $\lambda_a$ is the costate variable, and the Lagrangian is:

$$L = \alpha(n)\{(1-\delta)v_a+\delta\left[au(q_a) - c(q_a)\right]\}\tilde{g}(a;n)+\lambda_a u(q_a)+\mu_a\left[\frac{z}{\gamma} - au(q_a) + v_a\right]+\theta_a q_a+\eta_a v_a \tag{115}$$

where $\mu_a$, $\theta_a$ and $\eta_a$ are the Lagrangian multipliers associated with the liquidity constraint, non-negativity constraint, and IR constraint respectively.

The first-order conditions and the transversality condition are as follows:

$$\frac{\partial L}{\partial q_a} = \alpha(n)\delta\left[au'(q_a) - c'(q_a)\right]\tilde{g}(a;n) + \left(\lambda_a - \mu_a a\right)u'(q_a) + \theta_a = 0, \tag{116}$$

$$\frac{\partial L}{\partial v_a} = (1-\delta)\alpha(n)\tilde{g}(a;n) + \mu_a + \eta_a = -\dot{\lambda}_a, \tag{117}$$

$$\frac{\partial L}{\partial \lambda_a} = \dot{v}_a = u(q_a), \tag{118}$$

$$\lambda_{\bar{a}} v_{\bar{a}} = 0. \tag{119}$$

For the inequality constraints, the conditions are:

$$\mu_a \geq 0,\ \mu_a\left(\frac{z}{\gamma} - au(q_a) + v_a\right) = 0, \tag{120}$$

$$\theta_a \geq 0,\ \theta_a q_a = 0, \tag{121}$$

$$\eta_a \geq 0,\ \eta_a v_a = 0. \tag{122}$$

The following lemma provides expressions for $\lambda_a$ and $\Sigma_{a_c}$, where $\Sigma_a \equiv \int_a^{\bar{a}} \mu_x dx$.

**Lemma 14.** *For all* $a \in [a_0, a_c]$*, we have the following:*

$$\lambda_a = \alpha(n)(1-\delta)[1-\tilde{G}(a;n)] + \Sigma_{a_c} + \int_a^{\bar{a}} \eta_x dx \tag{123}$$

*and*

$$\Sigma_{a_c} = \frac{\alpha(n)}{\bar{a}} \int_{a_c}^{\bar{a}} [\delta(x-a_c)\tilde{g}(x;n) + (1-\delta)(\tilde{G}(a_c;n) - \tilde{G}(x;n)]dx. \tag{124}$$

The next lemma uses our assumption that $a_0 = 0$.

**Lemma 15.** *If* $a_0 = 0$*, we obtain the following:*

$$\delta = 1 + \frac{\Sigma_{a_c} + \int_{a_0}^{\bar{a}} \eta_x dx}{\alpha(n)}. \tag{125}$$

To determine $q_a$ for all $a \in A$, it remains only to determine $\delta$, $a_b$, and $a_c$.

By Lemma 7, there are three intervals to consider.
Case 1. For any $a \in [a_0, a_b]$, $v_a = 0$ for all $a$ and therefore $q_a = 0$.
Case 2. For any $a \in [a_b, a_c]$, we have $\theta_a = 0$ and $\mu_a = 0$, so $q_a$ solves

$$\alpha(n)\delta\left[au'(q_a) - c'(q_a)\right]\tilde{g}(a;n) \ = -\lambda_a u'(q_a). \tag{126}$$

Using the above two lemmas, plus the fact that $\int_a^{\bar{a}} \eta_x dx = \int_a^{a_b} \eta_x dx$ for all $a$ since $\eta_a = 0$ for $a > a_b$, and therefore $\int_a^{\bar{a}} \eta_x dx = \int_a^{a_b} \eta_x dx = 0$ for $a \geq a_b$, we can write

$$(a - \phi(a;n))u'(q_a) = c'(q_a) \tag{127}$$

where

$$\phi(a;n) = -\left(\frac{1-\delta}{\delta}\right)\left(\frac{1-\tilde{G}(a;n)}{\tilde{g}(a;n)}\right) - \frac{\Sigma_{a_c}}{\alpha(n)\delta\tilde{g}(a;n)}. \tag{128}$$

Case 3. For any $a \in [a_c, \bar{a}]$, we have $\theta_a = 0$ and $q_a = q_{a_c}$ by Lemma 7.

The following lemma will prove useful in deriving Proposition 6.

**Lemma 16.** *We have either* $a = \phi(a;n)$ *or* $a = 0$ *for all* $a \leq a_b$*.*

**Proof.** For $a = a_b$, we have $q_{a_b} = 0$. Using (127) above, we have

$$\lim_{a \to a_b} (a - \phi(a;n)) = \lim_{a \to a_b} \left[1 - \frac{\phi(a;n)}{a}\right] a = \lim_{q \to 0} \frac{c'(q)}{u'(q)} = 0 \tag{129}$$

since we have $\lim_{q \to 0} \frac{c'(q)}{u'(q)} = 0$ by assumption. Therefore, by continuity of the function $q_a$, we have either $\frac{\phi(a_b;n)}{a_b} = 1$, or equivalently $a_b = \phi(a_b;n)$, or $a_b = 0$. Similarly, we have $a = \phi(a;n)$ or $a = 0$ for all $a < a_b$. ■

**Stage 2. Outer maximization problem**

The outer maximization problem we solve next is

$$\max_{z,n,\delta} \left\{ J(n,z,\delta) - \delta nk - i\frac{z}{\gamma} \right\}, \tag{130}$$

where we define

$$J(n,z,\delta) \equiv \max_{\{(q_a,v_a)\}_{a \in A}} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \{(1-\delta)v_a + \delta\left[au(q_a) - c(q_a)\right]\} \tilde{g}(a;n)da \right\}, \tag{131}$$

subject to $v_0 = 0$ and, for all $a \in A$, constraints (110), (111), (112), and (113).

To solve the outer maximization problem, the function $J(n,z,\delta)$ is equivalent to

$$J(n,z,\delta) = \max_{\{(q_a,v_a)\}_{a \in A}} \left\{ \begin{array}{c} \int_{a_0}^{\bar{a}} \alpha(n)\{(1-\delta)v_a + \delta\left[au(q_a) - c(q_a)\right]\}\tilde{g}(a;n)da \\ + \int_{a_0}^{\bar{a}} \left[\mu_a\left(\frac{z}{\gamma} - au(q_a) + v_a\right) + \eta_a v_a + \lambda_a u(q_a) + \theta_a q_a\right] da \end{array} \right\}. \tag{132}$$

Define $\tilde{s}(n) \equiv \int_{a_0}^{\bar{a}} s_a d\tilde{G}(a;n)$ and $\tilde{v}(n) \equiv \int_{a_0}^{\bar{a}} v_a d\tilde{G}(a;n)$. Returning to our original formulation to eliminate $\delta$, the above problem is equivalent to

$$\max_{z,n} \left\{ \hat{J}(n,z) - i\frac{z}{\gamma} \right\}, \tag{133}$$

where

$$\hat{J}(n,z) = \max_{\{(q_a,v_a)\}_{a \in A}} \left\{ \alpha(n)\tilde{v}(n) + \int_{a_0}^{\bar{a}} \left[\mu_a\left(\frac{z}{\gamma} - au(q_a) + v_a\right) + \eta_a v_a + \lambda_a u(q_a) + \theta_a q_a\right] da \right\} \tag{134}$$

subject to the constraint

$$\frac{\alpha(n)}{n}[\tilde{s}(n) - \tilde{v}(n)] \leq K \tag{135}$$

and $n \geq 0$ with complementary slackness.

Using the envelope theorem, the first-order conditions for $z$ and $n$ respectively are

$$\int_{a_0}^{\bar{a}} \mu_a da = i \tag{136}$$

and

$$\alpha'(n)\tilde{v}(n) + \alpha(n)\tilde{v}'(n) = 0. \tag{137}$$

Using the fact that $\mu_a = 0$ for all $a < a_c$, by definition of $a_c$, we have $\int_{a_0}^{\bar{a}} \mu_a da = \Sigma_{a_c}$. The first-order condition for $z$ given by (136) thus becomes:

$$\Sigma_{a_c} = i, \tag{138}$$

Substituting $\Sigma_{a_c} = i$ into expression (125) in Lemma 15, the above yields

$$\delta = 1 + \frac{i + \int_{a_0}^{\bar{a}} \eta_x dx}{\alpha(n)}. \tag{139}$$

Finally, we verify that the condition $q'(a) \geq 0$ is indeed satisfied.

**Lemma 17.** *The function $q(.)$ is weakly increasing for all $a \in A$ and $q'(a) > 0$ for all $a \in (a_b, a_c)$.*

### Proof of Parts 1 to 8

*Part 1.* Follows from the definition of $a_b$.

*Part 2.* From above, for any $a \in [a_b, a_c]$, we have

$$(a - \phi(a; n))u'(q_a) = c'(q_a) \tag{140}$$

where, using $\Sigma_{a_c} = i$ plus expression (128) for $\phi(a; n)$, we have

$$\phi(a;n) = -\left(\frac{1-\delta}{\delta}\right)\left(\frac{1-\tilde{G}(a;n)}{\tilde{g}(a;n)}\right) - \frac{i}{\alpha(n)\delta\tilde{g}(a;n)}. \tag{141}$$

The expression for $\delta$ can be derived as follows. Using $[1 - \frac{\phi(a_b;n)}{a_b}]a_b = 0$ from Lemma 16 plus expression (141) for $\phi(a; n)$, we have

$$\left[1 + \left(\frac{1-\delta}{\delta}\right)\left(\frac{1-\tilde{G}(a_b;n)}{a_b\tilde{g}(a_b;n)}\right) + \frac{i}{\alpha(n)\delta a_b\tilde{g}(a_b;n)}\right] a_b = 0 \tag{142}$$

If $a_b \to 0$ then $\delta \to 1 + \frac{i}{\alpha(n)}$ from (125). If $a_b > 0$, the above implies that

$$i = -\alpha(n)[\delta a_b \tilde{g}(a_b;n) + (1-\delta)(1-\tilde{G}(a_b;n))]. \tag{143}$$

For any $a_b \geq 0$, the value of $\delta$ is given by the following expression:

$$\delta = \frac{1-\tilde{G}(a_b;n) + \frac{i}{\alpha(n)}}{1-\tilde{G}(a_b;n) - a_b\tilde{g}(a_b;n)} \tag{144}$$

which is equivalent to (42) using expression (37).

Also, $\dot{v}_a = u(q_a)$ implies $v_a - v_0 = \int_{a_0}^{a} u(q_x)dx$, so $v_a = \int_{a_0}^{a} u(q_x)dx$ since $v_0 = 0$. We can derive $d_a/\gamma$ from $v_a$ using the fact that $v_a \equiv au(q_a) - d_a/\gamma$.

*Part 3.* Clear from Lemma 7.

*Part 4.* Using $\Sigma_{a_c} = i$ and expression (124), and then using integration by parts twice, we obtain (43).

*Part 5.* Clear from the definition of $a_c$.

*Part 6.* The first-order condition for $n > 0$ given by (137) can be written as

$$\alpha'(n)\tilde{s}(n) + \alpha(n)\tilde{s}'(n) = K, \tag{145}$$

using the ZPC constraint (135) at equality. More precisely, this is equivalent to

$$\alpha'(n)\tilde{s}(n;\{q_a\}_{a\in A}) + \alpha(n)\tilde{s}'(n;\{q_a\}_{a\in A}) = K. \tag{146}$$

The fact that $n$ is strictly decreasing in $K$ is proven in Lemma 20 below.

*Part 7.* The zero profit condition is given by (135), using the definition of $v_a$.

*Part 8.* Since $v_a$ is increasing in $a$, the highest draw is always chosen by buyers. Therefore the cdf of chosen goods is given by (6). ■

**Proof of existence and uniqueness**

We first prove existence and uniqueness of the solution to the inner maximization problem and then prove the same for the outer maximization problem.

**Inner maximization.** We prove that, given $z$ and $n$ from the outer maximization problem, the solution to the inner maximization problem exists and is unique.

*Existence.* A solution to the problem exists because the set of admissible paths is non-empty and compact, and there exists an admissible path for which the objective is finite. For example, the path $q_a = 0$ and $v_a = (a-1)u(q_a)$ for all $a \in A$ is admissible (since $v_0 = 0$, $au(q_a) - v_a \leq z/\gamma$, $q_a \geq 0$, $v_a \geq 0$, and $\dot{v}_a = u(q_a) + (a-1)u'(q_a)q'(a) = u(q_a)$, and $q'(a) \geq 0$). Also, the objective is finite under this path. Finally, the set of feasible paths is compact since $q_a \in [0, q^*_{\bar{a}}]$ where $q^*_{\bar{a}}$ solves $\bar{a}u'(q_{\bar{a}}) = c'(q_{\bar{a}})$ and $v_a \in [0, v_{\bar{a}}]$ where $v_{\bar{a}} = u(q^*_{\bar{a}})[\bar{a} - a_0]$ since $v_a = \int_{a_0}^{a} u(q_x)dx$.

*Uniqueness.* The Hamiltonian $H(q_a, v_a, \lambda_a)$, where $\lambda_a$ is the co-state variable given by the Maximum Principle, is strictly concave in the control and state variables $(q_a, v_a)$ for all $a$. Therefore, the solution is an optimum that solves the inner maximization problem and it is unique. To establish strict concavity, differentiating $H(q_a, v_a, \lambda_a)$ with respect to $q_a$ yields

$$\begin{aligned}
\frac{\partial H}{\partial q_a} &= \alpha(n)\delta[u'(q_a) - c'(q_a)]\tilde{g}(a;n) + \lambda_a u'(q_a), \\
\frac{\partial^2 H}{\partial q_a^2} &= \alpha(n)\delta[u''(q_a) - c''(q_a)]\tilde{g}(a;n) + \lambda_a u''(q_a) \equiv -X,
\end{aligned}$$

where $X > 0$, since $u''(q_a) < 0$ and $c''(q_a) > 0$. Differentiating $H(q_a, v_a, \lambda_a)$ with respect to $v_a$, we obtain $\frac{\partial H}{\partial v_a} = \alpha(n)(1-\delta)\tilde{g}(a;n)$ and $\frac{\partial^2 H}{\partial v_a^2} = 0$. Finally, $\frac{\partial^2 H}{\partial v_a \partial q_a} = 0$, so we get the Hessian matrix, $\mathbb{H} = \begin{bmatrix} -X & 0 \\ 0 & 0 \end{bmatrix}$. Since $\mathbf{x}^T\mathbb{H}\mathbf{x} < 0$ for all $\mathbf{x} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$, the Hessian $\mathbb{H}$ is negative definite and the Hamiltonian is strictly concave in $(q_a, v_a)$.

**Outer maximization.** We prove that, given $\{(q_a, v_a)\}_{a \in A}$ from the inner maximization problem, the solution $(n, z)$ to the outer maximization problem exists and is unique, and $n, z$ are interior solutions with $n, z > 0$ if Assumption 6 holds. To establish this result, we first prove that there exists a non-empty set of solutions $n$, denoted by $N(K)$, that solves the problem. We then show that equilibrium is unique if $n > 0$ for all $n \in N(K)$, and finally we prove that $n > 0$ for any $n \in N(K)$.

Taking $\{(q_a, v_a)\}_{a \in A}$ as given by the inner maximization problem, and ignoring constants, the outer maximization problem is equivalent to

$$\max_{z,n} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[ au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a; n) + (\Sigma_{a_c} - i) \frac{z}{\gamma} \right\}, \tag{147}$$

subject to

$$\frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[ -c(q_a) + \frac{d_a}{\gamma} \right] d\tilde{G}(a; n) \leq K \tag{148}$$

and $n \geq 0$ with complementary slackness, where $\{(q_a, v_a)\}_{a \in A}$ solves the inner maximization problem.

**Lemma 18.** *The set of solutions $N(K)$ is nonempty and upper hemicontinuous.*

**Proof.** Since $\alpha(n)$ is a bijection, we can rewrite (147) in terms of $\alpha$ as follows:

$$\max_{z,\alpha} \left\{ \alpha \int_{a_0}^{\bar{a}} \left[ au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a; \alpha) + (\Sigma_{a_c} - i) \frac{z}{\gamma} \right\}. \tag{149}$$

The objective function is continuous and, without loss of generality, we can restrict $(z, \alpha)$ to the following compact set:

$$\Delta = \{(z, \alpha) : \alpha \in [0, 1],\ z/\gamma \in [0, \bar{a}u(q_{\bar{a}})]\} \tag{150}$$

since $q \in [0, q_{\bar{a}}^*]$ where $q_{\bar{a}}^*$ solves $\bar{a}u'(q_{\bar{a}}) = c'(q_{\bar{a}})$, and we have $z/\gamma < \bar{a}u(q_{\bar{a}})$. The constraint (148) can therefore be written as $(z, \alpha) \in \Gamma(K)$ for all $K \geq 0$, where $\Gamma(K)$ is a continuous and compact-valued correspondence. Applying the Theorem of the Maximum (Theorem 3.6 in Stokey, Lucas, and Prescott, 1989), the correspondence that gives the set of solutions for $\alpha$ is nonempty and upper hemicontinuous, and therefore also $N(K)$ is nonempty and upper hemicontinuous. ■

The following lemma establishes that any strictly positive solution $n \in N(K)$ must be unique. Since we know that $z = d_{a_c} > 0$ where $d_a/\gamma = au(q_a) - v_a$, and

$\{(q_a, v_a)\}_{a\in A}$ is given by the inner maximization problem, Lemma 19 implies that any solution $(n, z)$ where $n > 0$ is unique.

**Lemma 19.** *If* $N^+ \subseteq N(K)$ *and* $N^+ \subseteq \mathbb{R}_+\backslash\{0\}$, *then* $N^+ = \{n\}$.

**Proof.** Consider any solution $n \in N(K)$ such that $n > 0$. Defining $\Phi(n) \equiv \alpha(n)\tilde{v}(n)$, the solutions $n$ satisfy the first-order condition (137), which says $\Phi'(n) = 0$. We show that $\Phi''(n) < 0$ and thus any solution is unique. Using (81), we have

$$\Phi(n) = -\int_{a_0}^{\bar{a}} \mathbb{P}_0'(n(1-G(a)))v_a g(a)da. \tag{151}$$

Using Leibniz's integral rule, plus the envelope theorem,

$$\Phi'(n) = \int_{a_0}^{\bar{a}} -\mathbb{P}_0'(n(1-G(a)))v_a g(a)da - \int_{a_0}^{\bar{a}} n(1-G(a))\mathbb{P}_0''(n(1-G(a)))v_a g(a)da.$$

By integration by parts on the second integral in $\Phi'(n)$ above, we obtain

$$\Phi'(n) = \int_{a_0}^{\bar{a}} -\mathbb{P}_0'(n(1-G(a)))(1-G(a))v'(a)da - \mathbb{P}_0'(n)v(a_0) > 0. \tag{152}$$

Differentiating (152), we find that

$$\Phi''(n) = -\left(\int_{a_0}^{\bar{a}} \mathbb{P}_0''(n(1-G(a))(1-G(a))^2 v'(a)da + \mathbb{P}_0''(n)v(a_0)\right) < 0. \tag{153}$$

The fact that $\Phi''(n) < 0$ follows from the fact that $\mathbb{P}_0''(x) > 0$ by Lemma 12, plus the fact that $v'(a) = u(q_a) \geq 0$ for all $a$ and $v'(a) > 0$ for some $a$ and also $v(a_0) = 0$. Therefore, any solution $n > 0$ is unique. ■

From Lemma 18, we know that, for any given $K \geq 0$, there exists a non-empty set of solutions $N(K)$ that solves problem (147). We also know that any solution $z$ is interior, since $z/\gamma = \bar{a}u(q_{\bar{a}})$ implies $v_{\bar{a}} = \bar{a}u(q_{\bar{a}}) - \bar{z}/\gamma = 0$ and therefore $v_a = 0$ for all $a \in A$. We now prove that, for any $n \in N(K)$, we have $n \in \mathbb{R}_+\backslash\{0\}$ provided that Assumption 6 holds. Also, the function $n(K)$ is strictly decreasing in $K$.

**Lemma 20.** *Any solution* $n \in N(K)$ *is interior, i.e.* $n \in \mathbb{R}_+\backslash\{0\}$. *The function* $n(K)$ *is strictly decreasing in* $K$.

**Proof.** First, we show there exists an interior solution $n > 0$. Define $\Lambda(n) \equiv \alpha(n)\tilde{s}(n)$. The first-order condition (145) says $\Lambda'(n) = K$. We prove there exists $n > 0$ such that $\Lambda'(n) = K$ if Assumption 6 holds. We have $\lim_{n\to\infty} \Lambda'(n) = 0$, and

$$\lim_{n\to 0} \Lambda'(n) = \int_{a_0}^{\bar{a}} \lim_{n\to 0} s(a; q_a(n))dG(a) \tag{154}$$

where $\lim_{n\to 0} s(a; q_a(n)) = s(a; \lim_{n\to 0} q_a(n))$. If the following condition holds:

$$E_G[au(q_a^0) - c(q_a^0)] > K \tag{155}$$

where $q_a^0 \equiv \lim_{n\to 0} q_a(n)$, there exists $n > 0$ that satisfies $\Lambda'(n) = K$ provided that $\Lambda''(n) < 0$ (which we prove below).

Next, any interior solution $n > 0$ is better than $n = 0$. Define the value function:

$$V(K,\gamma) \equiv \max_{z,n} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[ au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a;n) + (\Sigma_{a_c} - i)\frac{z}{\gamma} \right\}. \tag{156}$$

Since we know that $z$ is interior, we have $V(K,\gamma) \equiv \max_n \{\alpha(n)\tilde{v}(n)\}$ since $\int_{a_0}^{\bar{a}} \mu_a = i$. If $n = 0$ then $V(K,\gamma) = 0$. If $n > 0$, $V(K,\gamma) \equiv \max_n \{\alpha(n)\tilde{s}(n) - nk\}$ using constraint (148) with equality. Letting $\Lambda(n) = \alpha(n)\tilde{s}(n)$, we have $V(K,\gamma) > 0$ if $\Lambda(n) - nk > 0$. Thus the candidate solution $n > 0$ is better than $n = 0$ if $\Lambda(n) > nk$ for $n > 0$. Using the fact that $\Lambda'(n) = K$, it suffices to show that $\Lambda''(n) < 0$ and $\frac{\Lambda'(n)n}{\Lambda(n)} < 1$ for $n > 0$. Similarly to Lemma 19, using (81), we have

$$\Lambda(n) = -\int_{a_0}^{\bar{a}} n\mathbb{P}_0'(n(1-G(a)))s_a g(a)da \tag{157}$$

and, using Leibniz's integral rule, plus the envelope theorem, yields

$$\Lambda'(n) = \int_{a_0}^{\bar{a}} -\mathbb{P}_0'(n(1-G(a)))s_a g(a)da - \int_{a_0}^{\bar{a}} n(1-G(a))\mathbb{P}_0''(n(1-G(a)))s_a g(a)da.$$

Therefore, letting $x = n(1-G(a))$, we have

$$\frac{\Lambda'(n)n}{\Lambda(n)} = 1 + \frac{\int_{a_0}^{\bar{a}} x\mathbb{P}_0''(x)s_a g(a)da}{\int_{a_0}^{\bar{a}} \mathbb{P}_0'(x)s_a g(a)da}. \tag{158}$$

Because $\mathbb{P}_0''(x) > 0$ and $\mathbb{P}_0'(x) < 0$ by Lemma 12, we have $\frac{\Lambda'(n)n}{\Lambda(n)} < 1$ for $n > 0$.

Finally, $\Phi(n) = \Lambda(n) - nk$ for $n > 0$, so $\Phi'(n) = \Lambda'(n) - K$ and $\Phi''(n) = \Lambda''(n)$. Since $\Phi''(n) < 0$ from the proof of Lemma 19, we have $\Lambda''(n) < 0$. It follows that, for any $n \in N(K)$, we have $n > 0$. Since we assume $K > 0$, this implies $n \in \mathbb{R}_+\backslash\{0\}$. Since $n$ is unique by Lemma 19, there is a function $n : \mathbb{R}_+\backslash\{0\} \to \mathbb{R}_+\backslash\{0\}$ such that $n(K)$ solves $\Lambda'(n) = K$. Clearly, $n$ is strictly decreasing in $K$ since $\Lambda''(n) < 0$. ■

**Proof of Lemma 14**

Start with the fact that

$$(1-\delta)\alpha(n)\tilde{g}(a;n) + \mu_a + \eta_a = -\dot{\lambda}_a \tag{159}$$

from the first-order condition (117) above. Integrating both sides over $[a, \bar{a}]$, we obtain

$$-\int_a^{\bar{a}} \dot{\lambda}_x dx = \int_a^{\bar{a}} (1-\delta)\alpha(n)\tilde{g}(x;n)dx + \int_a^{\bar{a}} \mu_x dx + \int_a^{\bar{a}} \eta_x dx \tag{160}$$

and therefore

$$-(\lambda_{\bar{a}} - \lambda_a) = \alpha(n)(1-\delta)\int_a^{\bar{a}} \tilde{g}(x;n)dx + \int_a^{\bar{a}} \mu_x dx + \int_a^{\bar{a}} \eta_x dx. \tag{161}$$

The transversality condition $\lambda_{\bar{a}} v_{\bar{a}} = 0$ implies $\lambda_{\bar{a}} = 0$ since $v_{\bar{a}} > 0$. Substituting $\Sigma_a \equiv \int_a^{\bar{a}} \mu_x dx$ into the above, and setting $\lambda_{\bar{a}} = 0$ yields

$$\lambda_a = \alpha(n)(1-\delta)\int_a^{\bar{a}} \tilde{g}(x;n)dx + \Sigma_a + \int_a^{\bar{a}} \eta_x dx. \tag{162}$$

Now, $\mu_a = 0$ for all $a \in [a_0, a_c]$, thus $\Sigma_a = \int_a^{\bar{a}} \mu_x dx = \int_{a_c}^{\bar{a}} \mu_x dx = \Sigma_{a_c}$ for all $a \in [a_0, a_c]$. Substituting into (162), and using the fact that $\int_a^{\bar{a}} \tilde{g}(x;n)dx = [\tilde{G}(x;n)]_a^{\bar{a}} = 1 - \tilde{G}(a;n)$, we obtain (123).

For the second part, using (116) and Lemma 7, for all $a \in [a_c, \bar{a}]$ we have

$$\alpha(n)\delta\left[au'(\bar{q}) - c'(\bar{q})\right]\tilde{g}(a;n) + (\lambda_a - \mu_a a)\,u'(\bar{q}) = 0 \tag{163}$$

where $\bar{q} \equiv q_{a_c}$, and, for all $a \in [a_c, \bar{a}]$, we also have

$$\alpha(n)\delta\left[a_c u'(\bar{q}) - c'(\bar{q})\right]\tilde{g}(a;n) + \lambda_{a_c} u'(\bar{q}) = 0. \tag{164}$$

Using the above two equations, and dividing both sides by $u'(\bar{q})$, we obtain

$$\alpha(n)\delta(a-a_c)\tilde{g}(a;n) = -\lambda_a + \mu_a a + \lambda_{a_c}. \tag{165}$$

Substituting (162) for both $\lambda_a$ and $\lambda_{a_c}$ into the above, and simplifying, yields

$$\alpha(n)[\delta(a-a_c)\tilde{g}(a;n) + (1-\delta)(\tilde{G}(a_c;n) - \tilde{G}(a;n)] = -\Sigma_a + \mu_a a + \Sigma_{a_c}. \tag{166}$$

Finally, $\Sigma_a = \int_a^{\bar{a}} \mu_x dx$ implies that $\dot{\Sigma}_a = -\mu_a$ and thus we obtain

$$\alpha(n)[\delta(a-a_c)\tilde{g}(a;n) + (1-\delta)(\tilde{G}(a_c;n) - \tilde{G}(a;n)] = -\Sigma_a - \dot{\Sigma}_a a + \Sigma_{a_c}. \tag{167}$$

Integrating both sides over $[a_c, \bar{a}]$, we have

$$\alpha(n)\int_{a_c}^{\bar{a}} [\delta(x-a_c)\tilde{g}(x;n) + (1-\delta)(\tilde{G}(a_c;n) - \tilde{G}(x;n)]dx = \int_{a_c}^{\bar{a}} \left(-\Sigma_x - \dot{\Sigma}_x x + \Sigma_{a_c}\right) dx \tag{168}$$

where $\int_{a_c}^{\bar{a}} \left(-\Sigma_x - \dot{\Sigma}_x x + \Sigma_{a_c}\right) dx = -\left(\int_{a_c}^{\bar{a}} \Sigma_x + \dot{\Sigma}_x x \; dx\right) + [\Sigma_{a_c} x]_{a_c}^{\bar{a}}$. Using integration by parts, $\int_{a_c}^{\bar{a}} \Sigma_x + \dot{\Sigma}_x x \; dx = [\Sigma_x x]_{a_c}^{\bar{a}} = \Sigma_{\bar{a}}\bar{a} - \Sigma_{a_c}a_c = -\Sigma_{a_c}a_c$, and $[\Sigma_{a_c} x]_{a_c}^{\bar{a}} = \Sigma_{a_c}\bar{a} - \Sigma_{a_c}a_c$. Substituting $\int_{a_c}^{\bar{a}} \left(-\Sigma_x - \dot{\Sigma}_x x + \Sigma_{a_c}\right) dx = \Sigma_{a_c}\bar{a}$ into the above yields

$$\alpha(n)\int_{a_c}^{\bar{a}} [\delta(x-a_c)\tilde{g}(x;n) + (1-\delta)(\tilde{G}(a_c;n) - \tilde{G}(x;n)]dx = \Sigma_{a_c}\bar{a} \tag{169}$$

and we therefore obtain (124). ■

**Proof of Lemma 15**

To start with, we have

$$\alpha(n)\delta\left[au'(q_a) - c'(q_a)\right]\tilde{g}(a;n) + (\lambda_a - \mu_a a)\,u'(q_a) + \theta_a = 0 \tag{170}$$

from the first-order condition (116) for $q_a$. Dividing both sides by $q_a$, we obtain

$$\alpha(n)\delta\left[a - \frac{c'(q_a)}{u'(q_a)}\right]\tilde{g}(a;n) + (\lambda_a - \mu_a a) = \frac{-\theta_a}{u'(q_a)}. \tag{171}$$

Taking the limit as $q_a \to 0$, and using $\lim_{q\to 0} u'(q) = +\infty$ and $\lim_{q\to 0} \frac{c'(q)}{u'(q)} = 0$ yields

(172)
$$\lim_{q\to 0} \alpha(n)\delta \left[a - \frac{c'(q)}{u'(q)}\right] \tilde{g}(a;n) + (\lambda_a - \mu_a a) + \frac{\theta_a}{u'(q)} = \alpha(n)\delta a \tilde{g}(a;n) + (\lambda_a - \mu_a a) = 0$$

for any $a \leq a_b$ and therefore

$$\lambda_a = -\alpha(n)\delta a \tilde{g}(a;n) - \mu_a a \tag{173}$$

for any $a \leq a_b$. In particular, we have

$$\lambda_{a_0} = -\alpha(n)\delta a_0 \tilde{g}(a_0;n) - \mu_{a_0} a_0. \tag{174}$$

If $a_0 = 0$, then the above implies that $\lambda_{a_0} = 0$. Next, applying Lemma 14 to the special case $a = a_0$, we have

$$\lambda_{a_0} = \alpha(n)(1-\delta) + \Sigma_{a_c} + \int_{a_0}^{\bar{a}} \eta_x dx. \tag{175}$$

Therefore, if $a_0 = 0$, we have $\lambda_{a_0} = \alpha(n)(1-\delta) + \Sigma_{a_c} + \int_{a_0}^{\bar{a}} \eta_x dx = 0$. ■

**Proof of Lemma 17**

For all $a \leq a_b$, we have $q_a = 0$ and $q'(a) = 0$. For all $a$ greater than or equal to $a_c$, $q_a$ is constant and thus $q'(a) = 0$. For $a \in (a_b, a_c)$, implicit differentiation of

$$(a - \phi(a;n))u'(q_a) = c'(q_a) \tag{176}$$

yields

$$q'(a) = \frac{-[1-\phi'(a)]u'(q_a)}{[a-\phi(a;n)]u''(q_a) - c''(q_a)} \tag{177}$$

where $\phi(a;n)$ can be simplified to:

$$\phi(a;n) = -\left(\frac{(1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}}{\delta \tilde{g}(a;n)}\right). \tag{178}$$

Differentiating the above with respect to $a$ yields

$$\phi'(a) = \frac{1-\delta}{\delta} + \frac{\left[(1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}\right]\tilde{g}'(a;n)}{\delta \tilde{g}(a;n)^2}, \tag{179}$$

Since $u'(q_a) > 0$ and $u''(q_a) < 0$ and $c''(q_a) > 0$ and $a - \phi(a;n) > 0$, to establish $q'(a) \geq 0$ it suffices to show that $\phi'(a) < 1$. That is, we require

$$\phi'(a) = \frac{1-\delta}{\delta} - \frac{\phi(a;n)\tilde{g}'(a;n)}{\tilde{g}(a;n)} < 1. \tag{180}$$

Since $\delta \geq 1$, we have $\phi'(a) < 0$ if either $\tilde{g}'(a;n) < 0$ and $\phi(a;n) < 0$, or otherwise $\tilde{g}'(a;n) > 0$. and $\phi(a;n) > 0$. So, we need only consider two cases: $(i)$ $\tilde{g}'(a;n) < 0$ and $\phi(a;n) > 0$ and $(ii)$ $\tilde{g}'(a;n) > 0$ and $\phi(a;n) < 0$.

For case $(i)$, assume that $\tilde{g}'(a;n) < 0$ and $\phi(a;n) > 0$. Writing

$$\phi(a;n) = \left(1 - \frac{1}{\delta}\right)\left(\frac{1-\tilde{G}(a;n)}{\tilde{g}(a;n)}\right) - \frac{i}{\alpha(n)\delta\tilde{g}(a;n)}, \tag{181}$$

The distribution $\tilde{G}(a;n)$ has an increasing hazard rate, i.e. $h'_{\tilde{G}}(a;n) \equiv \frac{\partial h_{\tilde{G}}(a;n)}{\partial a} \geq 0$ by Lemma 5, and $\delta \geq 1$ so the first term is weakly decreasing in $a$. Also, since $\tilde{g}'(a;n) < 0$, the second term (including the minus sign) is strictly decreasing in $a$. So, we have $\phi'(a) < 0$ and thus $\phi'(a) < 1$.

For case $(ii)$, assume that $\tilde{g}'(a;n) > 0$ and $\phi(a;n) < 0$. Rearranging (179), we have $\phi'(a) < 1$ if and only if

$$\left(\frac{(1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}}{\tilde{G}(a;n)}\right)\left(\frac{\tilde{g}'(a;n)\tilde{G}(a;n)}{\tilde{g}(a;n)^2}\right) < 2\delta - 1. \tag{182}$$

First, using (6) and (81), and differentating (81),

$$\frac{\tilde{g}'(a;n)\tilde{G}(a;n)}{\tilde{g}(a;n)^2} = \left(1 - \frac{\mathbb{P}_0(n)}{\mathbb{P}_0(x)}\right)\left(\frac{\mathbb{P}_0''(x)\mathbb{P}_0(x)}{\mathbb{P}_0'(x)^2} - \frac{\mathbb{P}_0(x)}{\mathbb{P}_0'(x)}\frac{G''(a)}{ng(a)^2}\right),$$

where $x = n(1-G(a))$ and $\frac{\mathbb{P}_0(n)}{\mathbb{P}_0(x)} \le 1$, and therefore

$$\frac{\tilde{g}'(a;n)\tilde{G}(a;n)}{\tilde{g}(a;n)^2} \le \frac{\mathbb{P}_0''(x)\mathbb{P}_0(x)}{\mathbb{P}_0'(x)^2} - \frac{\mathbb{P}_0(x)}{\mathbb{P}_0'(x)} \frac{G''(a)}{ng(a)^2}. \tag{183}$$

Notice that the above is equivalent to

$$\frac{\tilde{g}'(a;n)\tilde{G}(a;n)}{\tilde{g}(a;n)^2} \le \frac{\mathbb{P}_0''(x)\mathbb{P}_0(x)}{\mathbb{P}_0'(x)^2} - \frac{\mathbb{P}_0(x)}{\mathbb{P}_0'(x)x} \frac{G''(a)(1-G(a))}{g(a)^2}. \tag{184}$$

Given that $\mathbb{P}_0'(x) \le 0$ by Lemma 12, and we assume $G''(a) \le 0$, the entire second term on the right side of the inequality (including the minus sign) is negative, so

$$\frac{\tilde{g}'(a;n)\tilde{G}(a;n)}{\tilde{g}(a;n)^2} \le \frac{\mathbb{P}_0''(x)\mathbb{P}_0(x)}{\mathbb{P}_0'(x)^2}. \tag{185}$$

Now, if Assumption 7 holds then $\frac{\mathbb{P}_0''(x)\mathbb{P}_0(x)}{\mathbb{P}_0'(x)^2} \le 2$, which implies that

$$\frac{\tilde{g}'(a;n)\tilde{G}(a;n)}{\tilde{g}(a;n)^2} \le 2. \tag{186}$$

Given inequality (186), to prove (182) it suffices to show that

$$\frac{(1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}}{\tilde{G}(a;n)} < \delta - \frac{1}{2}. \tag{187}$$

Rearranging the above and simplifying, this is equivalent to

$$\delta + \frac{1}{2}\tilde{G}(a;n) > 1 + \frac{i}{\alpha(n)}. \tag{188}$$

For any $a \in (a_b, a_c)$, this is true if $\delta \ge 1 + \frac{i}{\alpha(n)}$, which is true since

$$\delta = \frac{1-\tilde{G}(a_b;n) + \frac{i}{\alpha(n)}}{1-\tilde{G}(a_b;n) - a_b\tilde{g}(a_b;n)} \ge 1 + \frac{i}{\alpha(n)(1-\tilde{G}(a_b;n))}. \tag{189}$$

Therefore, for both cases we have $q'(a) > 0$ for all $a \in (a_b, a_c)$. ■

## Other proofs for Section 6

### Proof of Lemma 7

Define $v_a \equiv au(q_a) - d_a/\gamma$, the buyer's ex post trading surplus, and $\dot{v}_a \equiv v'(a)$. First, it follows from the facts that $v_a \geq 0$ for all $a$ and $\dot{v}_a = u(q_a) \geq 0$ that there exists a unique $a_b \in A$ such that $q_a = 0$ and $d_a = 0$ if and only if $a \leq a_b$.

Next, let $f(a) = \frac{z}{\gamma} - au(q_a) + v_a$. Constraint (110) binds if and only if $f(a) = 0$. Differentiating, we have $f'(a) = -(u(q_a) + au'(q_a)q'(a)) + \dot{v}_a$. Using $\dot{v}_a = u(q_a)$, this implies that $f'(a) = -au'(q_a)q'(a)$. Since $u'(q_a) > 0$ and $q'(a) \geq 0$ is a constraint, we have $f'(a) \leq 0$. Therefore, there exists a unique $a_c \in A$ such that $f(a) = 0$ and constraint (110) binds if and only if $a \in [a_c, \bar{a}]$, so $d_a = z$ and thus $\frac{z}{\gamma} = au(q_a) - v_a$. Differentiating, we have $au'(q_a)q'(a) = 0$ for all $a \in [a_c, \bar{a}]$. Since $u'(q_a) > 0$ and $q'(a) \geq 0$ is a constraint, this requires $q'(a) = 0$ and thus $q_a = q_{a_c}$ on $[a_c, \bar{a}]$.

Finally, it is clear that $a_0 \leq a_b$ and $a_c \leq \bar{a}$. It remains only to show that $a_b \leq a_c$. We have $q_{a_b} = 0$ while $q_{a_c} > 0$, so $q_{a_b} \leq q_{a_c}$ and thus $a_b \leq a_c$ because $q'(a) \geq 0$. ■

### Proof of Lemma 8

In the limit as $n \to 0$, we have $\tilde{G}(a;n) \to G(a)$ by Lemma 3. As $n \to 0$, we have $i/\alpha(n) \to \infty$ so $\delta \to \infty$. Also, $1/\delta \to 0$ implies that $\phi(a;n) \to \frac{1-G(a)}{g(a)}$ on $(a_b, a_c]$. From Lemma 16, we have $[1 - \frac{\phi(a_b;n)}{a_b}]a_b = 0$, which is equivalent to $\psi_G(a_b) = 0$ where $\psi_G(a) \equiv a - \frac{1-G(a)}{g(a)}$. It follows from Assumption 2 that $\psi_G'(a) > 0$. Also, we have $\psi_G(a_0) = \psi_G(0) \leq 0$ and $\psi_G(\bar{a}) = \bar{a} > 0$. Therefore, there exists a unique solution $a_b \in [a_0, \bar{a})$ to $\psi_G(a) = 0$. Finally, as $n \to 0$, the condition for $a_c$ reduces to

$$(\bar{a} - a_c)[1 - G(a_c)] = \bar{a}\left[\frac{-\psi_G(a_b)}{a_b - \psi_G(a_b)}\right](1 - G(a_b)). \tag{190}$$

Given that $\psi_G(a_b) = 0$, if $a_b > 0$ then the right side is zero, which implies $a_c = \bar{a}$. ■

### Proof of Corollary 2

Starting with Proposition 6, we can directly replace the distribution of chosen goods is with the exogenous distribution of available goods, $G(a)$. We can also replace $\rho(a_b;n)$ with $\rho(a_b)$ and $\varepsilon_\rho(a_b;n)$ with $\varepsilon_\rho(a_b)$ because these no longer depend directly on $n$. Aside from these changes, the only part that is different from 6 is Part 6, which we derive here. Starting with (132) and replacing $\tilde{G}(a;n)$ with $G(a)$, the first-order condition for $n$ is given by

$$\alpha'(n)\int_{a_0}^{\bar{a}}(1-\delta)v_a+\delta s_a dG(a)+\alpha(n)\frac{\partial}{\partial n}\int_{a_0}^{\bar{a}}(1-\delta)v_a+\delta s_a dG(a)=\delta K. \tag{191}$$

Now, the second term on the left is zero because it does not depend directly on $n$, so

$$\alpha'(n)\int_{a_0}^{\bar{a}}(1-\delta)v_a+\delta s_a dG(a)=\delta K. \tag{192}$$

Rearranging, this is equivalent to

$$\alpha'(n)\int_{a_0}^{\bar{a}}\frac{1-\delta}{\delta}\left(au(q_a)-\frac{d_a}{\gamma}\right)+[au(q_a)-c(q_a)]dG(a)=K. \tag{193}$$

Since the zero profit condition implies $\int_{a_0}^{\bar{a}}\frac{d_a}{\gamma}dG(a)=\int_{a_0}^{\bar{a}}c(q_a)dG(a)+\frac{nK}{\alpha(n)}$, we obtain

$$\alpha'(n)\int_{a_0}^{\bar{a}}\frac{1-\delta}{\delta}\left(au(q_a)-c(q_a)-\frac{nK}{\alpha(n)}\right)+[au(q_a)-c(q_a)]dG(a)=K \tag{194}$$

and simplifying further yields

$$\alpha'(n)\int_{a_0}^{\bar{a}}[au(q_a)-c(q_a)]dG(a)=K[\delta+(1-\delta)\eta_\alpha(n)] \tag{195}$$

where $\eta_\alpha(n)\equiv\frac{\alpha'(n)n}{\alpha(n)}$. This is equivalent to (50) where $\tilde{s}(\{q_a\}_{a\in A})\equiv\int_{a_0}^{\bar{a}}s_a dG(a)$. ■

## Proofs for Section 9

We first present a lemma that is used to prove Proposition 7.

**Lemma 21.** *In any equilibrium where $i>0$,*

1. *There exists a unique cutoff $a_p\in[a_b,\bar{a}]$ such that (i) if $a_p\le a_c$, there is underconsumption for all $a\in(a_0,a_p)$ and overconsumption for all $a\in(a_p,a_c)$, and (ii) if $a_p>a_c$, there is underconsumption for all $a\in(a_0,a_p)$.*

2. *There exists a unique cutoff $a_d\in[a_b,\bar{a}]$ such that (i) if $a_c\le a_d$, there is overconsumption for all $a\in[a_c,a_d)$ and underconsumption for all $a\in(a_d,\bar{a}]$, and (ii) if $a_c>a_d$, there is underconsumption for all $a\in[a_c,\bar{a}]$.*

**Proof.** *Part 1. (i)* For $a \in (a_0, a_b]$, there is underconsumption, i.e. $q_a < q_a^*$, since $q_a = 0$ but $q_a^* > 0$. For $a \in (a_b, a_c]$, we have $a - \phi(a; n) = c'(q_a)/u'(q_a)$ and $a = c'(q_a^*)/u'(q_a^*)$, where $c'(q)/u'(q)$ is increasing in $q$, so $q_a < q_a^*$ (i.e. underconsumption) for $a \in (a_b, a_c]$ if and only if $\phi(a; n) > 0$. Rearranging (41) yields

$$\phi(a; n) = -\left(\frac{(1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}}{\delta \tilde{g}(a;n)}\right), \tag{196}$$

and therefore $\phi(a; n) > 0$ if and only if

$$-\left((1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}\right) > 0. \tag{197}$$

Rearranging, $\phi(a; n) > 0$ if and only if

$$\tilde{G}(a;n) < 1 + \frac{i}{\alpha(n)(1-\delta)}. \tag{198}$$

Since $\tilde{G}'(a; n) = \tilde{g}(a; n) \geq 0$, and $\tilde{G}(a_0; n) = 0$ and $\tilde{G}(\bar{a}; n) = 1$, while $1 + \frac{i}{\alpha(n)(1-\delta)} \in [0, 1]$, there exists a unique cut-off $a_p \in (a_b, \bar{a}]$ such that $\phi(a; n) > 0$ and there is underconsumption for all $a \in (a_0, a_p)$ where $a_p$ satisfies

$$\delta = 1 + \frac{i}{\alpha(n)[1-\tilde{G}(a_p;n)]} \tag{199}$$

provided that $a_p \leq a_c$. If $a \in (a_p, a_c)$ then $\phi(a; n) < 0$ and there is overconsumption. *(ii)* If $a_p > a_c$, the range of overconsumption $(a_p, a_c)$ is empty and we have underconsumption for all $a \in (a_0, a_p)$.

*Part 2. (i)* For $a \in [a_c, \bar{a}]$, $q_a = q_{a_c}$ where $a_c - \phi(a_c) = c'(q_{a_c})/u'(q_{a_c})$ and $a = c'(q_a^*)/u'(q_a^*)$. Since $c'(q)/u'(q)$ is increasing in $q$, we have $q_a > q_a^*$ (i.e. overconsumption) if and only if $a < a_c - \phi(a_c)$. Defining $a_d \equiv a_c - \phi(a_c)$, we have overconsumption for $a \in [a_c, a_d)$ and underconsumption for $a \in (a_d, \bar{a}]$. *(ii)* If $a_d < a_c$, the interval $[a_c, a_d)$ is empty and we have underconsumption for all $a \in [a_c, \bar{a}]$. ■

## Proof of Proposition 7

*Part 1.* Suppose that $a_d = \max\{a_c, a_d\}$. Follows from combining Parts 1 and 2 of Lemma 21 if $a_p \leq a_c \leq a_d$. Suppose that $a_c = \max\{a_c, a_d\}$. Follows from combining

Parts 1 and 2 of Lemma 21 if $a_p \leq a_c$ and $a_d < a_c$.

*Part 2.* If $a_p > a_c$, then $\phi(a;n) > 0$ for all $a \in (a_0, a_p)$ from Part 1 *(ii)* in Lemma 21. In particular, $\phi(a_c) > 0$, so we get $a_c > a_d$. The rest follows from combining Parts 1 and 2 in Lemma 21. If $a_p = a_c$, the result follows from Part 1.

*Part 3.* If $a_b = a_0$ then $\delta = 1 + \frac{i}{\alpha(n)}$ and (199) implies $\tilde{G}(a_p; n) = 0$ and thus $a_p = a_b = a_0$. Since $a_p \leq a_c$, Part 1 implies there is overconsumption on $(a_0, a_u)$ and underconsumption on $(a_u, \bar{a}]$ where $a_u = \max\{a_c, a_d\}$. Since $\phi(a_c) < 0$ by (**??**), we have overconsumption at $a_c$. Therefore, $a_c < a_u$ and $a_u = a_d$. ■

### Proof of Proposition 8

We know from Proposition 6 that the equilibrium $n$ satisfies

$$\alpha'(n)\tilde{s}(n; \{q_a\}_{a \in A}) + \alpha(n)\tilde{s}'(n; \{q_a\}_{a \in A}) = K \tag{200}$$

and the efficient $n^*$ satisfies

$$\alpha'(n^*)\tilde{s}(n^*; \{q_a^*\}_{a \in A}) + \alpha(n^*)\tilde{s}'(n^*; \{q_a^*\}_{a \in A}) = K. \tag{201}$$

We know from Proposition 7 that there are various possibilities for ranges of over-consumption and underconsumption, but we cannot infer anything about whether there is under-entry ($n < n^*$), over-entry ($n > n^*$), or efficient entry ($n = n^*$) overall. We can find examples of equilibria for each of these three possibilities. ■

## Proofs for Section 10

### Proof of Corollary 3

*Part 1.* Follows from Part 1 of Proposition 6.

*Part 2.* Setting $i = 0$ in expression (42) for $\delta$ in Proposition 6, we obtain

$$\delta = \frac{1}{1 - \varepsilon_\rho(a_b; n)}. \tag{202}$$

Setting $i = 0$ in expression (41) for $\phi(a; n)$ in Proposition 6, and substituting (202) into (41), we obtain (53).

*Part 3.* Starting with equation (43) in Proposition 6, setting $i = 0$ implies $a_c = \bar{a}$.

*Part 4.* Parts 5-8 from Proposition 6 also hold. ■

**Proof of Proposition 9**

First, given that $a_b > a_0$, we know that the Friedman rule does not give the efficient allocation. This follows from the lemma below. For any $a \in (a_0, a_b]$, the efficient quantity is not traded even when $i \to 0$ since $q_a^* > 0$ but $q_a = 0$. The efficient quantity is traded at $a_0 = 0$ since $q_0 = q_0^* = 0$. To get the efficient quantity as $i \to 0$ for any $a \in (a_b, \bar{a}]$, Corollary 3 implies that we require either $\varepsilon_\rho(a_b; n) = 0$ or $a = \bar{a}$, which is true only if $a_b = a_0 = 0$ or $a = \bar{a}$. In general, for any $a \in (a_0, \bar{a})$ we have $q_a \neq q_a^*$. Since $\varepsilon_\rho(a_b; n) > 0$, there is underconsumption for all $a \in (a_0, \bar{a})$. At the Friedman rule, the equilibrium $n$ satisfies

$$\alpha'(n)\tilde{s}(n; \{q_a\}_{a\in A}) + \alpha(n)\tilde{s}'(n; \{q_a\}_{a\in A}) = K \tag{203}$$

and the efficient $n^*$ satisfies

$$\alpha'(n^*)\tilde{s}(n^*; \{q_a^*\}_{a\in A}) + \alpha(n^*)\tilde{s}'(n^*; \{q_a^*\}_{a\in A}) = K. \tag{204}$$

We know from above that $q_a^* > q_a$ for any $a \in (a_0, \bar{a})$, but we cannot infer anything about whether there is under-entry ($n < n^*$), over-entry ($n > n^*$), or efficient entry ($n = n^*$). We can find examples of equilibria for each of these three possibilities. ■

**Lemma 22.** *At the Friedman rule, we have efficiency, i.e. $n = n^*$ and $q_a = q_a^*$ for all $a$, if and only if $a_b = a_0$.*

**Proof.** First, it is clear from (53) that the efficient quantities are traded at the Friedman rule if and only if $\varepsilon_\rho(a_b; n) = 0$, which is true if and only if $a_b = a_0 = 0$. Second, the equilibrium condition (44) is equivalent to the planner's condition given the same function $q_a$, i.e. *given the quantities traded are efficient.*

If $a_b = a_0$, letting $i \to 0$ gives same allocation as planner. If $a_b = a_0$, then $\varepsilon_\rho(a_b; n) = \varepsilon_\rho(a_0; n) = 0$ and Corollary 3 implies that $q_a$ satisfies $au'(q_a) = c'(q_a)$ for all $a \in A$, which is equivalent to the planner's first-order condition (13). Also, we know that $\alpha'(n)\tilde{s}(n) + \alpha(n)\tilde{s}'(n) = K$, which is equivalent to the planner's first-order condition (14). Finally, buyers always choose the highest utility seller in any meeting and therefore the distribution of chosen goods is equal to the distribution of the maximum, given by (6), which is the same as the distribution of goods chosen by the planner. Therefore, $i \to 0$ gives the same allocation as the planner. Conversely, $q_a = q_a^*$ for all $a$ requires that $a_b = a_0 = 0$. ■

**Proof of Corollary 4**

*Part 1.* Follows from Part 1 of Corollary 2.

*Part 2.* Setting $i = 0$ in expression (48) for $\delta$ in Corollary 2, we obtain

$$\delta = \frac{1}{1 - \varepsilon_\rho(a_b)}. \tag{205}$$

Setting $i = 0$ in expression (47) for $\phi(a;n)$ in Corollary 2, and then substituting into $(a - \phi(a;n))u'(q_a) = c'(q_a)$, we obtain (54).

*Part 3.* Setting $i = 0$ in (49) implies $a_c = \bar{a}$.

*Part 4.* Parts 5-8 from Corollary 2 also hold. ■

**Proof of Proposition 10**

Without consumer choice, the equilibrium $n$ satisfies

$$\frac{\alpha'(n)\tilde{s}(\{q_a\}_{a\in A})}{\delta + (1-\delta)\eta_\alpha(n)} = K \tag{206}$$

and the efficient $n^*$ satisfies

$$\alpha'(n^*)\tilde{s}(\{q_a^*\}_{a\in A}) = K. \tag{207}$$

At the Friedman rule, $q_a^* > q_a$ for any $a \in (a_0, \bar{a})$. Thus $\tilde{s}(\{q_a^*\}_{a\in A}) > \tilde{s}(\{q_a\}_{a\in A})$ and therefore, since $K$ is constant, we have

$$\alpha'(n^*) < \frac{\alpha'(n)}{\delta + (1-\delta)\eta_\alpha(n)}. \tag{208}$$

Now, $\delta + (1-\delta)\eta_\alpha(n) = \delta(1 - \eta_\alpha(n)) + \eta_\alpha(n) \geq 1$ since $\delta \geq 1$, so $\alpha'(n^*) < \alpha'(n)$. Since $\alpha''(n) < 0$, we have $n^* > n$ and there is under-entry of sellers. ■

## Proofs for Section 11

### Proof of Proposition 11

Let $a : \mathbb{R}_+ \to A$. Given that $I_{\tilde{G}}(a;n) = 1/h_{\tilde{G}}(a;n)$, to derive a necessary and sufficient condition for $\frac{d}{dn} I_{\tilde{G}}(a(n);n) < 0$, we derive a condition for $\frac{d}{dn} h_{\tilde{G}}(a(n);n) > 0$. From Lemma 5, we know that

$$h_{\tilde{G}}(a(n);n) = \eta_\alpha(x(n)) h_G(a(n)) \tag{209}$$

where $x(n) = n(1 - G(a(n)))$. Differentiating with respect to $n$, we have

$$\frac{d}{dn} h_{\tilde{G}}(a(n);n) = \eta'_\alpha(x(n)) x'_a(n) h_G(a(n)) + \eta_\alpha(x(n)) h'_G(a(n)) a'(n). \tag{210}$$

Now, we have

$$x'(n) = 1 - G(a(n)) - n g(a(n)) a'(n). \tag{211}$$

Therefore,

$$\begin{aligned} \frac{d}{dn} h_{\tilde{G}}(a(n);n) &= \eta'_\alpha(x) h_G(a)[1 - G(a) - n g(a) a'(n)] + \eta_\alpha(x) h'_G(a) a'(n) \\ &= \eta'_\alpha(x) h_G(a)(1 - G(a)) - \eta'_\alpha(x) h_G(a) n g(a) a'(n) + \eta_\alpha(x) h'_G(a) a'(n) \\ &= \eta'_\alpha(x) g(a) + a'(n)[\eta_\alpha(x) h'_G(a) - \eta'_\alpha(x) h_G(a) n g(a)]. \end{aligned}$$

So, we have $\frac{d}{dn} h_{\tilde{G}}(a(n);n) > 0$ if and only if

$$a'(n) > \frac{-\eta'_\alpha(x) g(a)}{\eta_\alpha(x) h'_G(a) - \eta'_\alpha(x) h_G(a) n g(a)}, \tag{212}$$

which is equivalent to

$$\hat{a}'(n) > \frac{1}{\frac{\eta_\alpha(x) h'_G(\hat{a})}{-\eta'_\alpha(x) g(\hat{a})} + n h_G(\hat{a})}, \tag{213}$$

which, using $\hat{x} = n(1 - G(\hat{a}))$ and $h_G(a) = g(a)/(1 - G(a))$, is equivalent to

$$a'(n) > \frac{1}{\frac{1}{\eta_{\eta_\alpha}(\hat{x})}\frac{nh'_G(a)}{h_G(a)} + nh_G(a)}, \tag{214}$$

where $\eta_{\eta_\alpha}(x) \equiv \frac{-\eta'_\alpha(x)x}{\eta_\alpha(x)}$. In terms of elasticities, this is equivalent to

$$\frac{a'(n)n}{a(n)} > \frac{1}{\frac{1}{\eta_{\eta_\alpha}(\hat{x})}\frac{h'_G(a)a}{h_G(a)} + h_G(a)a}, \tag{215}$$

which is equivalent to (57). ■

**Proof of Lemma 9**

First, by differentiating $\eta_\alpha(x)$, it is straightforward to show that

$$\eta_{\eta_\alpha}(x) \equiv \frac{-\eta'_\alpha(x)x}{\eta_\alpha(x)} = \frac{-\alpha''(x)x}{\alpha'(x)} - (1 - \eta_\alpha(x)). \tag{216}$$

By Assumption 7, we have $\frac{\mathbb{P}''_0(x)\mathbb{P}_0(x)}{\mathbb{P}'_0(x)^2} \leq 2$. Rearranging (216) and using this fact, it can be shown that $\eta_{\eta_\alpha}(x) < 1$. ■

**Proof of Lemma 10**

Suppose that $\mathbb{P}_k$ satisfies Assumption 4 and $n > 0$. It is shown in the proof of Lemma 4 that there exists a unique $\hat{a}(n)$ such that $\frac{\partial}{\partial n}\tilde{g}(a;n) = 0$ and we have $\frac{\partial}{\partial n}\tilde{g}(a;n) > 0$ if and only if $a > \hat{a}(n)$. Moreover, it can be shown that $\hat{a}'(n) > 0$ and $\lim_{n\to\infty} \hat{a}(n) = \bar{a}$. Also, $\hat{a}(n) > a_f(n)$, i.e. it is an over-represented good. Note that at $a_f$ we have $\frac{\partial}{\partial n}\tilde{g}(a;n) < 0$, so $a_f < \hat{a}$. ■

**Proof of Lemma 11**

Suppose that $\mathbb{P}_k$ satisfies Assumption 4 and $n > 0$. We can calculate $\hat{a}'(n)$ as follows. We know from the Proof of Lemma 4 that, for any given $n > 0$, the value $\hat{a}$

satisfies

$$\frac{-\mathbb{P}_0''(\hat{x})\hat{x}}{\mathbb{P}_0'(\hat{x})} = 1 - \eta_\alpha(n) \tag{217}$$

where $\hat{x} = n(1 - G(\hat{a}))$. Now define $A(x) \equiv \frac{-\mathbb{P}_0''(x)x}{\mathbb{P}_0'(x)}$ so $A(\hat{x}) = 1 - \eta_\alpha(n)$. Letting $F(\hat{a}, n) = A(n(1 - G(\hat{a}))) - (1 - \eta_\alpha(n))$ and differentiating, we obtain

$$\hat{a}'(n) = \frac{-\partial F/\partial n}{\partial F/\partial a} = \frac{-\left(A'(\hat{x})(1 - G(\hat{a})) + \eta_\alpha'(n)\right)}{-A'(\hat{x})ng(\hat{a})} \tag{218}$$

and therefore

$$\hat{a}'(n) = \frac{1 - G(\hat{a})}{ng(\hat{a})}\left(1 + \frac{\eta_\alpha'(n)}{A'(\hat{x})(1 - G(\hat{a}))}\right), \tag{219}$$

which is equivalent to

$$\hat{a}'(n) = \frac{1 - G(\hat{a})}{ng(\hat{a})}\left(1 + \frac{\eta_\alpha'(n)\hat{x}}{\frac{A'(\hat{x})\hat{x}}{A(\hat{x})}(1 - G(\hat{a}))A(\hat{x})}\right) \tag{220}$$

and therefore, using the fact that $\hat{x} = n(1 - G(\hat{a}))$ and $A(\hat{x}) = 1 - \eta_\alpha(n)$,

$$\hat{a}'(n) = \frac{1 - G(\hat{a})}{ng(\hat{a})}\left(1 + \frac{\eta_\alpha'(n)n}{\frac{A'(\hat{x})\hat{x}}{A(\hat{x})}(1 - \eta_\alpha(n))}\right), \tag{221}$$

which is equivalent to

$$\frac{\hat{a}'(n)n}{\hat{a}(n)} = \frac{1 - G(\hat{a})}{g(\hat{a})\hat{a}}\left(1 + \frac{\eta_\alpha'(n)n}{\eta_A(\hat{x})(1 - \eta_\alpha(n))}\right)$$

where $\hat{a} = \hat{a}(n)$, $\eta_A(x) \equiv \frac{A'(x)x}{A(x)}$and $A(x) \equiv \frac{-\mathbb{P}_0''(x)x}{\mathbb{P}_0'(x)}$ and $\hat{x} = n(1 - G(\hat{a}))$. ■

**Proof of Proposition 12**

Suppose that $\mathbb{P}_k$ satisfies Assumption 4 and $n > 0$. Then (62) follows immediately from Proposition 11 and Lemma 11. Moreover, in the limit as $n \to \infty$, we have $\lim_{n\to\infty} \hat{a}(n) = \bar{a}$ from Lemma 10, and thus $I_{\tilde{G}}(\hat{a}(n); n) \to 0$.

## Proofs for Section 12

### Proof of Proposition 13

*Part 1.* The first-order condition for $n$ is given by $\alpha'(n)\tilde{s}(n) + \alpha(n)\tilde{s}'(n) = K$. In the limit as $K \to 0$, we must have $\alpha'(n)\tilde{s}(n) \to 0$ because $\tilde{s}'(n) \geq 0$. Therefore, since $\tilde{s}(n) \geq 0$, we must have $\alpha'(n) \to 0$, which implies that $n \to \infty$ by Lemma 12.

*Part 2.* Lemma 3 says that, in the limit as $n \to \infty$, the distribution of chosen goods converges to a degenerate distribution with support $A = \{\bar{a}\}$.

*Part 3.* First, by Lemma 25 below, we have $a_c \to \bar{a}/(1+i)$ in the limit as $n \to \infty$. Next, by Lemma 24 below, we have $q_{a_c} = q^*_{a_c}$ so $a_c u'(q_{a_c}) = c'(q_{a_c})$. Thus we obtain

$$\bar{a}u'(q_{a_c}) = (1+i)c'(q_{a_c}). \tag{222}$$

Given that $\bar{a} \geq a_c$, we have $q_{\bar{a}} = q_{a_c}$, so we get

$$\bar{a}u'(q_{\bar{a}}) = (1+i)c'(q_{\bar{a}}). \tag{223}$$

*Part 4.* By Lemma 25 below, we have $a_c \to \bar{a}/(1+i)$ in the limit as $n \to \infty$. Also, from Proposition 6 we know that $d_a/\gamma = au(q_a) - \int_{a_0}^{a} u(q_x)dx$ for all $a \leq a_c$ and thus

$$d_{a_c}/\gamma = a_c u(q_{a_c}) - \int_{a_0}^{a_c} u(q_x)dx. \tag{224}$$

We also know that, given $\bar{a} \geq a_c$, we have $q_{\bar{a}} = q_{a_c}$ and $d_{\bar{a}} = d_{a_c} = z$. So, we obtain

$$d_{\bar{a}}/\gamma = a_c u(q^*_{a_c}) - \int_{a_0}^{a_c} u(q^*_x)dx \tag{225}$$

because Lemma 24 implies that $q_a = q^*_a$ for all $a \leq a_c$. ■

**Lemma 23.** *In the limit as $n \to \infty$, we have $\delta \to 1+i$.*

**Proof.** Writing $\delta(n)$ to emphasize the dependence on $n$, and using $[1-\frac{\phi(a_b;n)}{a_b}]a_b = 0$ from Lemma 16 plus expression (141) for $\phi(a;n)$, we have either $a_b = 0$ or

$$\delta(n)a_b\tilde{g}(a_b;n) + (1-\delta(n))\left(1-\tilde{G}(a_b;n)\right) + \frac{i}{\alpha(n)} = 0. \tag{226}$$

Taking the limit as $n \to \infty$, we have $\tilde{g}(a_b; n) \to 0$ and $\tilde{G}(a_b; n) \to 0$ for any $a_b < \bar{a}$. So, using the fact that $\lim_{n\to\infty} \alpha(n) = 1$, we have $\lim_{n\to\infty} \delta(n) = 1 + i$. ■

**Lemma 24.** *In the limit as $n \to \infty$, we have $q_a = q_a^*$ for all $a \in A$ such that $a \leq a_c$.*

**Proof.** Consider any $a$ such that $a \leq a_c$. Rewrite (41) from Proposition 6:

$$\phi(a;n)\tilde{g}(a;n) = \left(1 - \frac{1}{\delta(n)}\right)\left(1 - \tilde{G}(a;n)\right) - \left(1 - \frac{1}{\delta(n)} - (\delta(n) - \delta^{FT}(n))\right), \tag{227}$$

where $\delta^{FT}(n) \equiv 1 + \frac{i}{\alpha(n)}$, the value for $\delta$ when there is full-trade (i.e. all meetings result in trade). Taking the limit as $n \to \infty$, we have $\delta(n) - \delta^{FT}(n) \to 0$ by Lemma 23, and also $1 - \frac{1}{\delta(n)} \to \lim_{n\to\infty} \phi^{RW}$. So, we have

$$\lim_{n\to\infty} \phi(a;n)\tilde{g}(a;n) = \lim_{n\to\infty} \phi^{RW}\left(1 - \tilde{G}(a;n)\right) - \lim_{n\to\infty} \phi^{RW} \tag{228}$$

Now, $\tilde{G}(a;n) \to 0$ for any $a < \bar{a}$, so we have

$$\lim_{n\to\infty} \phi(a;n)\tilde{g}(a;n) = \lim_{n\to\infty} \phi^{RW} - \lim_{n\to\infty} \phi^{RW} = 0. \tag{229}$$

So, in the limit as $n \to \infty$, we have $au'(q_a) = c'(q_a)$ for all $a \in A$ such that $a \leq a_c$. ■

**Lemma 25.** *In the limit as $n \to \infty$, we have $a_c \to \bar{a}/(1 + i)$.*

**Proof.** First, we know from Proposition 6 that $a_c$ solves

$$\int_{a_c}^{\bar{a}} (x - a_c)\tilde{g}(x;n)dx = (1 - \delta)(\bar{a} - a_c)(1 - \tilde{G}(a_c;n)) + \frac{i\bar{a}}{\alpha(n)}. \tag{230}$$

Writing $\delta(n)$, and taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \int_{a_c}^{\bar{a}} (x - a_c)\tilde{g}(x;n)dx = \lim_{n\to\infty} (1 - \delta(n))(\bar{a} - a_c)(1 - \tilde{G}(a_c;n)) + \lim_{n\to\infty} \frac{i\bar{a}}{\alpha(n)}, \tag{231}$$

if all limits exist and are finite. For any $a_c < \bar{a}$, we have $\lim_{n\to\infty} \tilde{G}(a;n) \to 0$ and thus $\lim_{n\to\infty} \tilde{G}(a_c;n) \to 0$. Also, we have $\lim_{n\to\infty} \alpha(n) = 1$. Therefore,

$$\lim_{n\to\infty} \int_{a_c}^{\bar{a}} (x - a_c)\tilde{g}(x;n)dx = (\bar{a} - a_c)(1 - \lim_{n\to\infty} \delta(n)) + i\bar{a}. \tag{232}$$

Next, by Lemma 23 we have $\lim_{n\to\infty}\delta(n) = 1+i$. So, we obtain

(233)
$$\lim_{n\to\infty}\int_{a_c}^{\bar{a}}(x-a_c)\tilde{g}(x;n)dx = i\bar{a}.$$

Finally, $\lim_{n\to\infty}\int_{a_c}^{\bar{a}}(x-a_c)\tilde{g}(x;n)dx = \bar{a}-a_c$ because $\tilde{G}$ converges to a degenerate distribution with support $A = \{\bar{a}\}$. So, $a_c \to \bar{a}/(1+i)$ in the limit as $n \to \infty$. ■