Consumer Choice and Private Information in Monetary Exchange*

Sephorah Mangin†

March 15, 2024

Abstract

We introduce consumer choice into a competitive search model of monetary exchange. In contrast to standard search models featuring bilateral meetings, there is a general meeting technology which allows consumers to meet multiple sellers and *choose* a seller with whom to trade. Consumer choice is influenced by random utility shocks which are private information. Competitive search equilibrium delivers the efficient allocation at the Friedman rule, but it cannot decentralize the first-best allocation due the presence of private information, which distorts the quantities traded in equilibrium. Our main insight is that greater seller entry and more competition between sellers to attract consumers can alleviate the quantity distortion due to private information, bringing the economy closer to the first-best. In fact, in the competitive limit, we find that consumer choice eliminates the effects of private information and competitive search can deliver the first-best allocation at the Friedman rule.

JEL codes: D82, E31, E40, E50, E52

Keywords: Money; Inflation; Choice; Directed search; Private information

*I would like to thank Ayushi Bajaj, co-author of the companion paper "Consumer Choice and the Cost of Inflation," for her useful insights in our numerous discussions regarding both papers. I would also like to thank Guillaume Rocheteau and Randy Wright for particularly useful feedback, as well as Jonathan Chiu, Michael Choi, Mohammad Davoodalhosseini, Lucas Herrenbrueck, Janet Jiang, Mario Silva, and Yu Zhu for very useful comments. I also thank seminar participants at the Bank of Canada, U.C. Irvine, Simon Fraser University, the National University of Singapore, and the Bank of Canada Summer Workshop on Money, Banking, and Payments.

†Research School of Economics, The Australian National University, HW Arndt Building 25A, Kingsley Place, Acton ACT 2601 Australia. Email: sephorah.mangin@anu.edu.au.

1 Introduction

Consumer choice is an important feature of monetary exchange. When consumers purchase goods, they typically choose from a number of goods that are available simultaneously from a range of competing sellers. Choice is often idiosyncratic: different consumers might make different choices when faced with the same range of goods, and the same consumer might make different choices at different points in time. Discrete choice models with random utility shocks have been used extensively to study these kinds of choices in the large literature following Anderson, De Palma, and Thisse (1992), but these models do not feature monetary exchange.

Search-theoretic models have become the standard way of modelling the microfoundations of monetary exchange.[1] However, meetings are typically one-to-one or bilateral: each buyer meets at most one seller during a single period of time and chooses to either trade or wait. While many papers have incorporated random utility shocks into the process of monetary exchange, such as Lagos and Rocheteau (2005), these shocks typically influence only the quantities traded and the payments – not the choice of seller – because meetings are bilateral. In this way, these models do not feature what we call *consumer choice*, i.e. buyers' choice of seller.

This paper develops a new model that features both consumer choice and monetary exchange. There are two main contributions.

The first main contribution is to provide a way of modelling consumer purchases in retail trade that is more realistic than standard monetary search models with bilateral meetings. To do this, we introduce consumer choice into the search-theoretic monetary framework of Rocheteau and Wright (2005). Buyers can meet more than one seller at a time and buyers' choice of seller is driven by seller-specific utility shocks. We assume these utility shocks are buyers' private information because it is realistic to assume sellers cannot directly observe buyers' preferences. We also assume sellers post contracts that are realistic for an environment of retail trade.

The second main contribution is to show that consumer choice delivers a novel mechanism for alleviating the distortions in quantities traded due to buyers' private information. These informational distortions prevent the decentralized market from achieving the first-best allocation at the Friedman rule. With consumer choice, we find that greater competition through higher seller entry leads sellers to reduce the quantity distortion from private information, bringing us closer to the first-best.

[1]For a survey of the literature, see Lagos, Rocheteau, and Wright (2017).

We focus on *competitive search equilibrium.* Directed or competitive search is a natural alternative to bargaining in our environment because buyers can meet many sellers in a single meeting. We assume that sellers post terms of trade or contracts that specify quantities and payments. Buyers choose which submarket to enter, where submarkets correspond to potentially different contracts offered by sellers. After entering a submarket, agents commit to trading at the terms posted in that submarket. Within each submarket, there are search frictions that govern how agents meet.

Our model has two main features that are necessary for consumer choice.

First, search frictions within submarkets are modelled using a meeting technology that features multilateral meetings.[2] Following Shi and Delacroix (2018), we call these meetings *one-to-many meetings* because sellers meet only one buyer at a time while buyers may meet many sellers. In particular, a buyer can meet either no sellers, one seller, or more than one seller. A *meeting* is an opportunity for buyer to purchase from a seller, although buyers can only trade with one seller in each period.

Second, after meetings takes place, the buyer draws an i.i.d. preference or utility shock specific to each seller they meet. We interpret this as buyers learning their preferences *after* observing the goods offered by the sellers they meet, but before purchasing. Sellers cannot observe the buyer's utility shocks, which are private information. Buyers choose to purchase from the seller that maximizes their net utility. The pair consisting of a buyer and their chosen seller is called a *match*.

After choosing a seller, buyers choose the quantity of the good to purchase and make their payment. We focus on incentive-compatible direct revelation mechanisms that induce buyers to reveal their private information. We derive a sufficient condition on the meeting technology under which existence and uniqueness of competitive search equilibrium is guaranteed. In equilibrium, there is only one active submarket and all sellers offer the same non-linear price schedule that specifies both the quantity traded and the payment in real dollars for any given realization of the buyer's utility shock.

If the planner is subject to the same information frictions as sellers, competitive search equilibrium decentralizes the efficient allocation at the Friedman rule. This is not surprising since competitive search typically delivers efficiency.[3]

Neither the planner nor the decentralized equilibrium can deliver the first-best allocation due to the presence of information frictions. Outside the Friedman rule,

[2]We focus on *invariant* meeting technologies, see Lester, Visschers, and Wolthoff (2015).

[3]Directed or competitive search is often used to decentralize the constrained efficient allocation in search-theoretic environments, as discussed in Wright, Kircher, Julien, and Guerrieri (2021).

there are various possibilities for ranges of underconsumption and overconsumption relative to the first-best quantity. There may also be either under-entry or over-entry of sellers relative to the first-best. At the Friedman rule, there is underconsumption of all goods and there may be either under-entry or over-entry relative to the first-best.

While neither of these welfare results is particularly surprising, the key novelty in our paper is the interaction between private information and consumer choice in competitive search equilibrium. With consumer choice, sellers know that any buyer with whom they trade must have chosen their good, so the distribution of "types" is the endogenous distribution of buyers' *choices*, which depends on the equilibrium seller-buyer ratio. In particular, the distribution of choices for a higher seller-buyer ratio first-order stochastically dominates the distribution for a lower seller-buyer ratio. More sellers per buyer leads to greater choice for buyers, which leads to a higher average utility for chosen goods and a higher average surplus for trades.

Importantly, any shift in the distribution of choices affects the extent of the quantity distortion due to buyers' private information, which is also endogenous. In particular, we find that an increase in the seller-buyer ratio leads to a *decrease* in the overall quantity distortion due to private information. This is because, under competitive search, more competition to attract buyers leads sellers to reduce the quantity distortions in the contracts they offer. Greater seller entry can thus alleviate the private information distortion, bringing the economy closer to the first best.

In the competitive limit where the seller-buyer ratio becomes large, the informational distortion is eliminated altogether and the Friedman rule delivers the first best. Consumer choice is necessary for this result. In the absence of consumer choice (e.g. in standard models with bilateral meetings), competitive search equilibrium cannot achieve the first-best allocation at the Friedman rule when there is private information – even in the competitive limit. This is because the distribution of buyers' private information or "types" is not endogenous when there is no consumer choice.

2 Related literature

As discussed, our model builds on the environment in Rocheteau and Wright (2005) (hereafter denoted RW), which shares the alternating centralized and decentralized markets of Lagos and Wright (2005) but features endogenous seller entry, which is important for our results. In RW, the focus is on comparing different market structures (e.g. bargaining and competitive search) that feature bilateral meetings, while our paper examines the effect of consumer choice on monetary exchange in an environment featuring private information and competitive search.

We contribute to the literature on directed or competitive search and private information, including Faig and Jerez (2005), Menzio (2007), Guerrieri (2008), Guerrieri, Shimer, and Wright (2010), Moen and Rosen (2011), and Davoodalhosseini (2019). In our environment, both buyers and sellers are ex ante identical and buyers' private utility shocks are realized *after* buyers enter submarkets and meetings take place. In that sense, our environment is closest to the model of retail trade in Faig and Jerez (2005), which builds on the theory of non-linear pricing for continuous types developed in Mussa and Rosen (1978) and Maskin and Riley (1984).

A number of related papers consider monetary environments featuring buyer preference shocks that are private information. Ennis (2008) incorporates two types of private utility shocks into the monetary framework of Lagos and Wright (2005). Faig and Jerez (2006) and Dong and Jiang (2014) examine the effect of inflation on the extent of quantity discounts when buyers' valuations are continuous and private information. Faig and Jerez (2006), which builds on the model of retail trade in Faig and Jerez (2005), is effectively a special case of our model in which there is no consumer choice, no individual rationality constraint, and the distribution of utility shocks is uniform. Dong and Jiang (2014) considers a similar environment to Faig and Jerez (2006) but it features price posting with undirected search and an IR constraint. More recently, Choi and Rocheteau (2023) develops a search model of retail banking in which consumers' liquidity needs are private information.

All of these papers feature bilateral meetings without consumer choice. The key novelty of our paper is that meetings are one-to-many, which allows buyers to *choose* sellers within meetings. The sequential nature of search in our model, in which buyers first choose a submarket using directed or competitive search and then face a "noisy" process of choosing or matching among the random subset of sellers they meet, shares some similarities with the model of sequentially mixed search developed in Shi (2023). However, in our model, buyers' choice of seller within meetings is driven by buyers' private information about their preferences rather than prices.

Related papers that feature multilateral meetings in monetary environments include Julien, Kennes, and King (2008) and Galenianos and Kircher (2008). Julien et al. (2008) introduces multilateral meetings and directed search into the framework of Shi (1995) and Trejos and Wright (1995) with divisible goods and indivisible money. Galenianos and Kircher (2008) features ex ante heterogeneity, private information, and multilateral meetings in which indivisible goods are allocated by auctions. In both Julien et al. (2008) and Galenianos and Kircher (2008), sellers can meet multiple buyers and either money or goods are indivisible. In our paper, by contrast, buyers can meet multiple sellers and both money and goods are divisible.[4]

Bajaj and Mangin (2023) generalizes the current framework to allow for the possibility of meetings in which consumers are uninformed, in addition to meetings in which consumers make an informed choice. In that paper, we calibrate the model and examine how the degree of informed choice affects the welfare cost of inflation.

3 Environment

Time is discrete and continues forever. Each period $t \in \{0, 1, 2, ...\}$ is divided into two subperiods as in Lagos and Wright (2005). During the day, there is a frictionless, centralized market and at night there is a frictional, decentralized market. As in RW, there is a continuum of agents divided into two types: *buyers* and *sellers*. Buyers are ex ante identical and sellers are ex ante identical. The sets of buyers and sellers are denoted B and S respectively. While all agents both produce and consume during the day, buyers and sellers differ at night: buyers wish to consume (but cannot produce) and sellers do not wish to consume (but can produce).

There is a fixed measure of buyers and we normalize $|B| = 1$. All buyers participate in the night market at zero cost, but there is an entry decision by sellers. Only a subset $\bar{S}_t \subseteq S$ of sellers of measure n_t enter the night market. Sellers may or may not choose to enter the night market at cost $\kappa > 0$ and thus $n_t \in \mathbb{R}_+$ is endogenous.[5] Since $|B| = 1$, the measure of sellers who enter, n_t, is also the seller-buyer ratio.

Money is perfectly divisible. The aggregate money supply at date t is $M_t \in \mathbb{R}_+$, which grows at a constant rate $\gamma \in \mathbb{R}_+$, i.e. $M_{t+1} = \gamma M_t$. Money is either injected into the economy ($\gamma > 1$) or withdrawn ($\gamma < 1$) by lump sum transfers during the day. We assume these transfers are to buyers only, and we restrict attention to policies where $\gamma \geq \beta$, where β is the discount factor. When $\gamma = \beta$ (the Friedman rule), we only consider equilibria obtained by taking the limit as $\gamma \to \beta$ from above.

In the day market, the price of goods is normalized to one and the relative price of

[4] An alternative approach is Head and Kumar (2005), which combines the monetary search framework of Shi (1997, 1999) with the price-posting mechanism of Burdett and Judd (1983), which allows buyers to observe a random sample of prices posted by sellers and choose the lowest price. See also Herrenbrueck (2017), which extends the framework of Head and Kumar (2005).

[5] We assume the set S is sufficiently large that $n_t \leq |S|$ always.

money is denoted by ϕ_t. The real value of a quantity of money m_t held by an agent at date t is defined as $z_t \equiv \phi_t m_t$ and the aggregate real money supply is $Z_t \equiv \phi_t M_t$. We focus on steady-state equilibria where all of the aggregate real variables are constant. Since $M_{t+1}/M_t = \gamma$, this implies that in steady state $\phi_{t+1}/\phi_t = 1/\gamma$.

Pricing in the night market will be described in Section 6. For now, we discuss the features of the environment that enable the possibility of consumer choice.

One-to-many meetings. Meetings are one-to-many. All sellers meet exactly one buyer, but buyers can meet zero, one, or many sellers. Specifically, the number of sellers a buyer i meets is a random variable N_i. For any given seller-buyer ratio n, the probability that a buyer meets $j \in \{0, 1, 2, ...\}$ sellers is given by a *meeting technology* $\mathbb{P}_j : \mathbb{N} \to [0, 1]$ where $\mathbb{P}_j(n) = \Pr(N_i = j)$ and $\sum_{j=0}^{\infty} j\mathbb{P}_j(n) = n$.

Let $\alpha : \mathbb{R}^+ \to [0, 1]$ be a function that represents the endogenous probability $\alpha(n)$ that a buyer meets at least one seller, i.e. $\alpha(n) \equiv 1 - \mathbb{P}_0(n)$. This is the probability that a buyer receives a meeting. We refer to the function α as the *meeting function*. The probability that a seller receives a meeting is one, but the probability a seller receives a match equals $\alpha(n)/n$, the probability that the seller is *chosen* by a buyer.

Buyer's choice of seller. After a meeting takes place, the buyer draws a seller-specific random utility shock a for each seller they meet. We assume that shocks are drawn *after* meetings take place because we interpret this as buyers learning their preferences after observing the goods offered by each seller they meet, but before purchasing. The buyer then chooses a single seller with whom to trade in that subperiod.[6] The pair consisting of a buyer and their chosen seller is called a *match*.

Distribution of utility shocks. The random utility shocks a are drawn from a bounded, continuous distribution with cdf G and pdf $g = G'$. This distribution is known to all agents. Importantly, the realizations of the utility shocks are not observed by sellers; they are private information for the buyer.

We assume the distribution G is not degenerate, and Assumption 1 is maintained throughout the paper.[7]

Assumption 1. *The distribution of utility shocks has twice-differentiable cdf G, where $G' > 0$ and $G'' \leq 0$, and bounded support $A = [a_0, \bar{a}] \subseteq \mathbb{R}_+$ where $0 < a_0 g(a_0) \leq 1$.*

[6] Similarly to discrete choice models, we assume that consumers choose to purchase from a single firm in each meeting. Alternatively, this could be endogenized through specification of preferences.

[7] The assumption that $a_0 g(a_0) \leq 1$ will be used in the proof of Lemma 5 in Section 8.1.

Buyer and seller utility. Sellers can produce on demand any quantity $q \in \mathbb{R}_+$ of a divisible good at cost $c(q)$, where $c : \mathbb{R}_+ \to \mathbb{R}_+$, and we assume that $c(0) = 0$, $c'(q) > 0$, and $c''(q) \geq 0$ for all $q > 0$. A buyer who consumes quantity q of a good with utility shock a receives utility given by a function $\tilde{u} : \mathbb{R}_+^2 \to \mathbb{R}_+$ defined by

$$\tilde{u}(q, a) \equiv au(q), \tag{1}$$

where $u : \mathbb{R}_+ \to \mathbb{R}_+$ and we assume that $u(0) = 0$, $u'(0) = \infty$, $u'(q) > 0$, and $u''(q) < 0$ for all $q > 0$. As we discuss later in Section 8.1, our assumption on the form of the utility function $\tilde{u}$ will ensure that the single crossing condition holds.

The instantaneous utility of a buyer who meets a seller at night is[8]

$$U^b = \nu(x) - y + \beta E_{\tilde{G}}(au(q_a)), \tag{2}$$

and the instantaneous utility of a seller who is chosen by a buyer at night is

$$U^s = \nu(x) - y - \beta E_{\tilde{G}}(c(q_a)), \tag{3}$$

where x is the quantity consumed and y is the quantity produced during the day, q_a is the quantity consumed at night, a is the utility shock of the good consumed, and $\tilde{G}$ is the *distribution of buyers' choices*, which we introduce in the next section.

4 Distribution of buyer choices

The distribution of buyers' choices $\tilde{G}$ is the distribution of the utility shocks of the goods actually *chosen* by buyers. With consumer choice, we can think of this distribution as the endogenous distribution of buyer "types" because it is the outcome of a buyer's choice that represents the private information that is unavailable to their chosen seller. For brevity, we refer to $\tilde{G}$ simply as the *distribution of choices.*[9]

The distribution of choices $\tilde{G}$ depends on the equilibrium seller-buyer ratio n and the equilibrium choices made by buyers regarding which seller to purchase from.

[8] We assume $\nu'(x) > 0$ and $\nu''(x) < 0$ for all x, and that there exists x^* such that $\nu'(x^*) = 1$. We normalize $\nu(x^*) - x^* = 0$.

[9] Note that the distribution of buyer choices $\tilde{G}$ is different than the distribution of purchases. In equilibrium with private information, some meetings may not result in trade because of the IR constraint, so the distribution of purchases will be a truncation of the distribution of choices at an endogenous trading cut-off that will be determined in equilibrium.

We will later prove that, in any equilibrium we consider, buyers always choose the highest utility seller they meet. Therefore, the distribution of choices equals the distribution across buyers of the highest utility shock a among the sellers a buyer meets, conditional on meeting a seller. It is this distribution that we discuss here.

Throughout the paper, we assume that the meeting technology $\mathbb{P}_j$ is *invariant*, as defined in Lester et al. (2015). The assumption of invariance is useful because the function $\mathbb{P}_0$ captures everything we need to know about the meeting technology. Examples of invariant meetings technologies include the family of negative binomial distributions, which includes the Geometric meeting technology as a special case and the widely-used Poisson meeting technology as a limiting case. For a discussion of the intuition behind Assumption 2, see Lester et al. (2015) and Mangin (2023).[10]

Assumption 2. *The meeting technology $\mathbb{P}_j$ is* invariant, *i.e. for all $y \in [0,1]$,*

$$\sum_{j=0}^{\infty} \mathbb{P}_j(n) y^j = \mathbb{P}_0(n(1-y)) \tag{4}$$

where $\mathbb{E}_{\mathbb{P}}(N_i) = n$ and $\mathbb{P}_0 : \mathbb{R}^+ \to [0,1]$ is continuous and infinitely differentiable.

Lemma 1 shows that invariant search technologies deliver standard properties for the meeting function α, defined by $\alpha(n) \equiv 1 - \mathbb{P}_0(n)$. We refer to $\eta_\alpha(n) \equiv \alpha'(n)n/\alpha(n)$, the elasticity of the meeting function $\alpha(n)$, as the *meeting elasticity.*

Lemma 1. *If $\mathbb{P}_j$ is invariant and $n > 0$, then $\alpha'(n) > 0$, $\alpha''(n) < 0$, $\lim_{n\to 0} \alpha(n) = 0$, $\lim_{n\to 0} \alpha'(n) = 1$, $\lim_{n\to\infty} \alpha(n) = 1$, $\lim_{n\to\infty} \alpha'(n) = 0$, and $\lim_{n\to\infty} \alpha''(n) = 0$. We have $\eta'_\alpha(n) < 0$, $\lim_{n\to 0} \eta_\alpha(n) = 1$, $\lim_{n\to\infty} \eta_\alpha(n) = 0$, $\eta_\alpha(n) \in (0,1)$, and*

$$\frac{d}{dx}\left(\frac{-\alpha''(x)x}{\alpha'(x)}\right) > 0. \tag{5}$$

Lemma 2 presents an expression for the distribution of choices. This expression depends only on the function $\mathbb{P}_0$, which gives the probability a buyer is unmatched. For example, if the search technology is Poisson then $\mathbb{P}_0(x) = e^{-x}$.

Parts 1 and 2 of Lemma 2 say that, in the limit as $n \to 0$, the distribution $\tilde{G}$ converges to the distribution of utility shocks G, and in the limit as $n \to \infty$, the distribution $\tilde{G}$ becomes degenerate at the upper bound.

[10] Any distribution which can be represented as a mixed Poisson distribution satisfies Assumption 2. This is a very wide class of discrete probability distributions. See Cai, Gautier, and Wolthoff (2023) and Becker and Mangin (2023) for further details.

Part 3 of Lemma 2 states that the distribution of choices $\tilde{G}$ first-order stochastically dominates the distribution G, and the average utility of a *chosen* good $\tilde{a}(n) \equiv E_{\tilde{G}}(a)$ is greater than the average utility shock, $E_G(a)$.

Lemma 2. *If $\mathbb{P}_j$ is invariant and $n > 0$, the distribution of choices has cdf*

$$\tilde{G}(a;n) = \frac{\mathbb{P}_0(n(1-G(a))) - \mathbb{P}_0(n)}{\alpha(n)}. \tag{6}$$

1. *In the limit as $n \to 0$, we have $\tilde{G} \to G$ and $\tilde{a}(n) \to E_G(a)$.*

2. *In the limit as $n \to \infty$, we have $\tilde{G}(a;n) \to 0$ for all $a \in [a_0, \bar{a})$, and $\tilde{a}(n) \to \bar{a}$.*

3. *The distribution of choices $\tilde{G}$ first-order stochastically dominates the distribution of utility shocks G and $\tilde{a}(n) > E_G(a)$.*

4. *If $n > n'$, the distribution $\tilde{G}(a;n)$ first-order stochastically dominates the distribution $\tilde{G}(a;n')$.*

5. *For any $f : A \to \mathbb{R}_+$ such that $f' > 0$, $\tilde{f}'(n) > 0$ where $\tilde{f}(n) \equiv \int_{a_0}^{\bar{a}} f(a) d\tilde{G}(a;n)$.*

Part 4 of Lemma 2 says that an increase in the seller-buyer ratio n leads to a first-order stochastic dominance shift in the distribution of choices $\tilde{G}(a;n)$. Roughly speaking, more seller entry leads to a "better" distribution of buyers' chosen goods. Part 5 of Lemma 2 implies the average utility of a chosen good $\tilde{a}(n)$ is strictly increasing in n, i.e. $\tilde{a}'(n) > 0$. Intuitively, average utility is increasing in the seller-buyer ratio because more sellers per buyer means greater choice of seller.

Figure 1 compares the density $\tilde{g}(a;n)$ of the distribution of choices (for $n = 3$ and $n = 10$) and the density of the distribution of utility shocks $g(a)$. In this example, G is uniform on $[0,1]$ and $\mathbb{P}_j$ is Poisson. For both distributions of choices, there exists a unique point where the density $\tilde{g}(a;n)$ crosses the density g.[11]

5 First-best allocation

Before we consider competitive search equilibrium, we determine the first-best allocation. To do so, we solve a planner's problem where the planner has complete

[11] We prove this result in Lemma 22 found in Online Appendix A.

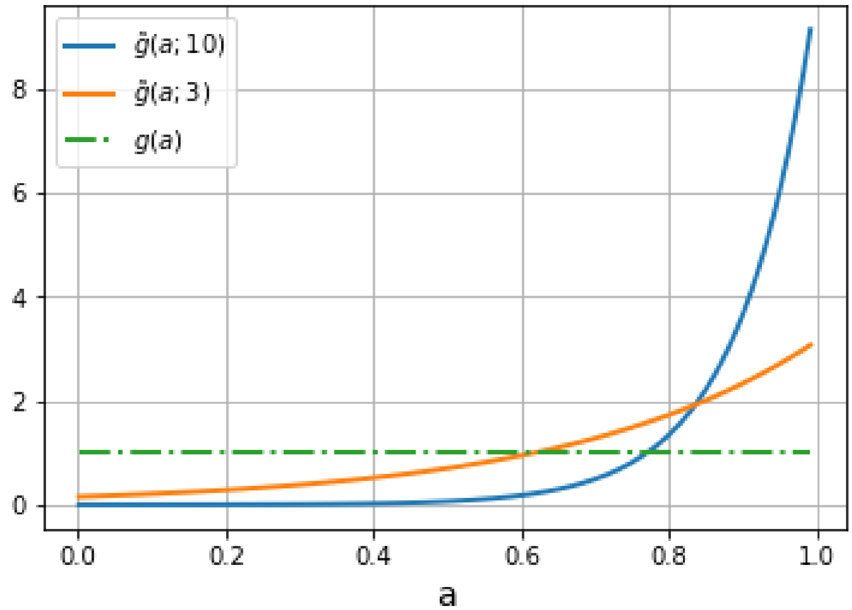

Figure 1: Example of densities of distributions of buyer choices for $n = 3$ and $n = 10$

information about buyers' utility shocks and is therefore not constrained by informational frictions but only by search frictions.[12]

The planner knows the meeting technology $\mathbb{P}_j$, the distribution of utility shocks G, and the cost of entry κ. The planner chooses a seller-buyer ratio n^*, a function $q^* : A \to \mathbb{R}_+$, and a distribution of choices with cdf $\tilde{G} : A \to [0, 1]$, to maximize the total surplus created minus the total cost of seller entry, subject to the search frictions. That is, the planner solves the following problem:

$$\max_{n \in \mathbb{R}_+, \{q_a\}_{a \in A}} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} [au(q_a) - c(q_a)]\, d\tilde{G}(a; n) - n\kappa \right\} \tag{7}$$

where $\tilde{G}$ represents the planner's optimal choice of seller for each buyer.[13]

Define $s_a \equiv au(q_a) - c(q_a)$, the trade surplus (or match surplus) for a good of utility a. Let q_a^* denote the first-best quantity of good a and define $s_a^* \equiv au(q_a^*) - c(q_a^*)$. Assume that $s_0^* \geq 0$ where $s_0^* \equiv a_0 u(q_0) - c(q_0)$ and $q_0 = q(a_0)$, so there is a (weakly)

[12] It is standard in the search literature to refer to this as the *first-best allocation* even though it is constrained by search frictions. For example, see Davoodalhosseini (2019).

[13] The planner's distribution of choices will turn out to be equal to the buyers' distribution of choices in equilibrium. Anticipating this, we use the same notation, $\tilde{G}$.

positive trade surplus for all goods. Define the *expected trade surplus* by

$$\tilde{s}(n; \{q_a\}_{a\in A}) \equiv \int_{a_0}^{\bar{a}} [au(q_a) - c(q_a)] d\tilde{G}(a; n). \tag{8}$$

For simplicity of notation, throughout the paper we sometimes suppress the dependence of the expected trade surplus $\tilde{s}(n; \{q_a\}_{a\in A})$ on the function $q : A \to \mathbb{R}_+$ and let $\tilde{s}(n)$ denote $\tilde{s}(n; \{q_a\}_{a\in A})$ and $\tilde{s}'(n)$ denote $\partial\tilde{s}(n)/\partial n$.

The following assumption ensures the existence of a social optimum where $n^* > 0$. We maintain Assumption 3 throughout the remainder of the paper. Intuitively, this condition says that the expected trade surplus in the limit as $n \to 0$, i.e. $\lim_{n\to 0} \tilde{s}(n)$, must be greater than κ.[14] It follows from our earlier assumptions that, for all $a \in A$, there exists a unique $q_a^* \in \mathbb{R}_+$ such that $au'(q_a^*) = c'(q_a^*)$.

Assumption 3. *The cost of entry is not too high:* $E_G[au(q_a^*) - c(q_a^*)] > \kappa$.

Proposition 1 states that there exists a unique first-best allocation $(n^*, \{q_a^*\}_{a\in A})$ with $n^* > 0$ and provides a characterization.

Proposition 1. *There exists a unique first-best allocation and it satisfies:*

1. *For any $a \in A$, the quantity $q_a^* > 0$ solves*

$$au'(q_a^*) = c'(q_a^*). \tag{9}$$

2. *The seller-buyer ratio $n^* > 0$ satisfies*

$$\alpha'(n^*)\tilde{s}(n^*; \{q_a^*\}_{a\in A}) + \alpha(n^*)\tilde{s}'(n^*; \{q_a^*\}_{a\in A}) = \kappa. \tag{10}$$

3. *The distribution of choices $\tilde{G}$ is given by (6).*

Equation (10) is a version of the *generalized Hosios condition* discussed in Mangin and Julien (2021). Defining the meeting elasticity by $\eta_\alpha(n) \equiv \alpha'(n)n/\alpha(n)$ and the surplus elasticity by $\eta_s(n) \equiv \tilde{s}'(n)n/\tilde{s}(n)$, condition (10) says

$$\underbrace{\eta_\alpha(n)}_{\text{meeting elasticity}} + \underbrace{\eta_s(n; \{q_a^*\}_{a\in A})}_{\text{surplus elasticity}} = \underbrace{\frac{n\kappa}{\alpha(n)\tilde{s}(n; \{q_a^*\}_{a\in A})}}_{\text{sellers' surplus share}}. \tag{11}$$

[14]Since $\tilde{G} \to G$ as $n \to 0$, as verified in Lemma 2, we have $\lim_{n\to 0} \tilde{s}(n) = E_G[au(q_a^*) - c(q_a^*)]$

We have not yet discussed equilibrium, but it is useful to refer to the term on the right of (11) as the sellers' surplus share. Given that our equilibrium features free entry of sellers at cost κ, sellers' total expected payoff will be equal to the total cost of seller entry $n\kappa$, and the total surplus created is $\alpha(n)\tilde{s}(n)$. Therefore, the term on the right will be sellers' surplus share in equilibrium. The generalized Hosios condition (11) says that constrained efficiency requires sellers' surplus share to be equal to the meeting elasticity plus the surplus elasticity.

Since s_a^* is increasing in a, Part 5 of Lemma 2 implies that the expected trade surplus $\tilde{s}(n)$ is increasing in the seller-buyer ratio, i.e. $\tilde{s}'(n) > 0$.[15] Therefore, the surplus elasticity $\eta_s(n)$ is positive. Intuitively, more sellers per buyer means greater choice for buyers, thus increasing the average trade surplus. Equivalently, there is a positive externality arising from the effect of seller entry on the average surplus when there is consumer choice. When the generalized Hosios condition (11) holds, both the search externalities and the "choice externality" are internalized.

6 Competitive search equilibrium

Competitive search is an equilibrium concept developed in Moen (1997) and Shimer (1996). The basic idea is that either buyers or sellers, or sometimes "market makers", post contracts that specify the terms of trade offered. Search is directed in the sense that buyers and sellers choose which *submarket* to enter, where each submarket corresponds to a particular specification of the terms of trade. Commitment is key: buyers and sellers who enter a submarket *commit* to trade at the terms specified within that submarket. Within each submarket, there are search frictions.

We assume that sellers post a menu of contracts.[16] Sellers take into account the expected relationship between the posted contracts and the seller-buyer ratio n. We restrict sellers to post contracts of the form (q_a, d_a) which specify the quantity of the good q_a and the payment in real dollars d_a contingent on the buyer's utility shock a for their chosen seller. These contracts (q_a, d_a) may depend on n, the expected number of sellers a buyer meets, but not the actual number of sellers a specific buyer meets (which is not observed by sellers).

Within each submarket, meetings take place, buyers choose sellers, and trade

[15] It is established in the proof of Proposition 1 that both q_a^* and s_a^* are increasing in a.

[16] Note that the equilibrium is equivalent if buyers post contracts, but it is more realistic to assume that sellers post contracts.

occurs as described in Section 3. In any meeting, the buyer chooses the seller that maximizes $v_a \equiv au(q_a) - d_a/\gamma$, the buyer's ex post trade surplus. The equilibrium distribution of choices $\tilde{G}$ is given by buyers' optimal choices of sellers.

Within meetings, buyers' utility shocks are private information and they cannot be observed directly by any seller (including their chosen seller). However, buyers may choose to reveal their private information within matches through their choice of contract (q_a, d_a) offered by the chosen seller. By the revelation principle, it is without loss of generality to focus on individually rational and incentive-compatible direct mechanisms that induce buyers to truthfully reveal their private information.

Sellers post contracts subject to the following constraints: an individual rationality (IR) constraint and an incentive compatibility (IC) constraint. The IR constraint is

$$au(q_a) - \frac{d_a}{\gamma} \geq 0 \tag{12}$$

for all $a \in A$. This condition states that buyers must receive a (weakly) positive ex post trade surplus, otherwise they will not trade. The IC constraint is given by

$$au(q_a) - \frac{d_a}{\gamma} \geq au(q_{a'}) - \frac{d_{a'}}{\gamma} \tag{13}$$

for all $a, a' \in A$. Intuitively, this condition states that a buyer with utility shock a cannot do better by choosing a contract $(q_{a'}, d_{a'})$ instead of (q_a, d_a).

Restriction of contract space. Our restriction on the contract space warrants a brief discussion. We restrict sellers to post contracts of the form (q_a, d_a) because it is more realistic in retail trade for sellers to post this form of contracts. Given that our model aims to achieve greater realism by introducing consumer choice, it is important to also maintain realism with respect to the form of contracts sellers post. Contracts of the form (q_a, d_a) are realistic for retail trade because they are equivalent to (q_a, p_a) where $p_a \equiv d_a/q_a$, the unit price. This is equivalent to sellers simply choosing a function $p(q)$ for unit prices based on the quantity. In retail trade, it is common for sellers to set different unit prices depending on the quantity purchased (e.g. through different package sizes and quantity discounts). By focusing on contracts of the form (q_a, p_a), or equivalently (q_a, d_a), we can apply the standard theory of non-linear pricing developed in Mussa and Rosen (1978) and Maskin and Riley (1984).[17]

[17] An alternative form of contract would be to allow payments and quantities to be contingent on

Within each period, the timing is as follows. At the start of each day, sellers enter and announce the submarkets $\{(q_a, d_a)\}_{a \in A}$ that will be open that night, implying an expected n for each submarket. During the day, agents trade in the centralized market and readjust their real balances, and then choose a submarket in which to trade at night, in a manner consistent with expectations. During the night, agents trade goods and money in the decentralized market in their chosen submarket.

Let Ω denote the set of open submarkets ω. A submarket ω is characterized by the menu of contracts and implied seller-buyer ratio, $(\{(q_a, d_a)\}_{a \in A}, n)_\omega$. Let W^b and W^s denote the value functions for buyers and sellers respectively in the day market, and let V^b and V^s denote the value functions for buyers and sellers in the night market.

Centralized market. In the day market, a buyer with real balance z solves:

$$W^b(z) = \max_{\hat{z}, x, y \in \mathbb{R}_+} \{\nu(x) - y + \beta V^b(\hat{z})\}, \tag{14}$$

subject to $\hat{z} + x = z + T + y$, where T is her real transfer and $\hat{z}$ is the real balances carried forward into that period's decentralized market. Substituting into (14) yields

$$W^b(z) = z + T + \max_{\hat{z}, x \in \mathbb{R}_+} \{\nu(x) - x - \hat{z} + \beta V^b(\hat{z})\}. \tag{15}$$

Thus, the buyer's $\hat{z}$ is independent of z, and $W^b(z) = z + W^b(0)$, which is linear.

Similarly, a seller with real balance z_s in the centralized market solves:

$$W^s(z_s) = \max_{\hat{z}, x, y \in \mathbb{R}_+} \left\{ \nu(x) - y + \beta \max \left[V^s(\hat{z}), W^s\left(\frac{\hat{z}}{\gamma}\right) \right] \right\}, \tag{16}$$

subject to $\hat{z} + x = z_s + y$. Substituting into (16), we obtain

$$W^s(z_s) = z_s + \max_{\hat{z}, x \in \mathbb{R}_+} \left\{ \nu(x) - x - \hat{z} + \beta \max \left[V^s(\hat{z}), W^s\left(\frac{\hat{z}}{\gamma}\right) \right] \right\}. \tag{17}$$

Thus, the seller's $\hat{z}$ is independent of z_s, and $W^s(z_s) = z_s + W^s(0)$.

both the utility shock a and the realization of the number of sellers j a buyer meets. While it may be realistic in some environments to assume that sellers observe the number of sellers j competing for the same buyer and set prices that are contingent on j, this is not typical in retail trade.

Decentralized market. For a seller in the decentralized or night market,

$$V^s(z_s) = \max_{\omega\in\Omega} \left\{ \begin{array}{c} \frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[-c(q_a) + W^s\left(\frac{z_s+d_a}{\gamma}\right)\right] d\tilde{G}(a;n) \\ + \left[1 - \frac{\alpha(n)}{n}\right] W^s\left(\frac{z_s}{\gamma}\right) \end{array} \right\} - \kappa \tag{18}$$

where each submarket $\omega \in \Omega$ is characterized by $(\{(q_a, d_a)\}_{a\in A}, n)$. A seller chooses ω and is matched with probability $\alpha(n)/n$. It is straightforward to verify that the seller's choice of real balances is $\hat{z} = 0$.[18] For a buyer in the decentralized market,

$$V^b(z) = \max_{\omega\in\Omega} \left\{ \begin{array}{c} \alpha(n) \int_{a_0}^{\bar{a}} \mathbf{1}_a \left[au(q_a) + W^b\left(\frac{z-d_a}{\gamma}\right)\right] d\tilde{G}(a;n) \\ + \left[1 - \alpha(n) \int_{a_0}^{\bar{a}} \mathbf{1}_a \, d\tilde{G}(a;n)\right] W^b\left(\frac{z}{\gamma}\right) \end{array} \right\} \tag{19}$$

where $\mathbf{1}_a$ is an indicator function that is equal to one if $z \geq d_a$ and zero otherwise. A buyer chooses ω among the set of open submarkets and gets the opportunity to trade if she meets at least one seller and has sufficient money z to pay the posted d_a for her chosen good. If she either fails to meet a seller or does not have sufficient money, she does not trade. Using $W^b(z) = z + W^b(0)$ we obtain

$$V^b(z) = \max_{\omega\in\Omega} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \mathbf{1}_a \left[au(q_a) - \frac{d_a}{\gamma}\right] d\tilde{G}(a;n) + \frac{z}{\gamma} + W^b(0) \right\}. \tag{20}$$

Thus, the buyer's choice of z from (15) is given by

$$\max_{z\in\mathbb{R}_+} \left\{ -z + \beta \max_{\omega\in\Omega} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[au(q_a) - \frac{d_a}{\gamma}\right] d\tilde{G}(a;n) + \frac{z}{\gamma} \right\} \right\} \tag{21}$$

subject to the liquidity constraint, $d_a \leq z$ for all $a \in A$.

Sellers' problem. Defining $i \equiv \frac{\gamma-\beta}{\beta}$, the nominal interest rate, sellers' problem is

$$\max_{z\in\mathbb{R}_+,\, \omega\in\Omega} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[au(q_a) - \frac{d_a}{\gamma}\right] d\tilde{G}(a;n) - i\frac{z}{\gamma} \right\}, \tag{22}$$

subject to $d_a \leq z$ for all $a \in A$, plus an additional constraint that a submarket with posted contracts $\{(q_a, d_a)\}_{a\in A}$ will attract measure n of sellers per buyer where n

[18] Using $W^s(z_s) = z_s + W^s(0)$, (18) simplifies to $V^s(z_s) = z_s/\gamma + V^s(0)$. Substituting into (17), the choice of $\hat{z}$ is given by the first order condition $-1 + \beta/\gamma \leq 0$, where $-1 + \beta/\gamma = 0$ if $\hat{z} > 0$. Since we only consider the case $\gamma = \beta$ by taking the limit as $\gamma \to \beta$ from above, $\hat{z} = 0$.

satisfies the following condition in all submarkets:

$$\frac{\alpha(n)}{n}\int_{a_0}^{\bar{a}}\left[-c(q_a)+\frac{d_a}{\gamma}\right]d\tilde{G}(a;n)\leq\kappa \tag{23}$$

and $n \geq 0$ with complementary slackness.[19] With private information, sellers also need to impose the IC and IR constraints (13) and (12) on this problem.

This complementary slackness condition deserves further discussion. It says that $n > 0$ and a submarket is active if and only if (23) holds with equality, i.e. sellers receive zero expected profit after paying the entry cost κ. On the other hand, it says that $n = 0$ and a submarket is inactive if and only if (23) is an inequality, i.e. sellers would receive negative expected profits from entering this submarket even if the probability of trade is one. We restrict attention to equilibria in which this complementary slackness condition holds for *all* submarkets, both active and inactive. This provides an important restriction on beliefs outside of the equilibrium path.[20]

7 Full information benchmark

We briefly present the full information equilibrium as a benchmark. By "full information," we refer to a version of the model in which, prior to trade occurring, the *chosen seller* can directly observe the buyer's utility shock for that seller. This means that there is no private information within the buyer-seller match.

With full information, the planner's solution is identical to the first-best allocation given in Section 5, so we do not need to solve the planner's problem separately. Proposition 1 gives both the first-best allocation and the efficient allocation.

7.1 Equilibrium

Our definition of equilibrium for full information is the same as Definition 1 for private information except there are no IR or IC constraints. Proposition 2 establishes the existence and uniqueness of equilibrium and provides a characterization.[21]

[19] This complementary slackness condition is really a restriction on n as a function of the menu of posted contracts in a submarket, i.e. $n(\{(q_a, d_a)\}_{a\in A})$. However, we omit the dependence on the submarket $\{(q_a, d_a)\}_{a\in A}$ in our notation both for notational simplicity and because we will later prove there is only one active submarket in equilibrium with one implied seller-buyer ratio, n.

[20] See, for example, Shi (2009) and Menzio and Shi (2010) for similar restrictions on beliefs.

[21] We omit the proof of Proposition 2 because it is straightforward but tedious to derive it from the proof of Proposition 3 by eliminating the IR and IC constraints.

Proposition 2. *For any $i > 0$, there exists a unique full-information competitive search equilibrium and it satisfies:*

1. *For any $a \in A$, the quantity $q_a > 0$ solves*

$$au'(q_a) = \left(1 + \frac{i}{\alpha(n)}\right) c'(q_a). \tag{24}$$

2. *The seller-buyer ratio $n > 0$ is strictly decreasing in κ and satisfies*

$$\alpha'(n)\tilde{s}(n; \{q_a\}_{a\in A}) + \alpha(n)\tilde{s}'(n; \{q_a\}_{a\in A}) = \kappa. \tag{25}$$

3. *We have $d_a = z$ for all $a \in A$ where $z > 0$ is given by*

$$\frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[-c(q_a) + \frac{z}{\gamma}\right] d\tilde{G}(a; n) = \kappa. \tag{26}$$

4. *The distribution of choices $\tilde{G}$ is given by (6).*

Condition (25) has the same form as the generalized Hosios condition for the first-best allocation. The only difference between the equilibrium condition (25) and condition (10) for the first-best allocation is that the equilibrium quantities q_a are different than the first-best quantities q_a^* for $i > 0$. We can write condition (24) as:

$$(a - \phi^m(a; n))u'(q_a) = c'(q_a), \tag{27}$$

where the distortion in quantities traded due to the monetary friction is

$$\phi^m(a; n) = \frac{ai}{\alpha(n) + i}. \tag{28}$$

When $i > 0$, quantities traded are distorted downwards because buyers hold less money, reducing the quantities they can purchase in equilibrium, as in RW.

Corollary 1 says there is underconsumption relative to the first best whenever $i > 0$ and there may be either under-entry or over-entry of sellers.

Corollary 1. *In any full-information competitive search equilibrium where $i > 0$, there may be either under-entry or over-entry of sellers relative to the first-best. There is underconsumption for all $a \in (a_0, \bar{a}]$.*

With full information, the Friedman rule delivers the first-best allocation, i.e. $n = n^*$ and $q_a = q_a^*$ for all $a \in A$. As $i \to 0$, it is clear that the equilibrium conditions (24) and (25) reduce to the planner's first-order conditions, (9) and (10).

8 Private information equilibrium

We now turn to the full model where buyers have private information about their utility shocks. As discussed in Section 6, we focus on incentive-compatible direct mechanisms that induce buyers to reveal their private information.

The first-best allocation is given by Proposition 1. While this is also the planner's solution with full information, this is not true under private information. For the planner's solution in an environment with private information, it is realistic to assume that the planner is subject to the same information frictions as sellers and therefore must satisfy the IC and IR constraints. Given the similarity to the equilibrium problem, we will describe the planner's solution after we characterize equilibrium.

8.1 Equilibrium

First, we define competitive search equilibrium under private information. We will later prove that there is a unique solution to this problem and thus all sellers post the same contracts and there is only one active submarket in equilibrium. Anticipating this result, we simply denote equilibrium by $(\{(q_a, d_a)\}_{a\in A}, z, n)$. We restrict attention to steady-state monetary equilibria where $z > 0$ and $n > 0$.

Definition 1. *A* competitive search equilibrium *is a list* $(\{(q_a, d_a)\}_{a\in A}, z, n)$ *and a distribution of choices* $\{\tilde{G}(a; n)\}_{a\in A}$ *where* $(q_a, d_a) \in \mathbb{R}_+^2$ *for all* $a \in A$ *and* $z, n \in \mathbb{R}_+ \backslash \{0\}$, *such that* $(\{(q_a, d_a)\}_{a\in A}, z, n)$ *maximizes (22) subject to (23), the liquidity constraint* $d_a \leq z$ *for all* $a \in A$, *plus the IR constraint (12) and the IC constraint (13), and* $\{\tilde{G}(a; n)\}_{a\in A}$ *represents buyers' optimal choices of sellers.*

In the lead up to presenting our main result, we first need to prepare the ground by making some assumptions and providing some definitions. In particular, our proof of the existence and uniqueness of equilibrium under private information requires some assumptions on the distribution of utility shocks G and the cost of entry κ, plus an additional assumption on the meeting technology $\mathbb{P}_j$. We discuss these below.

Single crossing condition. The definition $\tilde{u}(q,a) \equiv au(q)$ and the assumption $u'(q) > 0$ together ensure that single crossing property holds because

$$\frac{\partial^2 \tilde{u}(q,a)}{\partial q \partial a} = u'(q) \geq 0. \tag{29}$$

The fact that the single crossing property is satisfied in our environment is used in Lemma 3, which provides sufficient conditions for the IC constraints to hold. Versions of Lemma 3 are widely used in the literature on principal-agent problems and monopolistic screening, including in related papers featuring continuous types that are private information, such as Faig and Jerez (2005) and Dong and Jiang (2014).

Recalling that $v_a \equiv au(q_a) - d_a/\gamma$, the buyer's ex post trading surplus, Lemma 3 simplifies the IC constraints when there is a continuous distribution of types.

Lemma 3. *If the single crossing property holds, the incentive compatibility (IC) constraint holds for all $a \in A$ if (i) q_a is non-decreasing and (ii) $v'(a) = u(q_a)$.*

We use Lemma 3 to ensure the IC constraints hold when we solve for the equilibrium as an optimal control problem in the proof of Proposition 3 in the Appendix.

Virtual valuation function. The virtual valuation function $\psi_G : A \to \mathbb{R}$ is defined by $\psi_G(a) \equiv a - \frac{1-G(a)}{g(a)}$. It is common to assume that the virtual valuation function is weakly increasing as this is typically required to prove that q_a is non-decreasing. To ensure this, we assume the distribution of utility shocks G has an increasing hazard rate, $h_G : A \to \mathbb{R}$ defined by $h_G(a) \equiv \frac{g(a)}{1-G(a)}$. Assumption 4 is maintained throughout the remainder of the paper.

Assumption 4. *The distribution G has an increasing hazard rate, i.e. $h'_G(a) > 0$.*

In our environment, what is important is that the endogenous distribution of choices $\tilde{G}$ has an increasing hazard rate. This is because sellers know that whenever they are matched with a buyer, this must be because the buyer chose them from among the other sellers. Therefore, from the seller's perspective, the buyers' utility shock (which is private information) can be treated as a random draw from the distribution of choices, not the exogenous distribution G.

The distribution of choices has a hazard rate defined by $h_{\tilde{G}}(a;n) \equiv \frac{\tilde{g}(a;n)}{1-\tilde{G}(a;n)}$. Lemma 4 provides an expression for the hazard rate of $\tilde{G}$ in terms of the exogenous distribution G and establishes that $\tilde{G}$ has an increasing hazard rate for any invariant meeting technology $\mathbb{P}_j$ and any seller-buyer ratio n. This fact is used to prove that q_a is non-decreasing, which Lemma 3 uses to ensure the IC conditions hold.

Lemma 4. *The distribution of choices $\tilde{G}$ has an increasing hazard rate, i.e.*

$$h'_{\tilde{G}}(a;n) \equiv \frac{\partial h_{\tilde{G}}(a;n)}{\partial a} > 0. \tag{30}$$

Condition on meeting technology. Assumption 5 is a condition on the meeting technology. This is another assumption that is used to prove that the function q_a is non-decreasing, which Lemma 3 uses to ensure the IC conditions hold. We assume that Assumption 5 holds throughout the remainder of the paper.

Assumption 5. *The meeting technology $\mathbb{P}_j$ is invariant and, for all $x \in \mathbb{R}^+$,*

$$\frac{-\alpha''(x)(1-\alpha(x))}{\alpha'(x)^2} \leq 2. \tag{31}$$

Assumption 5 can be interpreted as saying that the function $\mathbb{P}_0$ is not *too* convex. This assumption will be true, for example, if the meeting technology $\mathbb{P}_j$ is in the negative binomial family, which includes the Geometric meeting technology as a special case and the widely-used Poisson meeting technology as a limiting case.[22]

Cost of entry. We maintain the following assumption throughout the paper.

Assumption 6. *The entry cost κ is not too high: $E_G[au(q_a^0) - c(q_a^0)] > \kappa$.*

Assumption 6 is necessary to ensure the existence of equilibrium with $n > 0$. It says the expected trade surplus $\tilde{s}(n)$ must be greater than κ in the limit as $n \to 0$, otherwise no sellers enter. This is more complex than Assumption 3 because q_a now depends on n. Since $\tilde{G} \to G$ as $n \to 0$, we have $\lim_{n\to 0} \tilde{s}(n) = E_G[au(q_a^0) - c(q_a^0)]$ where $q_a^0 \equiv \lim_{n\to 0} q_a(n)$. The value of q_a^0 can be calculated using Lemma 5.

Lemma 5. *For all $a \in [a_0, a_b]$, $q_a^0 = 0$ and, for all $a \in (a_b, \bar{a}]$, q_a^0 satisfies*

$$\left(a - \frac{1-G(a)}{g(a)}\right) u'(q_a) = c'(q_a) \tag{32}$$

where $a_b^0 \in A$ is the unique solution to $\psi_G(a) = 0$.

[22]For example, if $\mathbb{P}_j$ is Poisson, we have $\alpha(x) = 1 - e^{-x}$ and thus $\frac{-\alpha''(x)(1-\alpha(x))}{\alpha'(x)^2} = 1$.

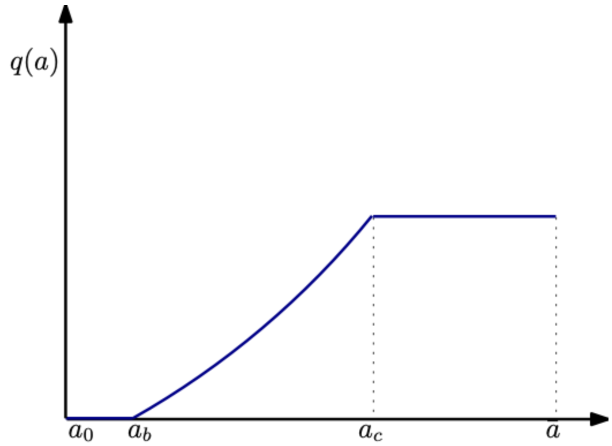

Figure 2: Example of no-trade, unconstrained, and liquidity constrained ranges

Before we present our main result, Lemma 6 defines the possibilities for trading ranges that may occur in equilibrium. There may exist a non-empty range of utility shocks a such that trade does not occur in equilibrium, i.e. $q_a = 0$. We call this type of equilibrium *partial trade*, i.e. not all chosen goods are purchased in equilibrium.[23] On the other hand, if we have $q_a > 0$ for all $a > a_0$, we call this type of equilibrium *full trade*, i.e. all chosen goods are purchased in equilibrium. There may also exist a non-empty range of utility shocks such that buyers are liquidity constrained.

Lemma 6. *In any competitive search equilibrium where $i > 0$,*

1. *There exists a unique* trading cut-off $a_b \geq a_0$ *such that $q_a = 0$ for all $a \in [a_0, a_b]$ and $q_a > 0$ for all $a \in (a_b, \bar{a}]$.*

2. *There exists a unique* liquidity cut-off $a_c \in [a_b, \bar{a})$ *such that $d_a < z$ for all $a \in [a_0, a_c)$ and $d_a = z$ and $q_a = q_{a_c} > 0$ for all $a \in [a_c, \bar{a}]$.*

Before presenting Proposition 3, it will be useful to define $\rho(a; n) \equiv 1 - \tilde{G}(a; n)$, the probability that a chosen good has utility greater than a. We also define $\varepsilon_\rho(a; n) \equiv -a\rho'(a; n)/\rho(a; n)$, the elasticity of $\rho(a; n)$ with respect to a, where $\rho'(a; n) \equiv \frac{\partial \rho(a;n)}{\partial a}$.

We can now present Proposition 3, which establishes the existence and uniqueness of equilibrium and provides a characterization.

Proposition 3. *For any $i > 0$, there exists a unique competitive search equilibrium and it satisfies:*

[23]In this case, the distribution of purchases is a truncation of the distribution of buyer choices at the equilibrium trading cut-off a_b.

1. No-trade range. We have $q_a = 0$ and $d_a = 0$ for all $a \in [a_0, a_b]$.

2. Unconstrained range. For any $a \in [a_b, a_c]$, the quantity $q_a > 0$ solves:

$$(a - \phi(a; n))u'(q_a) = c'(q_a) \tag{33}$$

where

$$\phi(a; n) = \left(1 - \frac{1}{\delta}\right) \frac{1 - \tilde{G}(a; n)}{\tilde{g}(a; n)} - \left(\frac{1}{\delta}\right) \frac{i}{\alpha(n)\tilde{g}(a; n)} \tag{34}$$

and

$$\delta = \frac{1}{1 - \varepsilon_\rho(a_b; n)} \left(1 + \frac{i}{\alpha(n)\rho(a_b; n)}\right) \tag{35}$$

and

$$\frac{d_a}{\gamma} = au(q_a) - \int_{a_0}^{a} u(q_x)dx. \tag{36}$$

3. Liquidity constrained range. For any $a \in [a_c, \bar{a}]$, $q_a = q_{a_c}$ and $d_a = d_{a_c}$.

4. The value of a_c satisfies

$$\frac{i\bar{a}}{\alpha(n)} = \int_{a_c}^{\bar{a}} (a - a_c)\tilde{g}(a; n)da + (\delta - 1)(\bar{a} - a_c)(1 - \tilde{G}(a_c; n)). \tag{37}$$

5. Real money holdings $z > 0$ is given by $z = d_{a_c}$.

6. The seller-buyer ratio $n > 0$ is strictly decreasing in κ and satisfies

$$\alpha'(n)\tilde{s}(n; \{q_a\}_{a \in A}) + \alpha(n)\tilde{s}'(n; \{q_a\}_{a \in A}) = \kappa. \tag{38}$$

7. The zero profit condition is satisfied:

$$\frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[-c(q_a) + \frac{d_a}{\gamma}\right] d\tilde{G}(a; n) = \kappa. \tag{39}$$

8. The distribution of choices $\tilde{G}$ is given by (6).

The equilibrium distribution of choices $\tilde{G}$ is the same as the first-best because buyers always choose the highest utility seller they meet. As before, a version of the generalized Hosios condition holds in our environment featuring competitive search since condition (38) is equivalent in form to the first-best condition (10).

In the next section, we discuss the distortion in quantities given by (34), which now incorporates the effects of private information as well as the monetary friction.

8.2 Consumption and entry

Consider the distortion in equilibrium quantities given by expression (34). Given that the first-best quantity q_a^* satisfies $au'(q_a^*) = c'(q_a^*)$, it is clear that we have *underconsumption* relative to the first-best quantity if $\phi(a;n) > 0$ and *overconsumption* if $\phi(a;n) < 0$. To better understand the term $\phi(a;n)$, we can decompose it as follows:

$$\phi(a;n) = \underbrace{\frac{1-\tilde{G}(a;n)}{\tilde{g}(a;n)}}_{\text{positive, >0}} - \frac{1}{\delta}\underbrace{\left(\frac{1-\tilde{G}(a;n)}{\tilde{g}(a;n)}\right)}_{\text{negative, <0}} - \frac{1}{\delta}\underbrace{\left(\frac{i}{\alpha(n)\tilde{g}(a;n)}\right)}_{\text{negative, <0}} \tag{40}$$

The first term represents the distortion due to private information, which we discuss in detail in Section 9.1. It is standard except the distribution $\tilde{G}$ is endogenous and depends on the seller-buyer ratio n due to the presence of consumer choice.

The second term represents the direct effect of competitive search in reducing the private information distortion, which we discuss in detail in Section 9.2. Competitive search reduces the informational distortion in quantities because sellers compete to offer contracts that are attractive to buyers, which induces them to offer better terms to buyers than they otherwise would, depending on the degree of competition.

The third term represents the direct effect of the monetary friction through the nominal interest rate $i > 0$. With private information, unlike with full information, the payments d_a vary across buyers. The direct effect of a higher nominal interest rate i is that firms offer buyers a menu of contracts with *less variability* in payments d_a in order to reduce the need for precautionary balances, which are costly. As a result, a higher i tends to reduce the quantity distortion due to private information, the opposite of what we see with full information. This effect of private information is not the focus of our paper, but it is discussed in Faig and Jerez (2006).[24]

[24] As we discussed in Section 2, Faig and Jerez (2006), which builds on Faig and Jerez (2005), is essentially a special case of our model in which there is no consumer choice, no IR constraint, and

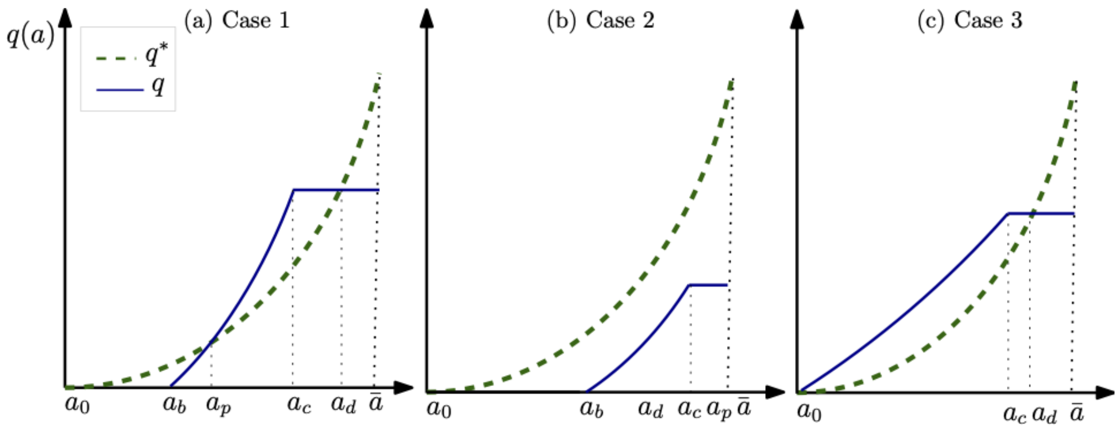

Figure 3: Examples of the three possible cases of under/over consumption

Proposition 4 states that there many be either under-entry or over-entry of sellers for $i > 0$ and describes the three possible equilibrium outcomes in terms of underconsumption or overconsumption ranges. In Proposition 4, we define $a_d \equiv a_c - \phi(a_c; n)$ and $a_u \equiv \max\{a_c, a_d\}$, while a_p solves $\delta = \frac{i}{\alpha(n)(1-\tilde{G}(a_p;n))}$.

Proposition 4. *For any $i > 0$, there may be either under-entry or over-entry of sellers relative to the first-best. There are three possible outcomes for consumption:*

1. *If $a_p \leq a_c$, there is underconsumption on (a_0, a_p), overconsumption on (a_p, a_u), and underconsumption on $(a_u, \bar{a}]$, relative to the first best.*

2. *If $a_p \geq a_c$, there is underconsumption on $(a_0, \bar{a}]$.*

3. *If $a_b = a_0$, there is overconsumption on (a_0, a_d) and underconsumption on $(a_d, \bar{a}]$.*

It is clear that the decentralized market cannot achieve the first-best allocation for any $i > 0$. There may be either underconsumption or overconsumption, and under-entry or over-entry, relative to the first-best.

8.3 Planner's problem

We assume the planner is constrained by the same search frictions and meeting technology as the decentralized market. It is also realistic to assume that the planner is subject to the same information frictions as sellers, i.e. they cannot observe buyers' utility shocks because they are private information. Given this, we focus on direct

the distribution of utility shocks is uniform with $a_0 = 0$.

mechanisms that enable the planner to elicit buyers' private information by posting a menu of contracts $\{(q_a, d_a)\}_{a\in A}$. As discussed in Section 6, we restrict attention to contracts of this form because this is realistic in the environment we consider.

The planner knows the meeting technology $\mathbb{P}_j$, the distribution of utility shocks G, and the cost of entry κ. The planner chooses a seller-buyer ratio n^P, a quantity function $q^P : A \to \mathbb{R}_+$, and a payment function $d^P : A \to \mathbb{R}_+$, plus a distribution of choices $\tilde{G} : A \to [0,1]$, to maximize the total surplus created minus the total cost of seller entry, subject to the constraints.[25] That is, the planner solves:

$$\max_{n\in\mathbb{R}_+,\{q_a,d_a\}_{a\in A}} \left\{\alpha(n)\int_{a_0}^{\bar{a}} [au(q_a) - c(q_a)]\, d\tilde{G}(a;n) - n\kappa\right\} \tag{41}$$

plus the IC and IR constraints given by (13) and (12).

There exists a unique solution to the planner's problem and it can be characterized as follows. We refer to this as the *efficient allocation.*

Proposition 5. *There exists a unique efficient allocation and it satisfies:*

1. *No-trade range. For any* $a \in [a_0, a_b)$, $q_a^P = 0$, *and* $d_a^P = 0$.
2. *Trading range. For all* $a \in [a_b, \bar{a}]$, *the quantity* q_a^P *satisfies*

$$\left(a - \varepsilon_\rho(a_b; n^P)\frac{1-\tilde{G}(a;n^P)}{\tilde{g}(a;n^P)}\right)u'(q_a^P) = c'(q_a^P) \tag{42}$$

   and the payment d_a^P *is given by (36).*

3. *Parts 6-8 from Proposition 3 hold.*

It is not surprising that the planner cannot achieve the first-best allocation due to the information frictions arising from buyers' private information. We see the same informational distortion as in equilibrium except the monetary distortion is absent.

8.4 Friedman rule

At the Friedman rule ($i \to 0$), there is no opportunity cost of holding money and no liquidity constrained meetings. Comparing Proposition 5 and Corollary 2 below,

[25]The payment function represents direct transfers from buyers to sellers that the planner can enforce without the need for money as a medium of exchange.

it is clear that competitive search equilibrium delivers the efficient allocation at the Friedman rule. This is not surprising because competitive search generally delivers efficiency and the planner is subject to the same information frictions as sellers.

Corollary 2. *At the Friedman rule, competitive search equilibrium satisfies:*

1. *No-trade range. For any* $a \in [a_0, a_b)$*,* $q_a = 0$*, and* $d_a = 0$*.*
2. *Unconstrained range. For all* $a \in [a_b, \bar{a}]$*, the quantity* q_a *satisfies*

$$\left(a - \varepsilon_\rho(a_b; n) \frac{1 - \tilde{G}(a; n)}{\tilde{g}(a; n)} \right) u'(q_a) = c'(q_a) \tag{43}$$

   and the payment d_a *is given by (36).*

3. *No meetings are liquidity constrained:* $a_c = \bar{a}$*.*
4. *Parts 6-8 from Proposition 3 hold.*

At the Friedman rule, the decentralized equilibrium cannot deliver the first-best allocation. This is again not surprising due buyers' private information, which distorts the quantities traded in equilibrium. In fact, at the Friedman rule, we recover two standard results in mechanism design: there is no distortion at the top, i.e. $q_a = q_a^*$ at $a = \bar{a}$, but there is downwards distortion below, i.e. $q_a < q_a^*$ for $a < \bar{a}$.

Corollary 3. *At the Friedman rule, there is underconsumption relative to the first-best for all* $a \in [a_0, \bar{a})$ *and there may be either under-entry or over-entry of sellers.*

In terms of entry, there may be either under-entry or over-entry relative to the first-best because the distortion in quantities traded due to buyers' private information induces a corresponding distortion in seller entry, which depends on quantities traded.

9 Informational frictions and choice

At the Friedman rule, the only reason for the deviation from the first-best is buyers' private information. In our model, however, the extent of the quantity distortion due to private information is *endogenous* and depends on the market tightness. In this section, we show that, in competitive search equilibrium, greater competition through higher seller entry can alleviate the quantity distortion due to private information.

9.1 Informational distortion

Consider the competitive search equilibrium at the Friedman rule where the monetary friction is eliminated. We can define the *informational distortion* by

$$I_{\tilde{G}}(a;n) \equiv \frac{1-\tilde{G}(a;n)}{\tilde{g}(a;n)}. \tag{44}$$

Except for the fact that the distribution of buyer types $\tilde{G}$ is endogenous with consumer choice, this is the "standard" informational distortion that would appear in the absence of competitive search if sellers were to simply take n as given. We know that the informational distortion $I_{\tilde{G}}(a;n)$ is decreasing in a because $\tilde{G}$ has an increasing hazard rate by Lemma 4. Thus, the standard distortion due to private information decreases for higher values of a, with no distortion at the top.

While the seller-buyer ratio n is endogenous in our model, we can still consider what happens as n increases because this is equivalent to simply decreasing the entry cost parameter, κ. To isolate the effect of the seller-buyer ratio n on the informational distortion, we can decompose (44) into two components:

$$I_{\tilde{G}}(a;n) = \frac{1}{\eta_\alpha(n(1-G(a)))}\left(\frac{1-G(a)}{g(a)}\right). \tag{45}$$

Given that $\eta'_\alpha < 0$ by Lemma 1, the informational distortion $I_{\tilde{G}}(a;n)$ is increasing in the seller-buyer ratio n (i.e. decreasing in the entry cost κ) for any given a.[26]

Let $I_{\tilde{G}}(n)$ denote the *average informational distortion*, defined by

$$I_{\tilde{G}}(n) \equiv \int_{a_0}^{\bar{a}} I_{\tilde{G}}(a;n)\tilde{g}(a;n)da. \tag{46}$$

The average informational distortion $I_{\tilde{G}}(n)$ is increasing in the seller-buyer ratio n (i.e. decreasing in the entry cost κ).[27] The basic idea is that, as n increases, there are fewer "low types" but these low types are subject to a greater informational distortion.[28] This means that, if we do not consider the effect of competitive search, higher

[26]Expression (45) follows directly from (77) in the proof of Lemma 4 in the Appendix.

[27]Note that substituting $I_{\tilde{G}}(a;n)$ from (44) into (46) yields $I_{\tilde{G}}(n) \equiv \int_{a_0}^{\bar{a}}(1-\tilde{G}(a;n))da$, which is equal to $\int_{a_0}^{\bar{a}} a\tilde{g}(a;n)da - a_0$ or $\tilde{a}(n) - a_0$, which is clearly increasing in n because $\tilde{a}'(n) > 0$.

[28]In Online Appendix A, we consider the "typical" informational distortion, i.e. the informational distortion at a "typical" realization of the utility shock, e.g. the average utility of a chosen good $\tilde{a}(n)$. We derive conditions under which the typical informational distortion is *decreasing* in the

seller entry increases the average informational distortion. As we will see, however, competitive search has a crucial effect on the extent of the quantity distortion.

9.2 Effect of competitive search

To determine the overall effect of greater competition on the quantity distortion, we need to also consider the impact of competitive search, which ensures that sellers compete to attract buyers through their choices of which menus of contracts to post. We can define the *quantity distortion* $D_{\tilde{G}}(a;n)$ by

$$1 - D_{\tilde{G}}(a;n) \equiv \frac{f(q_a)}{f(q_a^*)}, \tag{47}$$

where $f(q) \equiv c'(q)/u'(q)$ and $f' > 0$ follows from our assumptions in Section 3. To justify our measure of the quantity distortion, suppose that $c(q) = q$ and $u(q) = q^{1-b}/(1-b)$ for $b \in (0,1)$, which are standard assumptions. Then

$$1 - D_{\tilde{G}}(a;n) = f\left(\frac{q_a}{q_a^*}\right). \tag{48}$$

Given that f is an increasing function, $1 - D_{\tilde{G}}(a;n)$ moves in the same direction as q_a/q_a^* with respect to n. Thus we can interpret $D_{\tilde{G}}(a;n)$ as a measure of the quantity distortion. As $q_a \to q_a^*$, we have $D_{\tilde{G}}(a;n) \to 0$, and as $q_a \to 0$ we have $D_{\tilde{G}}(a;n) \to 1$.

At the Friedman rule, using expressions (43) for q_a and (9) for q_a^*, plus definition (44) for the informational distortion, we obtain

$$D_{\tilde{G}}(a;n) = \underbrace{\varepsilon_\rho(a_b(n);n)}_{\text{effect of competitive search}} \quad \underbrace{\frac{I_{\tilde{G}}(a;n)}{a}}_{\text{relative informational distortion}} \tag{49}$$

for all $a > a_b(n)$, and $D_{\tilde{G}}(a;n) = 1$ for $a \leq a_b(n)$.

Writing $a_b(n)$ to emphasize that the trading cut-off a_b itself depends on n, the term $\varepsilon_\rho(a_b(n);n)$ reflects the *effect of competitive search*, while the term $I_{\tilde{G}}(a;n)/a$ reflects the *relative informational distortion* in the absence of competitive search.

With competitive search, the quantity distortion $D_{\tilde{G}}(a;n)$ is strictly less than the relative informational distortion because $\varepsilon_\rho(a_b(n);n) < 1$ for any $n > 0$.[29] As

seller-buyer ratio n. These results capture the more intuitive idea that the typical informational distortion should be lower when n increases because the "types" are becoming higher.

[29]To see this, in the limit as $n \to 0$ we have $\varepsilon_\rho(a_b(n);n) = 1$ because a_b^0 solves $\psi_G(a_b) = 0$ by

mentioned in Section 8.2, the reason for the lower quantity distortion is the fact that sellers are competing to offer contracts that are attractive to buyers in order to attract buyers to their submarket. As a result, sellers offer higher quantities than they otherwise would have in the absence of competitive search.

Lemma 7 says that when the seller-buyer ratio n is higher and competition is greater, the term $\varepsilon_\rho(a_b(n); n)$ is lower, reducing the quantity distortion.

Lemma 7. *The term $\varepsilon_\rho(a_b(n); n)$ is decreasing in the seller buyer-ratio n (i.e. increasing in the entry cost κ).*

For example, in any full-trade equilibrium where $a_b = a_0$, the term reflecting the effect of competitive search is $\varepsilon_\rho(a_b(n); n) = a_0 g(a_0)\eta_\alpha(n)$. We know the elasticity $\eta_\alpha(n)$ is decreasing in n by Lemma 1. As the market becomes more competitive and firms compete more intensely to attract buyers to their submarket, $\eta_\alpha(n)$ decreases and thus $\varepsilon_\rho(a_b(n); n)$ decreases, thereby reducing the quantity distortion.

9.3 Quantity distortion

There are two opposing effects of the seller-buyer ratio n on the quantity distortion. We know the informational distortion $I_{\tilde{G}}(a; n)$ is increasing in the seller-buyer ratio n. At the same time, Lemma 7 says that the term $\varepsilon_\rho(a_b(n); n)$, which represents the effect of competitive search on the quantity distortion, is decreasing in n. Recall that

$$D_{\tilde{G}}(a; n) = \underbrace{\varepsilon_\rho(a_b(n); n)}_{\text{decreasing in } n} \underbrace{\frac{I_{\tilde{G}}(a; n)}{a}}_{\text{increasing in } n} \tag{50}$$

for all $a > a_b(n)$, and $D_{\tilde{G}}(a; n) = 1$ for $a \leq a_b(n)$.

It is unclear from (50) whether the overall quantity distortion $D_{\tilde{G}}(a; n)$ is increasing or decreasing in the degree of competition through n. In fact, Lemma 8 tells us that, for any utility shock $a \geq a_b(n)$, the effect of competitive search – which reduces the quantity distortion by more when n is higher and competition is stronger – always dominates. This competitive effect is sufficiently strong that the quantity distortion $D_{\tilde{G}}(a; n)$ is decreasing in the seller-buyer ratio n for all traded goods.

Lemma 8. *The quantity distortion $D_{\tilde{G}}(a; n)$ is decreasing in the seller-buyer ratio n (i.e. increasing in the entry cost κ) for all $a > a_b(n)$.*

Lemma 5. Also, we have $\varepsilon_\rho(a_b(n); n)$ decreasing in n by Lemma 7 so $\varepsilon_\rho(a_b(n); n) < 1$ for any $n > 0$.

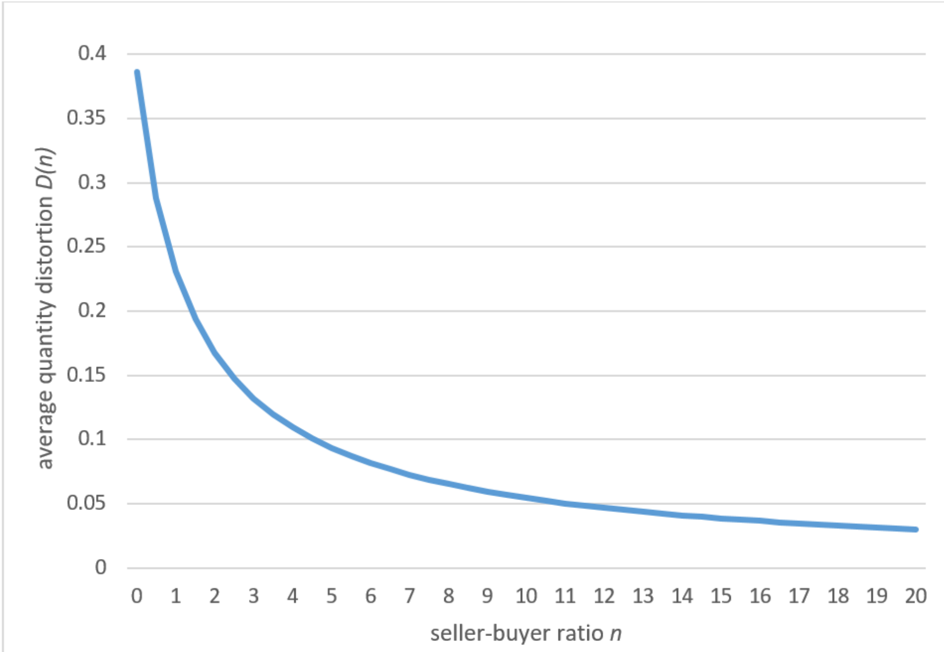

Figure 4: Example of average quantity distortion $D_{\tilde{G}}(n)$

For any given utility shock a, the quantity distortion $D_{\tilde{G}}(a;n)$ is decreasing in the seller-buyer ratio n. However, the distribution of choices $\tilde{G}$ is itself changing with n, so we still need to determine whether the average quantity distortion is increasing in n. To do this, we define the *average quantity distortion* by

$$D_{\tilde{G}}(n) \equiv \int_{a_0}^{\bar{a}} D_{\tilde{G}}(a;n)\tilde{g}(a;n)da. \tag{51}$$

Proposition 6 tells us that the effect of competitive search is sufficiently strong that the overall impact of greater entry on the average quantity distortion is negative.

Proposition 6. *The average quantity distortion $D_{\tilde{G}}(n)$ is decreasing in the seller-buyer ratio n (i.e. increasing in the entry cost κ).*

With competitive search, greater seller entry can therefore reduce the average quantity distortion due to buyers' private information. This is intuitive because sellers are competing to offer contracts that are attractive to buyers. As competition becomes more and more intense, sellers offer lower and lower quantity distortions.

Figure 4 illustrates an example of how the average quantity distortion $D_{\tilde{G}}(n)$ decreases with greater competition through a higher seller-buyer ratio n. For this example, we assume the meeting technology $\mathbb{P}_j$ is Geometric and the distribution G

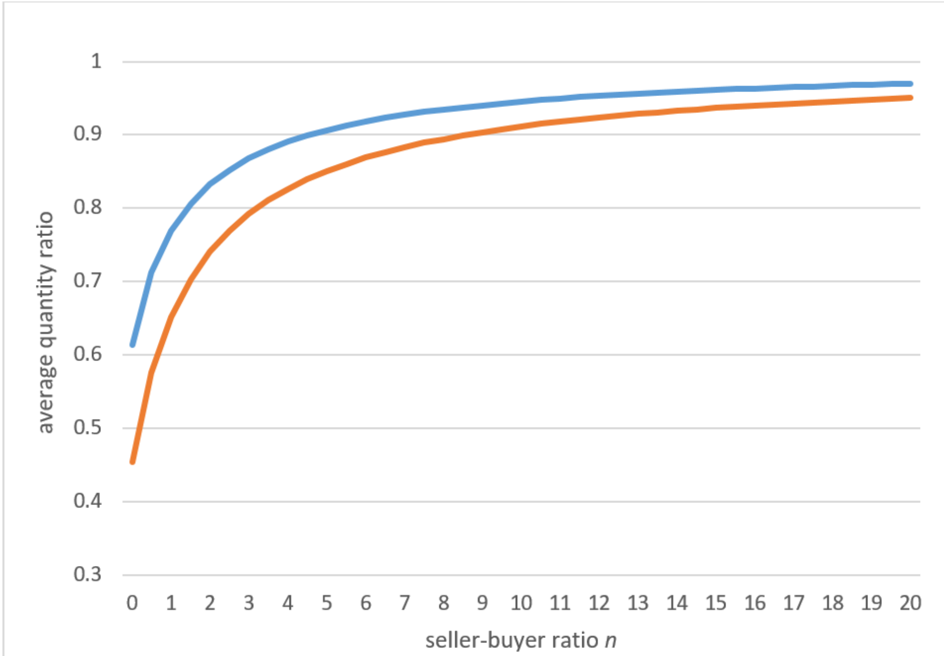

Figure 5: Example of average quantity ratio q_a/q_a^*

is uniform on $[1, 2]$. For example, if $n = 1$ the quantity distortion is 23.1%, while if $n = 5$ the quantity distortion is 9.3%. As we reach $n = 20$ sellers per buyer and the economy becomes highly competitive, the quantity distortion is only 3.0%.

The average ratio of equilibrium quantity to the first-best quantity is a monotonic transform of $1-D_{\tilde{G}}(n)$ that depends on the parameters of the functions $u(q)$ and $c(q)$. In Figure 5, the orange line depicts the average ratio of the equilibrium quantity to the first-best quantity if $c(q) = q$ and $u(q) = q^{1-b}/(1-b)$ where $b = 1/2$. For comparison, the blue line is $1 - D_{\tilde{G}}(n)$ where $D_{\tilde{G}}(n)$ is our measure of the quantity distortion depicted in Figure 4. For example, if $n = 1$ the average ratio of equilibrium quantity to first-best quantity is 65.2%, while if $n = 5$ the average ratio is 85.1%. As we reach a highly competitive economy with $n = 20$ sellers per buyer, the average ratio is 95.1% and the quantities traded are becoming very close to the first-best.

9.4 Competitive limit

With consumer choice, we have seen that greater seller entry can reduce the distortion in quantities due to buyers' private information. Given this is the *only* source of distortion that prevents the Friedman rule from achieving the first-best, this suggests that if the market is sufficiently competitive, i.e. the seller-buyer ratio is sufficiently high, consumer choice may be able to eliminate this distortion altogether.

Consider the limit as the entry cost $\kappa \to 0$ in competitive search equilibrium under private information. We know from Proposition 3 that n is strictly decreasing in κ. In the limiting case where $\kappa \to 0$, the equilibrium seller-buyer ratio $n \to \infty$. We refer to this as the *competitive limit* (or the "frictionless" limit) because all buyers meet a large number of sellers in each meeting. As $\kappa \to 0$ and $n \to \infty$, the distribution of choices $\tilde{G}$ converges to a degenerate distribution with support $A = \{\bar{a}\}$.

Proposition 7 says that, in this limiting case, we obtain an equilibrium in which all buyers are the "highest type" and all buyers trade. All meetings are liquidity constrained and the quantity traded is $q_{\bar{a}}$, which is below the first-best quantity. For all trades, the payment is $d_{\bar{a}} = z$, i.e. all money is used in trade.

Proposition 7. *For $i > 0$, in the competitive limit as $\kappa \to 0$, competitive search equilibrium satisfies:*

1. *The seller-buyer ratio $n \to \infty$ and all chosen goods have utility shock $\bar{a}$.*
2. *The quantity traded $q_{\bar{a}}$ satisfies*

$$\bar{a}u'(q_{\bar{a}}) = (1+i)c'(q_{\bar{a}}). \tag{52}$$

3. *We have $d_{\bar{a}} = z$ and*

$$\frac{d_{\bar{a}}}{\gamma} = a_c u(q^*_{a_c}) \tag{53}$$

where

$$a_c = \frac{\bar{a}}{1+i}. \tag{54}$$

With consumer choice, as the seller-buyer ratio n increases, the endogenous distribution of choices $\tilde{G}$ has greater mass concentrated at higher values that are closer to the upper bound $\bar{a}$. In the limit as $a \to \bar{a}$, we know that $I_{\tilde{G}}(\bar{a}; n) \to 0$, and therefore the quantity distortion $D_{\tilde{G}}(\bar{a}; n) \to 0$, because there is no informational distortion at the top. In the limit as $\kappa \to 0$, the seller-buyer ratio goes to infinity and all chosen goods are at the top, $\bar{a}$. Sellers know that all buyers have utility shock $\bar{a}$ for their chosen good and there is therefore no private information between buyers and their chosen seller. This eliminates the distortion in quantities due to private information.

While the monetary friction is not eliminated in the competitive limit as $\kappa \to 0$, it is clear that this friction can be eliminated at the Friedman rule as $i \to 0$.

Corollary 4. *In the competitive limit as $\kappa \to 0$, at the Friedman rule, competitive search equilibrium satisfies:*

1. *The seller-buyer ratio $n \to \infty$ and all chosen goods have utility shock $\bar{a}$.*
2. *The quantity traded $q_{\bar{a}}$ satisfies*

$$\bar{a}u'(q_{\bar{a}}) = c'(q_{\bar{a}}). \tag{55}$$

3. *We have $d_{\bar{a}} = z$ and $a_c = \bar{a}$, thus*

$$\frac{d_{\bar{a}}}{\gamma} = \bar{a}u(q_{\bar{a}}). \tag{56}$$

At the Friedman rule, both the informational distortion and the monetary friction are eliminated and we obtain the first-best allocation in the competitive limit, i.e. $q_{\bar{a}} = q^*_{\bar{a}}$ for all trades. At the same time, there is no longer any informational rents for buyers because there is no longer any private information for buyers, so $d_{\bar{a}}/\gamma \to \bar{a}u(q_{\bar{a}})$ and sellers are able to extract the full surplus in this limiting case.

10 Eliminating choice

Consider an environment which is identical to our main model except that buyers are randomly assigned to sellers within meetings, i.e. there is no consumer choice. This is effectively equivalent to an environment with bilateral (one-to-one) meetings in which buyers randomly meet one seller at at time. Studying this environment allows us to compare our results to what would happen in a standard model with bilateral meetings, and to show that consumer choice is necessary for our main results.[30]

Without choice, the main difference is that the distribution of choices $\tilde{G}$ is simply equal to the exogenous distribution G. Greater competition – as reflected in a higher seller-buyer ratio – has *no effect* on the distribution of choices. This difference is crucial. In particular, it means that seller entry cannot affect the quantity distortion by directly changing the distribution of choices. Without choice, we do not obtain the first-best allocation at the Friedman rule even in the competitive limit.

[30] Details of this exercise can be found in Online Appendix B.

11 Conclusion

This paper introduces consumer choice into a competitive search model of monetary exchange with the aim of providing greater realism as a model of retail trade. We allow consumers to meet multiple sellers and *choose* a seller with whom to trade. Consumer choice is influenced by random utility shocks that are private information and there are important interactions between consumer choice and private information.

With competitive search, the Friedman rule delivers the efficient allocation if the planner is subject the same information frictions as sellers. However, the Friedman rule cannot decentralize the first-best allocation due to the presence of buyers' private information. We find that the distortion in quantities traded – due to the presence of private information – can be alleviated through greater seller entry. In fact, in the competitive limit, consumer choice can eliminate the effects of private information altogether and deliver the first-best allocation at the Friedman rule.

Bajaj and Mangin (2023) extend the model and use it study the effect of consumer choice on the welfare cost of inflation. When we calibrate the model to U.S. money demand data, we find that a greater degree of consumer choice can significantly increase the welfare cost of inflation. This suggests that modelling consumer choice explicitly – instead of simply assuming meetings are bilateral – is important in a way that may be quantitatively significant for monetary policy.

In future work, it would be interesting to examine how monetary policy is affected by the nature of the meeting technology which governs the decentralized market. For example, does greater dispersion in the number of sellers each buyer meets have a positive or a negative effect on the welfare cost of inflation? Can reducing dispersion bring us closer to achieving the first-best allocation at the Friedman rule, even in the presence of private information? We leave these questions for future research.

References

S. P. Anderson, A. De Palma, and J.-F. Thisse. *Discrete choice theory of product differentiation*. MIT press, 1992.

A. Bajaj and S. Mangin. Consumer choice and the cost of inflation. *Working Paper*, 2023.

L. Becker and S. Mangin. The effect of search frictions on extreme outcomes. *Working paper*, 2023.

K. Burdett and K. L. Judd. Equilibrium price dispersion. *Econometrica*, pages 955–969, 1983.

X. Cai, P. Gautier, and R. Wolthoff. Spatial search. *Working paper*, 2023.

M. Choi and G. Rocheteau. A model of retail banking and the deposits channel of monetary policy. *Journal of Monetary Economics*, 139:127–147, 2023.

S. M. Davoodalhosseini. Constrained efficiency with adverse selection and directed search. *Journal of Economic Theory*, 183:568–593, 2019.

M. Dong and J. H. Jiang. Money and price posting under private information. *Journal of Economic Theory*, 150:740–777, 2014.

H. M. Ennis. Search, money, and inflation under private information. *Journal of Economic Theory*, 138(1):101–131, 2008.

M. Faig and B. Jerez. A theory of commerce. *Journal of Economic Theory*, 122(1): 60–99, 2005.

M. Faig and B. Jerez. Inflation, prices, and information in competitive search. *Advances in Macroeconomics*, 6(1):1–34, 2006.

M. Galenianos and P. Kircher. A model of money with multilateral matching. *Journal of Monetary Economics*, 55:1054–1066, 2008.

V. Guerrieri. Inefficient unemployment dynamics under asymmetric information. *Journal of Political Economy*, 116(4):667–708, 2008.

V. Guerrieri, R. Shimer, and R. Wright. Adverse selection in competitive search equilibrium. *Econometrica*, 78(6):1823–1862, 2010.

A. Head and A. Kumar. Price dispersion, inflation, and welfare. *International Economic Review*, 46(2):533–572, 2005.

L. Herrenbrueck. An open-economy model with money, endogenous search, and heterogeneous firms. *Economic Inquiry*, 55(4):1648–1670, 2017.

B. Julien, J. Kennes, and I. King. Bidding for Money. *Journal of Economic Theory*, 142:196–217, 2008.

R. Lagos and G. Rocheteau. Inflation, output, and welfare. *International Economic Review*, 46(2):495–522, 2005.

R. Lagos and R. Wright. A unified framework for monetary theory and policy analysis. *Journal of Political Economy*, 113(3):463–484, 2005.

R. Lagos, G. Rocheteau, and R. Wright. Liquidity: A new monetarist perspective. *Journal of Economic Literature*, 55(2):371–440, 2017.

B. Lester, L. Visschers, and R. Wolthoff. Meeting technologies and optimal trading mechanisms in competitive search markets. *Journal of Economic Theory*, 155:1–15, 2015.

S. Mangin. When is competition price-increasing? the impact of expected competition on prices. *The RAND Journal of Economics (forthcoming)*, 2023.

S. Mangin and B. Julien. Efficiency in search and matching models: A generalized hosios condition. *Journal of Economic Theory*, 193:105208, 2021.

E. Maskin and J. Riley. Monopoly with incomplete information. *The RAND Journal of Economics*, 15(2):171–196, 1984.

G. Menzio. A theory of partially directed search. *Journal of Political Economy*, 115 (5):748–769, 2007.

G. Menzio and S. Shi. Block recursive equilibria for stochastic models of search on the job. *Journal of Economic Theory*, 145(4):1453–1494, 2010.

E. Moen. Competitive Search Equilibrium. *Journal of Political Economy*, 105(2): 385–411, 1997.

E. R. Moen and A. Rosen. Incentives in competitive search equilibrium. *The Review of Economic Studies*, 78(2):733–761, 2011.

M. Mussa and S. Rosen. Monopoly and product quality. *Journal of Economic Theory*, 18(2):301–317, 1978.

G. Rocheteau and R. Wright. Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium. *Econometrica*, 73(1):175–202, 2005.

S. Shi. Money and prices: A model of search and bargaining. *Journal of Economic Theory*, 67(2):467–496, 1995.

S. Shi. A divisible search model of fiat money. *Econometrica*, pages 75–102, 1997.

S. Shi. Search, inflation and capital accumulation. *Journal of Monetary Economics*, 44(1):81–103, 1999.

S. Shi. Directed Search for Equilibrium Wage-Tenure Contracts. *Econometrica*, 77 (2):561–584, 2009.

S. Shi. Sequentially mixed search and equilibrium price dispersion. *Journal of Economic Theory*, 213:105735, 2023.

S. Shi and A. Delacroix. Should buyers or sellers organize trade in a frictional market? *The Quarterly Journal of Economics*, 133(4):2171–2214, 2018.

R. Shimer. PhD Thesis. *M.I.T.*, 1996.

N. Stokey, R. Lucas, and E. C. Prescott. *Recursive Methods in Economic Dynamics*. Harvard University Press, 1989.

A. Trejos and R. Wright. Search, bargaining, money, and prices. *Journal of Political Economy*, 103(1):118–141, 1995.

R. Wright, P. Kircher, B. Julien, and V. Guerrieri. Directed search and competitive search equilibrium: A guided tour. *Journal of Economic Literature*, 59(1):90–148, 2021.

Y. Zenou, A. Campbell, and P. Ushchev. The network origins of entry. Technical report, CEPR Discussion Papers, 2023.

Appendix

Proofs for Section 4

Before presenting the proofs for Section 4, we provide a lemma that will be useful. Lemma 9 summarizes some properties of invariant meeting technologies.

Given that $\mathbb{P}_j$ is invariant, we know that $\mathbb{P}_0$ is a completely monotone function and can therefore be represented as a Laplace transform. See, for example, Cai et al. (2023) and Becker and Mangin (2023) for further details and these properties.

Lemma 9. *If $\mathbb{P}_j$ is an invariant meeting technology, then*

1. *We have $\mathbb{P}_0'(x) < 0$ and $\mathbb{P}_0''(x) > 0$ for all $x \in \mathbb{R}^+ \backslash \{0\}$.*
2. *We have $\lim_{x \to 0} \mathbb{P}_0(x) = 1$ and $\lim_{x \to 0} \mathbb{P}_0'(x) = -1$.*
3. *We have $\lim_{x \to \infty} \mathbb{P}_0(x) = 0$, $\lim_{x \to \infty} \mathbb{P}_0'(x) = 0$, and $\lim_{x \to \infty} \mathbb{P}_0''(x) = 0$.*

Proof of Lemma 1

The properties of α follow immediately from $\alpha(n) = 1 - \mathbb{P}_0(n)$ and Lemma 9.

Given that $\mathbb{P}_j$ is invariant and $\mathbb{P}_0$ is a completely monotone function, we can apply Lemma 8 in the Appendix of Zenou, Campbell, and Ushchev (2023) to obtain the results that $\eta_\alpha'(n) < 0$ and $\frac{d}{dx}\left(\frac{-\alpha''(x)x}{\alpha'(x)}\right) > 0$. The fact that $\lim_{n \to 0} \eta_\alpha(n) = 1$ follows from L'Hopital's rule. We also have $\lim_{n \to \infty} \eta_\alpha(n) = \lim_{n \to \infty} \frac{\mathbb{P}_1(n)}{1 - \mathbb{P}_0(n)} = 0$. Finally, $\eta_\alpha'(n) < 0$ and $\lim_{n \to 0} \eta_\alpha(n) = 1$ implies $\eta_\alpha(n) < 1$ for any $n > 0$. ■

Proof of Lemma 2

The distribution of the maximum of $j \geq 1$ draws is $(G(a))^j$, and weighting by the probability $\mathbb{P}_j(n)$ that exactly j sellers meet a buyer, conditional on $j \geq 1$, yields

$$\tilde{G}(a; n) = \frac{\sum_{j=1}^{\infty} \mathbb{P}_j(n)(G(a))^j}{\alpha(n)}. \tag{57}$$

Given that we assume the meeting technology $\mathbb{P}_j$ is invariant, we have $\sum_{j=0}^{\infty} \mathbb{P}_j(n) y^j = \mathbb{P}_0(n(1-y))$ and substituting into the above yields (6).

Part 1. Taking the limit as $n \to 0$, we have

$$\lim_{n\to 0} \tilde{G}(a;n) = \lim_{n\to 0}\left(\frac{\mathbb{P}_0(n(1-G(a))) - \mathbb{P}_0(n)}{\alpha(n)}\right) = G(a) \tag{58}$$

using L'Hopital's rule and the fact that $\lim_{z\to 0}\mathbb{P}_0(z) = 1$ and $\lim_{z\to 0}\mathbb{P}_0'(z) = -1$ by Lemma 9. Therefore, $\tilde{a}(n) \to E_G(a)$.

Part 2. Taking the limit as $n \to \infty$, we have

$$\lim_{n\to \infty} \tilde{G}(a;n) = \lim_{n\to \infty}\left(\frac{\mathbb{P}_0(n(1-G(a))) - \mathbb{P}_0(n)}{\alpha(n)}\right) = 0 \tag{59}$$

for any $a \in [a_0, \bar{a})$ and $\lim_{n\to\infty}\tilde{G}(\bar{a};n) = 1$, using the fact that $\lim_{z\to\infty}\mathbb{P}_0(z) = 0$ by Lemma 9. Therefore, $\tilde{a}(n) \to \bar{a}$.

Part 3. For $n > 0$, we have $\tilde{G}(a;n) < G(a)$ for all $a \in A$. To see this, let $w_j(n) = \mathbb{P}_j(n)/\alpha(n)$. Using (57), $\tilde{G}(a;n) = \sum_{j=1}^{\infty} w_j(n)(G(a))^j$. Since $\tilde{G}(a;n)$ is a weighted average of the term $(G(a))^j$ for all $j \geq 1$, and $(G(a))^j < G(a)$ for all $j > 1$ and $a \in (a_0, \bar{a})$, and $G(a)^j = G(a)$ for $j = 1$ and $a = a_0$ or $a = \bar{a}$, we have $\tilde{G}(a;n) < G(a)$. So $\tilde{G}(a;n)$ first order stochastically dominates $G(a)$ and $\tilde{a}(n) > E_G(a)$.

Part 4. Consider any $f : A \to \mathbb{R}_+$ such that $f' > 0$. Re-stating Lemma 1 in terms of $\mathbb{P}_0(x)$ gives $\frac{d}{dx}\left(\frac{-\mathbb{P}_0''(x)x}{\mathbb{P}_0'(x)}\right) > 0$. For any n_1 and n_2 such that $n_1 > n_2$, Part 5 implies that $\tilde{f}(n_1) > \tilde{f}(n_2)$, i.e. $\int_{a_0}^{\bar{a}} f(a)d\tilde{G}(a;n_1) > \int_{a_0}^{\bar{a}} f(a)d\tilde{G}(a;n_2)$. Thus $\tilde{G}(a;n_1) \leq \tilde{G}(a;n_2)$ and $\tilde{G}(a;n_1)$ first order stochastically dominates $\tilde{G}(a;n_2)$.

Part 5. Applying Leibniz' integral rule gives us

$$\tilde{f}'(n) = \int_{a_0}^{\bar{a}} f(a)\frac{\partial \tilde{g}(a;n)}{\partial n}da. \tag{60}$$

First, we show that there exists a unique cutoff $\hat{a} \in A$ such that $\frac{\partial \tilde{g}(a;n)}{\partial n} = 0$, and we have $\frac{\partial \tilde{g}(a;n)}{\partial n} > 0$ for $a > \hat{a}$ and $\frac{\partial \tilde{g}(a;n)}{\partial n} < 0$ for $a < \hat{a}$. To start with, we have

$$\tilde{g}(a;n) = \frac{-ng(a)\mathbb{P}_0'(n(1-G(a)))}{\alpha(n)}. \tag{61}$$

Differentiating (61) with respect to n, we obtain

$$\frac{\partial \tilde{g}(a;n)}{\partial n} = \frac{g(a)\mathbb{P}_0'(x)}{\alpha(n)}\left(\frac{-x\mathbb{P}_0''(x)}{\mathbb{P}_0'(x)} - (1-\eta_\alpha(n))\right) \tag{62}$$

where $x = n(1 - G(a))$. Since $\mathbb{P}_0'(x) < 0$ by Lemma 9, $\frac{\partial \tilde{g}(a;n)}{\partial n} > 0$ if and only if

$$\frac{-\mathbb{P}_0''(x)x}{\mathbb{P}_0'(x)} < 1 - \eta_\alpha(n). \tag{63}$$

Re-stating Lemma 1 in terms of $\mathbb{P}_0(x)$ gives $\frac{d}{dx}\left(\frac{-\mathbb{P}_0''(x)x}{\mathbb{P}_0'(x)}\right) > 0$. So there exists a unique solution x, and therefore a unique solution a, such that the above holds with equality. Defining $\hat{a}$ as the solution to this equality, we have $\frac{\partial \tilde{g}(a;n)}{\partial n} > 0$ if and only if $a > \hat{a}$, so

$$\tilde{f}'(n) \equiv \int_{a_0}^{\hat{a}} f(a)\frac{\partial \tilde{g}(a;n)}{\partial n}da + \int_{\hat{a}}^{\bar{a}} f(a)\frac{\partial \tilde{g}(a;n)}{\partial n}da. \tag{64}$$

We therefore have $\tilde{f}'(n) > 0$ if and only if

$$\int_{\hat{a}}^{\bar{a}} f(a)\frac{\partial \tilde{g}(a;n)}{\partial n}da > -\int_{a_0}^{\hat{a}} f(a)\frac{\partial \tilde{g}(a;n)}{\partial n}da > 0. \tag{65}$$

Given that $f' > 0$, and both sides of (65) are positive, by definition of $\hat{a}$, a sufficient condition for $\tilde{f}'(n) > 0$ is

$$\int_{\hat{a}}^{\bar{a}} f(\hat{a})\frac{\partial \tilde{g}(a;n)}{\partial n}da \geq -\int_{a_0}^{\hat{a}} f(\hat{a})\frac{\partial \tilde{g}(a;n)}{\partial n}da, \tag{66}$$

which is true if and only if $\int_{\hat{a}}^{\bar{a}} \frac{\partial \tilde{g}(a;n)}{\partial n}da \geq -\int_{a_0}^{\hat{a}} \frac{\partial \tilde{g}(a;n)}{\partial n}da$, or equivalently $\int_{a_0}^{\bar{a}} \frac{\partial \tilde{g}(a;n)}{\partial n}da \geq 0$. Applying Leibniz' integral rule again, $\int_{a_0}^{\bar{a}} \frac{\partial \tilde{g}(a;n)}{\partial n}da = \frac{\partial}{\partial n}\int_{a_0}^{\bar{a}} \tilde{g}(a;n)da = 0$, since $\int_{a_0}^{\bar{a}} \tilde{g}(a;n)da = 1$, and thus $\tilde{f}'(n) > 0$. ■

Proofs for Section 5

Proof of Proposition 1

The first-order condition with respect to q_a is

$$\alpha(n)[au'(q_a) - c'(q_a)]\tilde{g}(a;n) = 0 \tag{67}$$

and the first order-condition with respect to n is

$$\alpha'(n)\tilde{s}(n;\{q_a\}_{a\in A}) + \alpha(n)\tilde{s}'(n;\{q_a\}_{a\in A}) = \kappa. \tag{68}$$

We can verify that $s_a^* = au(q_a^*) - c(q_a^*)$ is strictly increasing in a. Differentiating s_a^*,

$$\frac{ds_a^*}{da} = u(q_a^*) + [au'(q_a^*) - c'(q_a^*)] \frac{dq^*}{da}. \tag{69}$$

Since $au'(q_a^*) - c'(q_a^*) = 0$ by (67) if $n^* > 0$, we have $\frac{ds_a^*}{da} = u(q_a^*) > 0$ for all $a \in (a_0, \bar{a}]$. Given that s_a^* is strictly increasing in a and $s_0^* \geq 0$ where $s_0^* \equiv a_0 u(q_0) - c(q_0)$, we have $s_a^* \geq 0$ for all $a \in A$. Therefore, all chosen goods $a \in A$ are traded because we assume $a_0 > 0$. For all $a \in A$, the quantity q_a satisfies $au'(q_a) = c'(q_a)$.

Since s_a^* is strictly increasing in a, at the first-best the seller with the highest utility shock a is chosen. The distribution of choices $\tilde{G}$ is thus equal to (6).

Existence and uniqueness of the first-best allocation follows from Proposition 3, which is proven below. We know that $s_a^* \geq 0$ for all $a \in A$ and thus all chosen goods are traded. Setting $i = 0$ in Proposition 3 results in equilibrium conditions that are equivalent to the first-order conditions. It follows that there exists a unique first-best allocation with $n^* > 0$ provided that Assumption 6 holds, except that $q_a^0 = q_a^*$ since q_a^* does not depend directly on n. That is, Assumption 3 suffices. ■

Proofs for Section 7

Proof of Corollary 1

For any $a \in [a_0, \bar{a}]$, we have $(a - \phi^m(a; n))u'(q_a) = c'(q_a)$ where $\phi^m(a; n) \equiv \frac{ai}{i+\alpha(n)}$ is greater than zero, so there is underconsumption, i.e. $q_a < q_a^*$.

The equilibrium n satisfies

$$\alpha'(n)\tilde{s}(n; \{q_a\}_{a\in A}) + \alpha(n)\tilde{s}'(n; \{q_a\}_{a\in A}) = \kappa \tag{70}$$

and the first-best n^* satisfies

$$\alpha'(n^*)\tilde{s}(n^*; \{q_a^*\}_{a\in A}) + \alpha(n^*)\tilde{s}'(n^*; \{q_a^*\}_{a\in A}) = \kappa. \tag{71}$$

We know from above that $q_a^* > q_a$ for any $a \in (a_0, \bar{a})$, but we cannot infer anything about whether there is under-entry ($n < n^*$), over-entry ($n > n^*$), or first-best entry ($n = n^*$). We can find examples of equilibria for each of these three possibilities.

Proofs for Section 8

Proof of Lemma 4

Starting with (6) and letting $x = n(1 - G(a))$, we have

$$\tilde{G}(a; n) = \frac{\mathbb{P}_0(x) - \mathbb{P}_0(n)}{\alpha(n)} \tag{72}$$

and therefore, using $\alpha(n) = 1 - \mathbb{P}_0(n)$, we obtain

$$\tilde{G}(a; n) = \frac{\alpha(n) - \alpha(x)}{\alpha(n)} \tag{73}$$

Next, differentiating with respect to a yields

$$\tilde{g}(a; n) = \frac{ng(a)\alpha'(x)}{\alpha(n)}. \tag{74}$$

Therefore, we obtain

$$\frac{1 - \tilde{G}(a; n)}{\tilde{g}(a; n)} = \frac{\alpha(x)}{ng(a)\alpha'(x)}, \tag{75}$$

which, using the fact that $x = n(1 - G(a))$, is equivalent to

$$\frac{1 - \tilde{G}(a; n)}{\tilde{g}(a; n)} = \frac{\alpha(x)}{\alpha'(x)x}\left(\frac{1 - G(a)}{g(a)}\right). \tag{76}$$

The hazard rate of the distribution $\tilde{G}(a; n)$ is thus given by

$$h_{\tilde{G}}(a; n) = \eta_\alpha(n(1 - G(a))h_G(a) \tag{77}$$

where $\eta_\alpha(x) = \frac{\alpha'(x)x}{\alpha(x)}$ and $h_G(a) = \frac{1-G(a)}{g(a)}$. The result follows immediately from (77) plus Lemma 1, which implies that $\eta'_\alpha(x) < 0$ and therefore $\eta_\alpha(x)$ is increasing in a, plus our assumption that G has an increasing hazard rate, i.e. $h'_G(a) > 0$. ■

Proof of Lemma 5

In the limit as $n \to 0$, we have $\tilde{G}(a; n) \to G(a)$ by Lemma 2. As $n \to 0$, we have $i/\alpha(n) \to \infty$ so $\delta \to \infty$. Also, $1/\delta \to 0$ implies that $\phi(a; n) \to \frac{1-G(a)}{g(a)}$ on $(a_b, a_c]$.

Lemma 12 says a_b must solve $a_b = \phi(a_b; n)$ which is equivalent to $\psi_G(a_b) = 0$ where $\psi_G(a) \equiv a - \frac{1-G(a)}{g(a)}$. It follows from Assumption 4 that $\psi'_G(a) > 0$. Also, we have $\psi_G(\bar{a}) = \bar{a} > 0$ and $\psi_G(a_0) < 0$ if $a_0 g(a_0) \leq 1$, which is true by Assumption 1. So, there exists a unique a_b such that $\psi_G(a_b) = 0$. Finally, as $n \to 0$, the cut-off a_c solves

$$(\bar{a} - a_c)[1 - G(a_c)] = \bar{a}\left[\frac{-\psi_G(a_b)}{a_b - \psi_G(a_b)}\right](1 - G(a_b)). \tag{78}$$

Thus $\psi_G(a_b) = 0$ and $a_b > 0$, so the right side is zero, which implies $a_c = \bar{a}$. ■

Proof of Lemma 6

Part 1. Clear from the fact that q_a must be non-decreasing in equilibrium.

Part 2. Next, consider the trading interval $[a_b, \bar{a}]$. Recall that $v_a \equiv au(q_a) - d_a/\gamma$, the buyer's ex post trading surplus, and let $\dot{v}_a \equiv v'(a)$. Next, let $f(a) = \frac{z}{\gamma} - au(q_a) + v_a$. Constraint (86) binds if and only if $f(a) = 0$. Differentiating, we have $f'(a) = -(u(q_a) + au'(q_a)q'(a)) + \dot{v}_a$. Using $\dot{v}_a = u(q_a)$, this implies that $f'(a) = -au'(q_a)q'(a)$. Since $u'(q_a) > 0$ and $q'(a) \geq 0$ on this range, we have $f'(a) \leq 0$. Therefore, there exists a unique $a_c \in [a_b, \bar{a}]$ such that $f(a) = 0$ and constraint (86) binds if and only if $a \in [a_c, \bar{a}]$, so $d_a = z$ and thus $\frac{z}{\gamma} = au(q_a) - v_a$. Differentiating, we have $au'(q_a)q'(a) = 0$ for all $a \in [a_c, \bar{a}]$. Since $u'(q_a) > 0$ and $q'(a) \geq 0$, this requires $q'(a) = 0$ and thus $q_a = q_{a_c}$ on $[a_c, \bar{a}]$. ■

Proof of Proposition 3

Our strategy is to solve for the equilibrium in two stages. First, we take z and n as given and solve for sellers' posted contracts $\{(q_a, d_a)\}_{a \in A}$ (inner maximization problem). Second, we solve for z and n (outer maximization problem) given the solutions for the posted contracts $\{(q_a, d_a)\}_{a \in A}$.

We first solve the inner and outer maximization problems. Next, we prove Parts 1 to 8 of Proposition 3. Finally, we prove existence and uniqueness of equilibrium.

Stage 1. Inner maximization problem

In the first stage, taking $z > 0$ and $n > 0$ as given (we later prove this), the problem is to maximize (22) subject to (23) at equality, plus a liquidity constraint $d_a \leq z$ for all $a \in A$, the IC constraint (13), and the IR constraint (12). Ignoring constants, the inner maximization problem is:

$$\max_{\{(q_a,d_a)\}_{a\in A}} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a;n) - i\frac{z}{\gamma} \right\}, \tag{79}$$

subject to

$$\frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[-c(q_a) + \frac{d_a}{\gamma} \right] d\tilde{G}(a;n) = \kappa, \tag{80}$$

and, for all $a, a' \in A$,

$$\begin{aligned}
d_a &\leq z, && (81)\\
au(q_a) - \frac{d_a}{\gamma} &\geq au(q_{a'}) - \frac{d_{a'}}{\gamma}, && (82)\\
au(q_a) - \frac{d_a}{\gamma} &\geq 0, && (83)\\
d_a, q_a &\geq 0. && (84)
\end{aligned}$$

To solve the inner maximization problem (79), we transform the above problem using Lemma 3. Recalling that $v_a \equiv au(q_a) - d_a/\gamma$, the buyer's ex post trading surplus, Lemma 3 simplifies the IC constraints in our environment because the single crossing property (29) holds. In particular, Lemma 3 says the IC constraints hold if q_a is non-decreasing and $v'(a) = u(q_a)$.

We can use $v_a \equiv au(q_a) - d_a/\gamma$ and Lemma 3 to re-write the problem as an optimal control problem where q_a is the control variable, v_a is the state variable, and δ is the Lagrange multiplier associated with the seller entry constraint (80).

In the first stage, we take z, n, δ as given and later solve for these. Using $v_a \equiv au(q_a) - d_a/\gamma$ to eliminate d_a in the above, and substituting in the constraint (80), the inner maximization problem becomes

$$\max_{\{(q_a,v_a)\}_{a\in A}} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left\{ (1-\delta)v_a + \delta \left[au(q_a) - c(q_a) \right] \right\} \tilde{g}(a;n) da - \delta nk - i\frac{z}{\gamma} \right\}, \tag{85}$$

subject to $v_0 = 0$ and q_a is non-decreasing, and for all $a \in A$,

$$au(q_a) - v_a \leq \frac{z}{\gamma}, \tag{86}$$

$$\dot{v}_a = u(q_a), \tag{87}$$

$$q_a, v_a \geq 0. \tag{88}$$

The inner maximization problem is a standard optimal control problem with q_a as the control variable and v_a as the state variable. We can therefore apply the Maximum Principle to find the necessary conditions for the optimal path of the control and state variables. To solve the inner maximization problem, we ignore the condition that q_a is non-decreasing and later verify that it holds in Lemma 13.

Ignoring constants, the current value Hamiltonian for this problem is:

$$H = \alpha(n)\{(1-\delta)v_a + \delta\left[au(q_a) - c(q_a)\right]\}\tilde{g}(a;n) + \lambda_a u(q_a) \tag{89}$$

where λ_a is the costate variable, and the Lagrangian is:

$$L = \alpha(n)\{(1-\delta)v_a + \delta\left[au(q_a) - c(q_a)\right]\}\tilde{g}(a;n) + \lambda_a u(q_a) + \mu_a\left[\frac{z}{\gamma} - au(q_a) + v_a\right] + \theta_a q_a + \eta_a v_a \tag{90}$$

where μ_a, θ_a and η_a are the Lagrangian multipliers associated with the liquidity constraint, non-negativity constraint, and IR constraint respectively.

The first-order conditions and the transversality condition are as follows:

$$\frac{\partial L}{\partial q_a} = \alpha(n)\delta\left[au'(q_a) - c'(q_a)\right]\tilde{g}(a;n) + (\lambda_a - \mu_a a)\, u'(q_a) + \theta_a = 0, \tag{91}$$

$$\frac{\partial L}{\partial v_a} = (1-\delta)\alpha(n)\tilde{g}(a;n) + \mu_a + \eta_a = -\dot{\lambda}_a, \tag{92}$$

$$\frac{\partial L}{\partial \lambda_a} = \dot{v}_a = u(q_a), \tag{93}$$

$$\lambda_{\bar{a}} v_{\bar{a}} = 0. \tag{94}$$

For the inequality constraints, the conditions are:

$$\mu_a \geq 0,\ \mu_a(\frac{z}{\gamma} - au(q_a) + v_a) = 0, \tag{95}$$

$$\theta_a \geq 0,\ \theta_a q_a = 0, \tag{96}$$

$$\eta_a \geq 0,\ \eta_a v_a = 0. \tag{97}$$

The following lemma provides expressions for λ_a and Σ_{a_c}, where $\Sigma_a \equiv \int_a^{\bar{a}} \mu_x dx$.

Lemma 10. *For all $a \in [a_0, a_c]$, we have the following:*

$$\lambda_a = \alpha(n)(1-\delta)[1 - \tilde{G}(a;n)] + \Sigma_{a_c} + \int_a^{\bar{a}} \eta_x dx \tag{98}$$

and

$$\Sigma_{a_c} = \frac{\alpha(n)}{\bar{a}} \int_{a_c}^{\bar{a}} [\delta(x - a_c)\tilde{g}(x;n) + (1-\delta)(\tilde{G}(a_c;n) - \tilde{G}(x;n)]dx. \tag{99}$$

Proof. Start with the fact that

$$(1-\delta)\alpha(n)\tilde{g}(a;n) + \mu_a + \eta_a = -\dot{\lambda}_a \tag{100}$$

from the first-order condition (92) above. Integrating both sides over $[a, \bar{a}]$, we obtain

$$-\int_a^{\bar{a}} \dot{\lambda}_x dx = \int_a^{\bar{a}} (1-\delta)\alpha(n)\tilde{g}(x;n)dx + \int_a^{\bar{a}} \mu_x dx + \int_a^{\bar{a}} \eta_x dx \tag{101}$$

and therefore

$$-(\lambda_{\bar{a}} - \lambda_a) = \alpha(n)(1-\delta)\int_a^{\bar{a}} \tilde{g}(x;n)dx + \int_a^{\bar{a}} \mu_x dx + \int_a^{\bar{a}} \eta_x dx. \tag{102}$$

The transversality condition $\lambda_{\bar{a}} v_{\bar{a}} = 0$ implies $\lambda_{\bar{a}} = 0$ since $v_{\bar{a}} > 0$. Substituting $\Sigma_a \equiv \int_a^{\bar{a}} \mu_x dx$ into the above, and setting $\lambda_{\bar{a}} = 0$ yields

$$\lambda_a = \alpha(n)(1-\delta)\int_a^{\bar{a}} \tilde{g}(x;n)dx + \Sigma_a + \int_a^{\bar{a}} \eta_x dx. \tag{103}$$

Now, $\mu_a = 0$ for all $a \in [a_0, a_c]$, thus $\Sigma_a = \int_a^{\bar{a}} \mu_x dx = \int_{a_c}^{\bar{a}} \mu_x dx = \Sigma_{a_c}$ for all $a \in [a_0, a_c]$. Substituting into (103) and using $\int_a^{\bar{a}} \tilde{g}(x;n)dx = [\tilde{G}(x;n)]_a^{\bar{a}} = 1 - \tilde{G}(a;n)$ yields (98).

For the second part, using (91) and Lemma 6, for all $a \in [a_c, \bar{a}]$ we have

$$\alpha(n)\delta\left[au'(\bar{q}) - c'(\bar{q})\right]\tilde{g}(a;n) + (\lambda_a - \mu_a a)\, u'(\bar{q}) = 0 \tag{104}$$

where $\bar{q} \equiv q_{a_c}$, and, for all $a \in [a_c, \bar{a}]$, we also have

$$\alpha(n)\delta\left[a_c u'(\bar{q}) - c'(\bar{q})\right]\tilde{g}(a;n) + \lambda_{a_c} u'(\bar{q}) = 0. \tag{105}$$

Using the above two equations, and dividing both sides by $u'(\bar{q})$, we obtain

$$\alpha(n)\delta(a - a_c)\tilde{g}(a;n) = -\lambda_a + \mu_a a + \lambda_{a_c}. \tag{106}$$

Substituting (103) for both λ_a and λ_{a_c} into the above, and simplifying, yields

$$\alpha(n)[\delta(a - a_c)\tilde{g}(a;n) + (1-\delta)(\tilde{G}(a_c;n) - \tilde{G}(a;n)] = -\Sigma_a + \mu_a a + \Sigma_{a_c}. \tag{107}$$

Finally, $\Sigma_a = \int_a^{\bar{a}} \mu_x dx$ implies that $\dot{\Sigma}_a = -\mu_a$ and thus we obtain

$$\alpha(n)[\delta(a - a_c)\tilde{g}(a;n) + (1-\delta)(\tilde{G}(a_c;n) - \tilde{G}(a;n)] = -\Sigma_a - \dot{\Sigma}_a a + \Sigma_{a_c}. \tag{108}$$

Integrating both sides over $[a_c, \bar{a}]$, we have

$$\alpha(n)\int_{a_c}^{\bar{a}}[\delta(x-a_c)\tilde{g}(x;n)+(1-\delta)(\tilde{G}(a_c;n)-\tilde{G}(x;n)]dx = \int_{a_c}^{\bar{a}}\left(-\Sigma_x - \dot{\Sigma}_x x + \Sigma_{a_c}\right)dx \tag{109}$$

where $\int_{a_c}^{\bar{a}}\left(-\Sigma_x - \dot{\Sigma}_x x + \Sigma_{a_c}\right)dx = -\left(\int_{a_c}^{\bar{a}} \Sigma_x + \dot{\Sigma}_x x \; dx\right) + [\Sigma_{a_c} x]_{a_c}^{\bar{a}}$. Using integration by parts, $\int_{a_c}^{\bar{a}} \Sigma_x + \dot{\Sigma}_x x \; dx = [\Sigma_x x]_{a_c}^{\bar{a}} = \Sigma_{\bar{a}}\bar{a} - \Sigma_{a_c} a_c = -\Sigma_{a_c} a_c$, and $[\Sigma_{a_c} x]_{a_c}^{\bar{a}} = \Sigma_{a_c}\bar{a} - \Sigma_{a_c} a_c$. Substituting $\int_{a_c}^{\bar{a}}\left(-\Sigma_x - \dot{\Sigma}_x x + \Sigma_{a_c}\right)dx = \Sigma_{a_c}\bar{a}$ into the above yields

$$\alpha(n)\int_{a_c}^{\bar{a}}[\delta(x - a_c)\tilde{g}(x;n) + (1-\delta)(\tilde{G}(a_c;n) - \tilde{G}(x;n)]dx = \Sigma_{a_c}\bar{a} \tag{110}$$

and we therefore obtain (99). ■

Lemma 11. *We have the following:*

$$\delta = \frac{1}{1 - a_0\tilde{g}(a_0;n)}\left(1 + \frac{\Sigma_{a_c} + \int_{a_0}^{\bar{a}} \eta_x dx}{\alpha(n)}\right). \tag{111}$$

Proof. To start with, we have

$$\alpha(n)\delta\left[au'(q_a)-c'(q_a)\right]\tilde{g}(a;n)+(\lambda_a-\mu_a a)\,u'(q_a)+\theta_a=0 \tag{112}$$

from the first-order condition (91) for q_a. Dividing both sides by q_a, we obtain

$$\alpha(n)\delta\left[a-\frac{c'(q_a)}{u'(q_a)}\right]\tilde{g}(a;n)+(\lambda_a-\mu_a a)=\frac{-\theta_a}{u'(q_a)}. \tag{113}$$

Taking the limit as $q_a \to 0$, and using $\lim_{q\to 0} u'(q) = +\infty$ and $\lim_{q\to 0}\frac{c'(q)}{u'(q)} = 0$ yields

$$\lim_{q\to 0}\alpha(n)\delta\left[a-\frac{c'(q)}{u'(q)}\right]\tilde{g}(a;n)+(\lambda_a-\mu_a a)+\frac{\theta_a}{u'(q)}=\alpha(n)\delta a\tilde{g}(a;n)+(\lambda_a-\mu_a a)=0 \tag{114}$$

for any $a \le a_b$ and therefore

$$\lambda_a=-\alpha(n)\delta a\tilde{g}(a;n)-\mu_a a \tag{115}$$

for any $a \le a_b$. In particular, we have

$$\lambda_{a_0}=-\alpha(n)\delta a_0\tilde{g}(a_0;n)-\mu_{a_0}a_0. \tag{116}$$

The liquidity constraint doesn't bind at a_0 so $\mu_{a_0} = 0$. Thus, we have

$$\lambda_{a_0}=-\alpha(n)\delta a_0\tilde{g}(a_0;n). \tag{117}$$

Next, applying Lemma 10 to the special case $a = a_0$, we have

$$\lambda_{a_0}=\alpha(n)(1-\delta)+\Sigma_{a_c}+\int_{a_0}^{\bar{a}}\eta_x dx. \tag{118}$$

Therefore, we have

$$-\alpha(n)\delta a_0\tilde{g}(a_0;n)=\alpha(n)(1-\delta)+\Sigma_{a_c}+\int_{a_0}^{\bar{a}}\eta_x dx, \tag{119}$$

which is equivalent to (111). ■

To determine q_a for all $a \in A$, it remains only to determine δ, a_b, and a_c. By Lemma 6, there are three intervals to consider.

Case 1. We have $q_a = 0$ for all $a \in [a_0, a_b]$.

Case 2. For any $a \in (a_b, a_c]$, we have $\theta_a = 0$ and $\mu_a = 0$, so q_a solves

$$\alpha(n)\delta\left[au'(q_a) - c'(q_a)\right]\tilde{g}(a;n) \;= -\lambda_a u'(q_a). \tag{120}$$

Using Lemma 10, plus the fact that $\int_a^{\bar{a}} \eta_x dx = 0$ for all $a > a_b$ since $\eta_a = 0$ for $a > a_b$,

$$\lambda_a = \alpha(n)(1-\delta)[1 - \tilde{G}(a;n)] + \Sigma_{a_c}. \tag{121}$$

for any $a > a_b$. Substituting into (120), we obtain

$$(a - \phi(a;n))u'(q_a) = c'(q_a) \tag{122}$$

where

$$\phi(a;n) = \left(\frac{\delta - 1}{\delta}\right)\left(\frac{1 - \tilde{G}(a;n)}{\tilde{g}(a;n)}\right) - \frac{\Sigma_{a_c}}{\alpha(n)\delta\tilde{g}(a;n)}. \tag{123}$$

Case 3. For any $a \in [a_c, \bar{a}]$, we have $\theta_a = 0$ and $q_a = q_{a_c}$ by Lemma 6.

The following lemma will prove useful in deriving Proposition 3.

Lemma 12. *We have $a = \phi(a;n)$ for all $a \le a_b$.*

Proof. For $a = a_b$, we have $q_{a_b} = 0$. Using (122) above, we have

$$\lim_{a \to a_b}(a - \phi(a;n)) = \lim_{a \to a_b}\left[1 - \frac{\phi(a;n)}{a}\right]a = \lim_{q \to 0}\frac{c'(q)}{u'(q)} = 0 \tag{124}$$

since we have $\lim_{q\to 0}\frac{c'(q)}{u'(q)} = 0$ by assumption. Therefore, by continuity of the function q_a, we have either $\frac{\phi(a_b;n)}{a_b} = 1$, or equivalently $a = \phi(a_b;n)$, or $a_b = 0$. Since we assume $a_0 > 0$, we have $a = \phi(a;n)$ for all $a \le a_b$. ■

Stage 2. Outer maximization problem

The outer maximization problem we solve next is

$$\max_{z,n,\delta}\left\{J(n,z,\delta) - \delta nk - i\frac{z}{\gamma}\right\}, \tag{125}$$

where we define

$$(126)\quad J(n,z,\delta) \equiv \max_{\{(q_a,v_a)\}_{a\in A}} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \{(1-\delta)v_a + \delta\left[au(q_a) - c(q_a)\right]\} \tilde{g}(a;n)da \right\},$$

subject to $v_0 = 0$ and, for all $a \in A$, constraints (86), (87), (**??**), and (88).

To solve the outer maximization problem, the function $J(n,z,\delta)$ is equivalent to

(127)

$$J(n,z,\delta) = \max_{\{(q_a,v_a)\}_{a\in A}} \left\{ \begin{array}{c} \int_{a_0}^{\bar{a}} \alpha(n)\{(1-\delta)v_a + \delta\left[au(q_a) - c(q_a)\right]\}\tilde{g}(a;n)da \\ + \int_{a_0}^{\bar{a}} \left[\mu_a\left(\frac{z}{\gamma} - au(q_a) + v_a\right) + \eta_a v_a + \lambda_a u(q_a) + \theta_a q_a\right] da \end{array} \right\}.$$

Define $\tilde{s}(n) \equiv \int_{a_0}^{\bar{a}} s_a d\tilde{G}(a;n)$ and $\tilde{v}(n) \equiv \int_{a_0}^{\bar{a}} v_a d\tilde{G}(a;n)$. Returning to our original formulation to eliminate δ, the above problem is equivalent to

$$(128)\qquad \max_{z,n}\left\{\hat{J}(n,z) - i\frac{z}{\gamma}\right\},$$

where

(129)

$$\hat{J}(n,z) = \max_{\{(q_a,v_a)\}_{a\in A}} \left\{\alpha(n)\tilde{v}(n) + \int_{a_0}^{\bar{a}} \left[\mu_a\left(\frac{z}{\gamma} - au(q_a) + v_a\right) + \eta_a v_a + \lambda_a u(q_a) + \theta_a q_a\right] da\right\}$$

subject to the constraint

$$(130)\qquad \frac{\alpha(n)}{n}[\tilde{s}(n) - \tilde{v}(n)] \leq \kappa$$

and $n \geq 0$ with complementary slackness.

Using the envelope theorem, the first-order conditions for z and n respectively are

$$(131)\qquad \int_{a_0}^{\bar{a}} \mu_a da = i$$

and

$$(132)\qquad \alpha'(n)\tilde{v}(n) + \alpha(n)\tilde{v}'(n) = 0.$$

Using the fact that $\mu_a = 0$ for all $a < a_c$, by definition of a_c, we have $\int_{a_0}^{\bar{a}} \mu_a da = \Sigma_{a_c}$. The first-order condition for z given by (131) thus becomes:

$$(133)\qquad \Sigma_{a_c} = i,$$

Substituting $\Sigma_{a_c} = i$ into expression (111) in Lemma 11, the above yields

$$\delta = \frac{1}{1 - a_0 \tilde{g}(a_0; n)} \left(1 + \frac{i + \int_{a_0}^{\bar{a}} \eta_x dx}{\alpha(n)} \right) \tag{134}$$

Proof of Parts 1 to 8

Part 1. Follows from the definition of a_b.

Part 2. From above, for any $a \in [a_b, a_c]$, we have

$$(a - \phi(a; n))u'(q_a) = c'(q_a) \tag{135}$$

where, using $\Sigma_{a_c} = i$ plus expression (123) for $\phi(a; n)$, we have

$$\phi(a; n) = -\left(\frac{1 - \delta}{\delta} \right) \left(\frac{1 - \tilde{G}(a; n)}{\tilde{g}(a; n)} \right) - \frac{i}{\alpha(n)\delta\tilde{g}(a; n)}. \tag{136}$$

The expression for δ can be derived as follows.

We have $a_b - \phi(a_b; n) = 0$ from Lemma 12. Using this, plus expression (136),

$$a_b + \left(\frac{\delta - 1}{\delta} \right) \left(\frac{1 - \tilde{G}(a_b; n)}{\tilde{g}(a_b; n)} \right) + \frac{i}{\alpha(n)\delta\tilde{g}(a_b; n)} = 0. \tag{137}$$

This implies that

$$i = -\alpha(n)[\delta a_b \tilde{g}(a_b; n) + (1 - \delta)(1 - \tilde{G}(a_b; n))] \tag{138}$$

and thus the value of δ is given by the following expression:

$$\delta = \frac{1}{1 - \frac{a_b \tilde{g}(a_b; n)}{1 - \tilde{G}(a_b; n)}} \left(1 + \frac{i}{\alpha(n)(1 - \tilde{G}(a_b; n))} \right). \tag{139}$$

If $a_b = a_0$ then $\int_{a_0}^{\bar{a}} \eta_x dx = 0$ because $v_a > 0$ for all $a \in (a_0, \bar{a}]$ and thus (134) yields

$$\delta = \frac{1}{1 - a_0 \tilde{g}(a_0; n)} \left(1 + \frac{i}{\alpha(n)} \right). \tag{140}$$

Given that $a_b = a_0$ and therefore $1 - \tilde{G}(a_b; n) = 1$, we can write (140) as (139), which is equivalent to (35) in terms of $\rho(a; n) \equiv 1 - \tilde{G}(a; n)$.

Also, $\dot{v}_a = u(q_a)$ implies $v_a - v_0 = \int_{a_0}^{a} u(q_x)dx$, so $v_a = \int_{a_0}^{a} u(q_x)dx$ since $v_0 = 0$. We can derive d_a/γ from v_a using the fact that $v_a \equiv au(q_a) - d_a/\gamma$.

Part 3. Clear from Lemma 6.

Part 4. Using $\Sigma_{a_c} = i$ and expression (99), and then using integration by parts twice, we obtain (37).

Part 5. Clear from the definition of a_c.

Part 6. The first-order condition for $n > 0$ given by (132) can be written as

$$\alpha'(n)\tilde{s}(n) + \alpha(n)\tilde{s}'(n) = \kappa, \tag{141}$$

using the constraint (130) at equality. More precisely, this is equivalent to

$$\alpha'(n)\tilde{s}(n; \{q_a\}_{a\in A}) + \alpha(n)\tilde{s}'(n; \{q_a\}_{a\in A}) = \kappa. \tag{142}$$

The fact that n is strictly decreasing in κ is proven in Lemma 16 below.

Part 7. The zero profit condition is given by (130), using the definition of v_a.

Part 8. Since v_a is increasing in a, the highest draw is always chosen by buyers. Therefore the cdf of chosen goods is given by (6). ■

Proof that q_a is non-decreasing

Finally, we verify that q_a is non-decreasing, as required for Lemma 3.

Lemma 13. *The function $q(.)$ is non-decreasing for all $a \in A$. In addition, $q'(a) > 0$ for all $a \in (a_b, a_c)$.*

Proof. For all $a < a_b$, $q_a = 0$ and $q'(a) = 0$. For all a greater than or equal to a_c, q_a is constant and thus $q'(a) = 0$. For $a \in [a_b, a_c)$, implicit differentiation of

$$(a - \phi(a; n))u'(q_a) = c'(q_a) \tag{143}$$

yields

$$q'(a) = \frac{-[1 - \phi'(a)]u'(q_a)}{[a - \phi(a; n)]u''(q_a) - c''(q_a)} \tag{144}$$

where $\phi(a;n)$ can be simplified to:

$$\phi(a;n) = -\left(\frac{(1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}}{\delta \tilde{g}(a;n)}\right). \tag{145}$$

Differentiating the above with respect to a yields

$$\phi'(a) = \frac{1-\delta}{\delta} + \frac{\left[(1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}\right]\tilde{g}'(a;n)}{\delta \tilde{g}(a;n)^2}. \tag{146}$$

Since $u'(q_a) > 0$ and $u''(q_a) < 0$ and $c''(q_a) > 0$ and $a - \phi(a;n) > 0$, to establish $q'(a) \geq 0$ it suffices to show that $\phi'(a) < 1$. That is, we require

$$\phi'(a) = \frac{1-\delta}{\delta} - \frac{\phi(a;n)\tilde{g}'(a;n)}{\tilde{g}(a;n)} < 1. \tag{147}$$

Since $\delta \geq 1$, we have $\phi'(a) < 0$ if either $\tilde{g}'(a;n) < 0$ and $\phi(a;n) < 0$, or otherwise $\tilde{g}'(a;n) > 0$. and $\phi(a;n) > 0$. So, we need only consider two cases: (i) $\tilde{g}'(a;n) < 0$ and $\phi(a;n) > 0$ and (ii) $\tilde{g}'(a;n) > 0$ and $\phi(a;n) < 0$.

For case (i), assume that $\tilde{g}'(a;n) < 0$ and $\phi(a;n) > 0$. Write (145) as

$$\phi(a;n) = \left(1 - \frac{1}{\delta}\right)\left(\frac{1-\tilde{G}(a;n)}{\tilde{g}(a;n)}\right) - \frac{i}{\alpha(n)\delta\tilde{g}(a;n)}. \tag{148}$$

The distribution $\tilde{G}(a;n)$ has an increasing hazard rate, i.e. $h'_{\tilde{G}}(a;n) \equiv \frac{\partial h_{\tilde{G}}(a;n)}{\partial a} \geq 0$ by Lemma 4, and $\delta \geq 1$ so the first term is weakly decreasing in a. Also, since $\tilde{g}'(a;n) < 0$, the second term (including the minus sign) is strictly decreasing in a. So, we have $\phi'(a) < 0$ and thus $\phi'(a) < 1$.

For case (ii), assume that $\tilde{g}'(a;n) > 0$ and $\phi(a;n) < 0$. Rearranging (146), we have $\phi'(a) < 1$ if and only if

$$\left(\frac{(1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}}{\tilde{G}(a;n)}\right)\left(\frac{\tilde{g}'(a;n)\tilde{G}(a;n)}{\tilde{g}(a;n)^2}\right) < 2\delta - 1. \tag{149}$$

First, using (6) and (61), and differentiating (61),

$$\frac{\tilde{g}'(a;n)\tilde{G}(a;n)}{\tilde{g}(a;n)^2} = \left(1 - \frac{\mathbb{P}_0(n)}{\mathbb{P}_0(x)}\right)\left(\frac{\mathbb{P}_0''(x)\mathbb{P}_0(x)}{\mathbb{P}_0'(x)^2} - \frac{\mathbb{P}_0(x)}{\mathbb{P}_0'(x)}\frac{G''(a)}{ng(a)^2}\right),$$

where $x = n(1 - G(a))$ and $\frac{\mathbb{P}_0(n)}{\mathbb{P}_0(x)} \leq 1$, and therefore

$$\frac{\tilde{g}'(a;n)\tilde{G}(a;n)}{\tilde{g}(a;n)^2} \leq \frac{\mathbb{P}_0''(x)\mathbb{P}_0(x)}{\mathbb{P}_0'(x)^2} - \frac{\mathbb{P}_0(x)}{\mathbb{P}_0'(x)} \frac{G''(a)}{ng(a)^2}. \tag{150}$$

Notice that the above is equivalent to

$$\frac{\tilde{g}'(a;n)\tilde{G}(a;n)}{\tilde{g}(a;n)^2} \leq \frac{\mathbb{P}_0''(x)\mathbb{P}_0(x)}{\mathbb{P}_0'(x)^2} - \frac{\mathbb{P}_0(x)}{\mathbb{P}_0'(x)x} \frac{G''(a)(1-G(a))}{g(a)^2}. \tag{151}$$

Given that $\mathbb{P}_0'(x) \leq 0$ by Lemma 9, and we assume $G''(a) \leq 0$, the entire second term on the right side of the inequality (including the minus sign) is negative, so

$$\frac{\tilde{g}'(a;n)\tilde{G}(a;n)}{\tilde{g}(a;n)^2} \leq \frac{\mathbb{P}_0''(x)\mathbb{P}_0(x)}{\mathbb{P}_0'(x)^2}. \tag{152}$$

Now, if Assumption 5 holds then $\frac{-\alpha''(x)(1-\alpha(x))}{\alpha'(x)^2} \leq 2$. Stating this assumption in terms of $\mathbb{P}_0(x)$ using $\alpha(x) = 1 - \mathbb{P}_0(x)$, Assumption 5 is equivalent to

$$\frac{\mathbb{P}_0''(x)\mathbb{P}_0(x)}{\mathbb{P}_0'(x)^2} \leq 2. \tag{153}$$

We therefore obtain

$$\frac{\tilde{g}'(a;n)\tilde{G}(a;n)}{\tilde{g}(a;n)^2} \leq 2. \tag{154}$$

Given inequality (154), to prove (149) it suffices to show that

$$\frac{(1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}}{\tilde{G}(a;n)} < \delta - \frac{1}{2}. \tag{155}$$

Rearranging the above and simplifying, this is equivalent to

$$\delta + \frac{1}{2}\tilde{G}(a;n) > 1 + \frac{i}{\alpha(n)}. \tag{156}$$

For any $a \in (a_b, a_c)$, this is true if $\delta \geq 1 + \frac{i}{\alpha(n)}$, which is true since

$$\delta = \frac{1}{1 - \frac{a_b\tilde{g}(a_b;n)}{1-\tilde{G}(a_b;n)}} \left(1 + \frac{i}{\alpha(n)}\right) \geq 1 + \frac{i}{\alpha(n)(1-\tilde{G}(a_b;n))}. \tag{157}$$

Therefore, for both cases we have $q'(a) > 0$ for all $a \in (a_b, a_c)$. ■

Proof of existence and uniqueness

We first prove existence and uniqueness of the solution to the inner maximization problem and then prove the same for the outer maximization problem.

Inner maximization. We prove that, given z and n from the outer maximization problem, the solution to the inner maximization problem exists and is unique.

Existence. A solution to the problem exists because the set of admissible paths is non-empty and compact, and there exists an admissible path for which the objective is finite. For example, the path $q_a = 0$ and $v_a = (a-1)u(q_a)$ for all $a \in A$ is admissible (since $v_0 = 0$, $au(q_a) - v_a \leq z/\gamma$, $q_a \geq 0$, $v_a \geq 0$, and $\dot{v}_a = u(q_a) + (a-1)u'(q_a)q'(a) = u(q_a)$, and $q'(a) \geq 0$). Also, the objective is finite under this path. Finally, the set of feasible paths is compact since $q_a \in [0, q^*_{\bar{a}}]$ where $q^*_{\bar{a}}$ solves $\bar{a}u'(q_{\bar{a}}) = c'(q_{\bar{a}})$ and $v_a \in [0, v_{\bar{a}}]$ where $v_{\bar{a}} = u(q^*_{\bar{a}})[\bar{a} - a_0]$ since $v_a = \int_{a_0}^{a} u(q_x)dx$.

Uniqueness. The Hamiltonian $H(q_a, v_a, \lambda_a)$, where λ_a is the co-state variable given by the Maximum Principle, is strictly concave in the control and state variables (q_a, v_a) for all a. Therefore, the solution is an optimum that solves the inner maximization problem and it is unique. To establish strict concavity, differentiating $H(q_a, v_a, \lambda_a)$ with respect to q_a yields

$$\begin{aligned}
\frac{\partial H}{\partial q_a} &= \alpha(n)\delta[u'(q_a) - c'(q_a)]\tilde{g}(a; n) + \lambda_a u'(q_a), \\
\frac{\partial^2 H}{\partial q_a^2} &= \alpha(n)\delta[u''(q_a) - c''(q_a)]\tilde{g}(a; n) + \lambda_a u''(q_a) \equiv -X,
\end{aligned}$$

where $X > 0$, since $u''(q_a) < 0$ and $c''(q_a) > 0$. Differentiating $H(q_a, v_a, \lambda_a)$ with respect to v_a, we obtain $\frac{\partial H}{\partial v_a} = \alpha(n)(1-\delta)\tilde{g}(a; n)$ and $\frac{\partial^2 H}{\partial v_a^2} = 0$. Finally, $\frac{\partial^2 H}{\partial v_a \partial q_a} = 0$, so we get the Hessian matrix, $\mathbb{H} = \begin{bmatrix} -X & 0 \\ 0 & 0 \end{bmatrix}$. Since $\mathbf{x}^T\mathbb{H}\mathbf{x} < 0$ for all $\mathbf{x} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$, the Hessian $\mathbb{H}$ is negative definite and the Hamiltonian is strictly concave in (q_a, v_a).

Outer maximization. We prove that, given $\{(q_a, v_a)\}_{a \in A}$ from the inner maximization problem, the solution (n, z) to the outer maximization problem exists and is unique, and n, z are interior solutions with $n, z > 0$ if Assumption 6 holds. To establish this result, we first prove that there exists a non-empty set of solutions n, denoted by $N(\kappa)$, that solves the problem. We then show that equilibrium is unique if $n > 0$ for all $n \in N(\kappa)$, and finally we prove that $n > 0$ for any $n \in N(\kappa)$.

Taking $\{(q_a, v_a)\}_{a\in A}$ as given by the inner maximization problem, and ignoring constants, the outer maximization problem is equivalent to

$$\max_{z,n} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a;n) + (\Sigma_{a_c} - i) \frac{z}{\gamma} \right\}, \tag{158}$$

subject to

$$\frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[-c(q_a) + \frac{d_a}{\gamma} \right] d\tilde{G}(a;n) \leq \kappa \tag{159}$$

and $n \geq 0$ with complementary slackness, where $\{(q_a, v_a)\}_{a\in A}$ solves the inner maximization problem.

Lemma 14. *The set of solutions $N(\kappa)$ is nonempty and upper hemicontinuous.*

Proof. Since $\alpha(n)$ is a bijection, we can rewrite (158) in terms of α as follows:

$$\max_{z,\alpha} \left\{ \alpha \int_{a_0}^{\bar{a}} \left[au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a;\alpha) + (\Sigma_{a_c} - i) \frac{z}{\gamma} \right\}. \tag{160}$$

The objective function is continuous and, without loss of generality, we can restrict (z, α) to the following compact set:

$$\Delta = \{(z, \alpha) : \alpha \in [0, 1],\ z/\gamma \in [0, \bar{a}u(q_{\bar{a}})]\} \tag{161}$$

since $q \in [0, q^*_{\bar{a}}]$ where $q^*_{\bar{a}}$ solves $\bar{a}u'(q_{\bar{a}}) = c'(q_{\bar{a}})$, and we have $z/\gamma < \bar{a}u(q_{\bar{a}})$. The constraint (159) can therefore be written as $(z, \alpha) \in \Gamma(\kappa)$ for all $\kappa \geq 0$, where $\Gamma(\kappa)$ is a continuous and compact-valued correspondence. Applying the Theorem of the Maximum (Theorem 3.6 in Stokey, Lucas, and Prescott, 1989), the correspondence that gives the set of solutions for α is nonempty and upper hemicontinuous, and therefore also $N(\kappa)$ is nonempty and upper hemicontinuous. ■

The following lemma establishes that any strictly positive solution $n \in N(\kappa)$ must be unique. Since we know that $z = d_{a_c} > 0$ where $d_a/\gamma = au(q_a) - v_a$, and $\{(q_a, v_a)\}_{a\in A}$ is given by the inner maximization problem, Lemma 15 implies that any solution (n, z) where $n > 0$ is unique.

Lemma 15. *If $N^+ \subseteq N(\kappa)$ and $N^+ \subseteq \mathbb{R}_+\backslash\{0\}$, then $N^+ = \{n\}$.*

Proof. Consider any solution $n \in N(\kappa)$ such that $n > 0$. Defining $\Phi(n) \equiv \alpha(n)\tilde{v}(n)$, the solutions n satisfy the first-order condition (132), which says $\Phi'(n) = 0$. We show that $\Phi''(n) < 0$ and thus any solution is unique. Using (61), we have

$$\Phi(n) = -\int_{a_0}^{\bar{a}} \mathbb{P}_0'(n(1-G(a)))v_a g(a)da. \tag{162}$$

Using Leibniz's integral rule, plus the envelope theorem,

$$\Phi'(n) = \int_{a_0}^{\bar{a}} -\mathbb{P}_0'(n(1-G(a)))v_a g(a)da - \int_{a_0}^{\bar{a}} n(1-G(a))\mathbb{P}_0''(n(1-G(a)))v_a g(a)da.$$

By integration by parts on the second integral in $\Phi'(n)$ above, we obtain

$$\Phi'(n) = \int_{a_0}^{\bar{a}} -\mathbb{P}_0'(n(1-G(a)))(1-G(a))v'(a)da - \mathbb{P}_0'(n)v(a_0) > 0. \tag{163}$$

Differentiating (163), we find that

$$\Phi''(n) = -\left(\int_{a_0}^{\bar{a}} \mathbb{P}_0''(n(1-G(a))(1-G(a))^2 v'(a)da + \mathbb{P}_0''(n)v(a_0)\right) < 0. \tag{164}$$

The fact that $\Phi''(n) < 0$ follows from the fact that $\mathbb{P}_0''(x) > 0$ by Lemma 9, plus the fact that $v'(a) = u(q_a) \geq 0$ for all a and $v'(a) > 0$ for some a and also $v(a_0) = 0$. Therefore, any solution $n > 0$ is unique. ■

From Lemma 14, we know that, for any given $\kappa \geq 0$, there exists a non-empty set of solutions $N(\kappa)$ that solves problem (158). We also know that any solution z is interior, since $z/\gamma = \bar{a}u(q_{\bar{a}})$ implies $v_{\bar{a}} = \bar{a}u(q_{\bar{a}}) - \bar{z}/\gamma = 0$ and therefore $v_a = 0$ for all $a \in A$. We now prove that, for any $n \in N(\kappa)$, we have $n \in \mathbb{R}_+ \backslash \{0\}$ provided that Assumption 6 holds. Also, the function $n(\kappa)$ is strictly decreasing in κ.

Lemma 16. *Any solution $n \in N(\kappa)$ is interior, i.e. $n \in \mathbb{R}_+ \backslash \{0\}$. The function $n(\kappa)$ is strictly decreasing in κ.*

Proof. First, we show there exists an interior solution $n > 0$. Define $\Lambda(n) \equiv \alpha(n)\tilde{s}(n)$. The first-order condition (141) says $\Lambda'(n) = \kappa$. We prove there exists $n > 0$ such that $\Lambda'(n) = \kappa$ if Assumption 6 holds. We have $\lim_{n\to\infty} \Lambda'(n) = 0$, and

$$\lim_{n\to 0} \Lambda'(n) = \int_{a_0}^{\bar{a}} \lim_{n\to 0} s(a; q_a(n))dG(a) \tag{165}$$

where $\lim_{n\to 0} s(a; q_a(n)) = s(a; \lim_{n\to 0} q_a(n))$. If the following condition holds:

$$E_G[au(q_a^0) - c(q_a^0)] > \kappa \tag{166}$$

where $q_a^0 \equiv \lim_{n\to 0} q_a(n)$, there exists $n > 0$ that satisfies $\Lambda'(n) = \kappa$ provided that $\Lambda''(n) < 0$ (which we prove below).

Next, any interior solution $n > 0$ is better than $n = 0$. Define the value function:

$$V(\kappa, \gamma) \equiv \max_{z,n} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a; n) + (\Sigma_{a_c} - i) \frac{z}{\gamma} \right\}. \tag{167}$$

Since we know that z is interior, we have $V(\kappa, \gamma) \equiv \max_n \{\alpha(n)\tilde{v}(n)\}$ since $\int_{a_0}^{\bar{a}} \mu_a = i$. If $n = 0$ then $V(\kappa, \gamma) = 0$. If $n > 0$, $V(\kappa, \gamma) \equiv \max_n \{\alpha(n)\tilde{s}(n) - n\kappa\}$ using constraint (159) with equality. Letting $\Lambda(n) = \alpha(n)\tilde{s}(n)$, we have $V(\kappa, \gamma) > 0$ if $\Lambda(n) - n\kappa > 0$. Thus the candidate solution $n > 0$ is better than $n = 0$ if $\Lambda(n) > n\kappa$ for $n > 0$. Using the fact that $\Lambda'(n) = \kappa$, it suffices to show that $\Lambda''(n) < 0$ and $\frac{\Lambda'(n)n}{\Lambda(n)} < 1$ for $n > 0$. Similarly to Lemma 15, using (61), we have

$$\Lambda(n) = -\int_{a_0}^{\bar{a}} n\mathbb{P}_0'(n(1 - G(a)))s_a g(a)da \tag{168}$$

and, using Leibniz's integral rule, plus the envelope theorem, yields

$$\Lambda'(n) = \int_{a_0}^{\bar{a}} -\mathbb{P}_0'(n(1 - G(a)))s_a g(a)da - \int_{a_0}^{\bar{a}} n(1 - G(a))\mathbb{P}_0''(n(1 - G(a)))s_a g(a)da.$$

Therefore, letting $x = n(1 - G(a))$, we have

$$\frac{\Lambda'(n)n}{\Lambda(n)} = 1 + \frac{\int_{a_0}^{\bar{a}} x\mathbb{P}_0''(x)s_a g(a)da}{\int_{a_0}^{\bar{a}} \mathbb{P}_0'(x)s_a g(a)da}. \tag{169}$$

Because $\mathbb{P}_0''(x) > 0$ and $\mathbb{P}_0'(x) < 0$ by Lemma 9, we have $\frac{\Lambda'(n)n}{\Lambda(n)} < 1$ for $n > 0$.

Finally, $\Phi(n) = \Lambda(n) - n\kappa$ for $n > 0$, so $\Phi'(n) = \Lambda'(n) - \kappa$ and $\Phi''(n) = \Lambda''(n)$. Since $\Phi''(n) < 0$ from the proof of Lemma 15, we have $\Lambda''(n) < 0$. It follows that, for any $n \in N(\kappa)$, we have $n > 0$. Since we assume $\kappa > 0$, this implies $n \in \mathbb{R}_+\backslash\{0\}$. Since n is unique by Lemma 15, there is a function $n : \mathbb{R}_+\backslash\{0\} \to \mathbb{R}_+\backslash\{0\}$ such that $n(\kappa)$ solves $\Lambda'(n) = \kappa$. Clearly, n is strictly decreasing in κ since $\Lambda''(n) < 0$. ■

Proof of Proposition 4

Regarding entry, we know from Proposition 3 that the equilibrium n satisfies

$$\alpha'(n)\tilde{s}(n; \{q_a\}_{a\in A}) + \alpha(n)\tilde{s}'(n; \{q_a\}_{a\in A}) = \kappa \tag{170}$$

and the first-best n^* satisfies

$$\alpha'(n^*)\tilde{s}(n^*; \{q_a^*\}_{a\in A}) + \alpha(n^*)\tilde{s}'(n^*; \{q_a^*\}_{a\in A}) = \kappa. \tag{171}$$

There are various possibilities for ranges of over-consumption and underconsumption, so we know that $q_a \neq q_a^*$ in general, but we cannot infer anything about whether there is under-entry ($n < n^*$), over-entry ($n > n^*$), or first-best entry ($n = n^*$) overall. We can find examples of equilibria for each of these three possibilities.

Part 1. Suppose that $a_d = \max\{a_c, a_d\}$. Follows from combining Parts 1 and 2 of Lemma 17 below if $a_p \leq a_c \leq a_d$. Suppose that $a_c = \max\{a_c, a_d\}$. Follows from combining Parts 1 and 2 of Lemma 17 if $a_p \leq a_c$ and $a_d < a_c$.

Part 2. If $a_p > a_c$, then $\phi(a; n) > 0$ for all $a \in (a_0, a_p)$ from Part 1 *(ii)* in Lemma 17. In particular, $\phi(a_c) > 0$, so we get $a_c > a_d$. The rest follows from combining Parts 1 and 2 in Lemma 17. If $a_p = a_c$, the result follows from Part 1.

Part 3. If $a_b = a_0$ then $\delta = 1 + \frac{i}{\alpha(n)}$ and (175) implies $\tilde{G}(a_p; n) = 0$ and thus $a_p = a_b = a_0$. Since $a_p \leq a_c$, Part 1 implies there is overconsumption on (a_0, a_u) and underconsumption on $(a_u, \bar{a}]$ where $a_u = \max\{a_c, a_d\}$. Since $\phi(a_c) < 0$, we have overconsumption at a_c. Therefore, $a_c < a_u$ and $a_u = a_d$. ■

Lemma 17. *In any equilibrium where $i > 0$,*

1. *There exists a unique cutoff $a_p \in [a_b, \bar{a}]$ such that (i) if $a_p \leq a_c$, there is underconsumption for all $a \in (a_0, a_p)$ and overconsumption for all $a \in (a_p, a_c)$, and (ii) if $a_p > a_c$, there is underconsumption for all $a \in (a_0, a_p)$.*

2. *There exists a unique cutoff $a_d \in [a_b, \bar{a}]$ such that (i) if $a_c \leq a_d$, there is overconsumption for all $a \in [a_c, a_d)$ and underconsumption for all $a \in (a_d, \bar{a}]$, and (ii) if $a_c > a_d$, there is underconsumption for all $a \in [a_c, \bar{a}]$.*

Proof. *Part 1. (i)* For $a \in (a_0, a_b]$, there is underconsumption, i.e. $q_a < q_a^*$, since $q_a = 0$ but $q_a^* > 0$. For $a \in (a_b, a_c]$, we have $a - \phi(a; n) = c'(q_a)/u'(q_a)$ and $a =$

$c'(q_a^*)/u'(q_a^*)$, where $c'(q)/u'(q)$ is increasing in q, so $q_a < q_a^*$ (i.e. underconsumption) for $a \in (a_b, a_c]$ if and only if $\phi(a;n) > 0$. Rearranging (34) yields

$$\phi(a;n) = -\left(\frac{(1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}}{\delta\tilde{g}(a;n)}\right), \tag{172}$$

and therefore $\phi(a;n) > 0$ if and only if

$$-\left((1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}\right) > 0. \tag{173}$$

Rearranging, $\phi(a;n) > 0$ if and only if

$$\tilde{G}(a;n) < 1 + \frac{i}{\alpha(n)(1-\delta)}. \tag{174}$$

Since $\tilde{G}'(a;n) = \tilde{g}(a;n) \geq 0$, and $\tilde{G}(a_0;n) = 0$ and $\tilde{G}(\bar{a};n) = 1$, while $1 + \frac{i}{\alpha(n)(1-\delta)} \in [0,1]$, there exists a unique cut-off $a_p \in (a_b, \bar{a}]$ such that $\phi(a;n) > 0$ and there is underconsumption for all $a \in (a_0, a_p)$ where a_p satisfies

$$\delta = 1 + \frac{i}{\alpha(n)[1-\tilde{G}(a_p;n)]} \tag{175}$$

provided that $a_p \leq a_c$. If $a \in (a_p, a_c)$ then $\phi(a;n) < 0$ and there is overconsumption. *(ii)* If $a_p > a_c$, the range of overconsumption (a_p, a_c) is empty and we have underconsumption for all $a \in (a_0, a_p)$.

Part 2. *(i)* For $a \in [a_c, \bar{a}]$, $q_a = q_{a_c}$ where $a_c - \phi(a_c) = c'(q_{a_c})/u'(q_{a_c})$ and $a = c'(q_a^*)/u'(q_a^*)$. Since $c'(q)/u'(q)$ is increasing in q, we have $q_a > q_a^*$ (i.e. overconsumption) if and only if $a < a_c - \phi(a_c)$. Defining $a_d \equiv a_c - \phi(a_c)$, we have overconsumption for $a \in [a_c, a_d)$ and underconsumption for $a \in (a_d, \bar{a}]$. *(ii)* If $a_d < a_c$, the interval $[a_c, a_d)$ is empty and we have underconsumption for all $a \in [a_c, \bar{a}]$. ■

Proof of Proposition 5

The planner's problem is to solve the following:

$$\max_{n\in\mathbb{R}_+,\{q_a,d_a\}_{a\in A}} \left\{\alpha(n)\int_{a_0}^{\bar{a}} [au(q_a) - c(q_a)]\, d\tilde{G}(a;n) - n\kappa\right\}, \tag{176}$$

subject to the IC, IR, and non-negativity constraints:

$$au(q_a) - \frac{d_a}{\gamma} \geq au(q_{a'}) - \frac{d_{a'}}{\gamma}, \tag{177}$$

$$au(q_a) - \frac{d_a}{\gamma} \geq 0, \tag{178}$$

$$d_a, q_a \geq 0. \tag{179}$$

We can compare this to the sellers' problem for $i \to 0$ under competitive search:

$$\max_{n \in \mathbb{R}_+, \{q_a, d_a\}_{a \in A}} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a; n) \right\}, \tag{180}$$

subject to

$$\alpha(n) \int_{a_0}^{\bar{a}} \left[-c(q_a) + \frac{d_a}{\gamma} \right] d\tilde{G}(a; n) = n\kappa \tag{181}$$

and the same IC and IR constraints. The problem (180) is equivalent to

$$\max_{n \in \mathbb{R}_+, \{q_a, d_a\}_{a \in A}} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[au(q_a) - c(q_a) + c(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a; n) \right\}. \tag{182}$$

Substituting (181) into the above, this is equivalent to the planner's problem (176). Therefore, the planner's solution is the same as competitive search equilibrium at the Friedman rule, which is given by Corollary 2.

Proof of Corollary 2

Part 1. Follows from Part 1 of Proposition 3.

Part 2. Setting $i = 0$ in expression (35) for δ in Proposition 3, we obtain

$$\delta = \frac{1}{1 - \varepsilon_\rho(a_b; n)}. \tag{183}$$

Setting $i = 0$ in expression (34) for $\phi(a; n)$ in Proposition 3, and substituting (183) into (34), we obtain (43).

Part 3. Starting with equation (37) in Proposition 3, setting $i = 0$ implies $a_c = \bar{a}$.

Part 4. Parts 5-8 from Proposition 3 also hold. ■

Proof of Corollary 3

In general, we have $\varepsilon_\rho(a_b;n) > 0$ and there is underconsumption for all $a \in (a_0, \bar{a})$ relative to the first-best. At the Friedman rule, the equilibrium n satisfies

$$\alpha'(n)\tilde{s}(n;\{q_a\}_{a\in A}) + \alpha(n)\tilde{s}'(n;\{q_a\}_{a\in A}) = \kappa \tag{184}$$

and the first-best n^* satisfies

$$\alpha'(n^*)\tilde{s}(n^*;\{q_a^*\}_{a\in A}) + \alpha(n^*)\tilde{s}'(n^*;\{q_a^*\}_{a\in A}) = \kappa. \tag{185}$$

We know from above that $q_a^* > q_a$ for any $a \in (a_0, \bar{a})$, but we cannot infer anything about whether there is under-entry ($n < n^*$), over-entry ($n > n^*$), or first-best entry ($n = n^*$). We can find examples of equilibria for each of these three possibilities. ■

Proofs for Section 9

Proof of Lemma 7

Before we prove Lemma 7, we first prove the following lemma.

Lemma 18. *The trading cut-off $a_b(n)$ is strictly decreasing in n, i.e. $a_b'(n) < 0$.*

Consider the zero-profit condition (39), which can be written as

$$\int_{a_0}^{\bar{a}} \left(-c(q_a) + \frac{d_a}{\gamma}\right) \frac{\alpha(n)\tilde{g}(a;n)}{n} da = \kappa. \tag{186}$$

Using the fact that $q_a = d_a = 0$ for all $a \leq a_b(n)$, we can write

$$\int_{a_b(n)}^{\bar{a}} \pi_a \frac{\alpha(n)\tilde{g}(a;n)}{n} da = \kappa \tag{187}$$

where $\pi_a \equiv -c(q_a) + \frac{d_a}{\gamma}$. Using Leibniz' integral rule to differentiate with regard to n, we obtain

$$\int_{a_b(n)}^{\bar{a}} \frac{d}{dn}\left(\pi_a \frac{\alpha(n)\tilde{g}(a;n)}{n}\right) da - \lim_{a\to a_b} \pi_a \frac{\alpha(n)\tilde{g}(a(n);n)}{n} a_b'(n) = 0. \tag{188}$$

Rearranging, and using the envelope theorem, $a_b'(n)$ has the same sign as $\frac{d}{dn}\left(\frac{\alpha(n)\tilde{g}(a;n)}{n}\right)$.

Using expression (74) for $\tilde{g}(a;n)$, we have

$$\frac{d}{dn}\left(\frac{\alpha(n)\tilde{g}(a;n)}{n}\right) = \frac{d}{dn}\left[g(a)\alpha'(x(a;n))\right] \tag{189}$$

where $x(a;n) = n(1-G(a))$ and

$$\frac{d}{dn}\left[g(a)\alpha'(x(a;n))\right] = g(a)\alpha''(x(a;n))(1-G(a)). \tag{190}$$

Therefore, using the fact that $\alpha'' < 0$ by Lemma 1, we have $\frac{d}{dn}\left(\frac{\alpha(n)\tilde{g}(a;n)}{n}\right) < 0$ for $a < \bar{a}$ and thus $a_b'(n) < 0$. ■

We can now prove Lemma 7. Differentiating $\varepsilon_\rho(a_b(n);n)$ with respect to n,

$$\frac{d}{dn}\varepsilon_\rho(a_b(n);n) = a_b'(n)\frac{\partial}{\partial a}\varepsilon_\rho(a_b(n);n) + \frac{\partial}{\partial n}\varepsilon_\rho(a_b(n);n). \tag{191}$$

By definition of $\varepsilon_\rho(a;n)$, we have

$$\varepsilon_\rho(a;n) = \frac{a\tilde{g}(a;n)}{1-\tilde{G}(a;n)} = \frac{a}{I_{\tilde{G}}(a;n)}. \tag{192}$$

Using expression (45) gives us

$$\varepsilon_\rho(a;n) = \eta_\alpha(x(a;n))h_G(a) \tag{193}$$

where $x(a;n) = n(1-G(a))$. Thus, we have $\frac{\partial}{\partial n}\varepsilon_\rho(a;n) < 0$ because $\eta_\alpha' < 0$ by Lemma 1 and $\frac{\partial}{\partial n}x(a;n) > 0$. We also know that $a_b'(n) < 0$ from Lemma 18, so it suffices to show that $\frac{\partial}{\partial a}\varepsilon_\rho(a;n) > 0$. Differentiating $\varepsilon_\rho(a;n)$ with respect to a, we have

$$\frac{\partial}{\partial a}\varepsilon_\rho(a;n) = -\eta_\alpha'(x(a;n))ng(a)h_G(a) + \eta_\alpha(x)h_G'(a) > 0 \tag{194}$$

because $\eta_\alpha' < 0$ by Lemma 1 and $h_G'(a) > 0$ by Assumption 4.

Therefore, we have proven that $\frac{d}{dn}\varepsilon_\rho(a_b(n);n) > 0$, as required. ■

Proof of Lemma 8

We must show that $D(a;n)$ is decreasing in n where

$$D(a;n) = \varepsilon_\rho(a_b(n);n)\frac{I_{\tilde{G}}(a;n)}{a}. \tag{195}$$

Using expression (192) gives us $\frac{I_{\tilde{G}}(a;n)}{a} = \frac{1}{\varepsilon_\rho(a;n)}$, so we obtain

$$D(a;n) = \frac{\varepsilon_\rho(a_b(n);n)}{\varepsilon_\rho(a;n)}. \tag{196}$$

Next, expression (193) delivers

$$D(a;n) = \frac{h_G(a_b)}{h_G(a)}\frac{\eta_\alpha(x(a_b(n);n))}{\eta_\alpha(x(a;n))}. \tag{197}$$

where $x(a;n) = n(1-G(a))$. Differentiating with respect to n, we have

$$\frac{\partial}{\partial n}D(a;n) = \frac{h_G(a_b)}{h_G(a)}\frac{d}{dn}\frac{\eta_\alpha(x(a_b(n);n))}{\eta_\alpha(x(a;n))} \tag{198}$$

and

$$\frac{d}{dn}\frac{\eta_\alpha(x(a_b(n);n))}{\eta_\alpha(x(a;n))} = \frac{\eta'_\alpha(x_b)\frac{d}{dn}x(a_b(n);n)}{\eta_\alpha(x)} - \frac{\eta_\alpha(x_b)\eta'_\alpha(x)\frac{d}{dn}x(a;n)}{\eta_\alpha(x)^2} \tag{199}$$

where x denotes $x(a;n)$ and x_b denotes $x(a_b(n);n)$. Therefore, we have $\frac{\partial}{\partial n}D(a;n) < 0$ if and only if

$$\eta'_\alpha(x_b)\eta_\alpha(x)\frac{d}{dn}x(a_b(n);n) - \eta_\alpha(x_b)\eta'_\alpha(x)\frac{d}{dn}x(a;n) < 0. \tag{200}$$

Now, $\frac{d}{dn}x(a;n) = -ng(a)$, so we require the following:

$$\eta'_\alpha(x_b)\eta_\alpha(x)\frac{d}{dn}x(a_b(n);n) + \eta_\alpha(x_b)\eta'_\alpha(x)ng(a) < 0. \tag{201}$$

Given that $\eta'_\alpha < 0$ by Lemma 1, it suffices to show that $\frac{d}{dn}x(a_b(n);n) \geq 0$. Given that $x(a_b(n);n) = n(1-G(a_b(n)))$, differentiating yields

$$\frac{d}{dn}x(a_b(n);n) = -ng(a_b)a'_b(n) + 1 - G(a_b). \tag{202}$$

We have $a_b'(n) < 0$ by Lemma 18 and thus $\frac{d}{dn}x(a_b(n); n) > 0$, so $\frac{\partial}{\partial n}D(a; n) < 0$. ■

Proof of Proposition 6

Differentiating $\tilde{D}(n)$ using Leibniz' integral formula yields

$$\tilde{D}'(n) = \int_{a_0}^{\bar{a}} \tilde{g}(a; n)\frac{\partial}{\partial n}D(a; n)da + \int_{a_0}^{\bar{a}} D(a; n)\frac{\partial}{\partial n}\tilde{g}(a; n)da. \tag{203}$$

The first term on the right is negative because $\frac{\partial}{\partial n}D(a; n) < 0$ by Lemma 8. Therefore, it suffices to show that

$$\int_{a_0}^{\bar{a}} D(a; n)\frac{\partial}{\partial n}\tilde{g}(a; n)da < 0. \tag{204}$$

Holding n fixed, we can define a function $D_n(a) \equiv D(a; n)$ where $D_n'(a) < 0$ because $\frac{\partial}{\partial n}D(a; n) < 0$. Rearranging (204) above and applying Leibniz' integral rule, the sufficient condition (204) is equivalent to

$$\frac{\partial}{\partial n}\int_{a_0}^{\bar{a}} \left(-D_n(a)\right) d\tilde{G}(a; n) > 0. \tag{205}$$

By Part 5 of Lemma 2, we know that for any $f : A \to \mathbb{R}_+$ such that $f' > 0$, we have $\tilde{f}'(n) > 0$ where $\tilde{f}(n) \equiv \int_{a_0}^{\bar{a}} f(a)d\tilde{G}(a; n)$. Letting $f(a) = -D_n(a)$, it is clear that $f' > 0$ because $D_n'(a) < 0$. Thus inequality (205) follows. ■

Proof of Proposition 7

Part 1. The first-order condition for n is given by $\alpha'(n)\tilde{s}(n) + \alpha(n)\tilde{s}'(n) = \kappa$. In the limit as $\kappa \to 0$, we must have $\alpha'(n)\tilde{s}(n) \to 0$ because $\tilde{s}'(n) \geq 0$. Therefore, since $\tilde{s}(n) \geq 0$, we must have $\alpha'(n) \to 0$, which implies that $n \to \infty$ by Lemma 1. Also, Lemma 1 says that in the limit as $n \to \infty$, the distribution of choices converges to a degenerate distribution with support $A = \{\bar{a}\}$.

Part 2. First, by Lemma 21 below, we have $a_c \to \bar{a}/(1+i)$ in the limit as $n \to \infty$. Next, by Lemma 20 below, we have $q_{a_c} = q^*_{a_c}$ so $a_c u'(q_{a_c}) = c'(q_{a_c})$. Thus we obtain

$$\bar{a}u'(q_{a_c}) = (1 + i)c'(q_{a_c}). \tag{206}$$

Given that $\bar{a} \geq a_c$, we have $q_{\bar{a}} = q_{a_c}$, so we get equation (52).

Part 3. By Lemma 21 below, $a_c \to \bar{a}/(1+i)$ in the limit as $n \to \infty$. In the limit as $n \to \infty$, there are no informational rents so $d_a/\gamma = au(q_a)$ for all $a \leq a_c$ and thus

$$d_{a_c}/\gamma = a_c u(q_{a_c}). \tag{207}$$

We also know that, given $\bar{a} \geq a_c$, we have $q_{\bar{a}} = q_{a_c}$ and $d_{\bar{a}} = d_{a_c} = z$. So, we obtain (53) because Lemma 20 implies that $q_a = q_a^*$ for all $a \leq a_c$. ■

Lemma 19. *In the limit as $n \to \infty$, we have $\delta \to 1+i$.*

Proof. Writing $\delta(n)$ to emphasize the dependence on n, and using $a_b - \phi(a_b; n) = 0$ from Lemma 12 plus expression (136) for $\phi(a; n)$, we have

$$\delta(n) a_b \tilde{g}(a_b; n) + (1 - \delta(n)) \left(1 - \tilde{G}(a_b; n)\right) + \frac{i}{\alpha(n)} = 0. \tag{208}$$

Taking the limit as $n \to \infty$, we have $\tilde{g}(a_b; n) \to 0$ and $\tilde{G}(a_b; n) \to 0$ for any $a_b < \bar{a}$. So, using the fact that $\lim_{n\to\infty} \alpha(n) = 1$, we have $\lim_{n\to\infty} \delta(n) = 1 + i$. ■

Lemma 20. *In the limit as $n \to \infty$, we have $q_a = q_a^*$ for all $a \in A$ such that $a \leq a_c$.*

Proof. Consider any a such that $a \leq a_c$. Rewrite (34) from Proposition 3:

$$\phi(a; n)\tilde{g}(a; n) = \left(1 - \frac{1}{\delta(n)}\right) \left(1 - \tilde{G}(a; n)\right) - \left(1 - \frac{1}{\delta(n)} - (\delta(n) - \bar{\delta}(n))\right) \tag{209}$$

where $\bar{\delta}(n) \equiv 1 + \frac{i}{\alpha(n)}$. Taking the limit as $n \to \infty$, we have $\delta(n) - \bar{\delta}(n) \to 0$ by Lemma 19, and also $1 - 1/\delta(n) \to \lim_{n\to\infty} \frac{i}{\alpha(n)+i}$. So, we have

$$\lim_{n\to\infty} \phi(a; n)\tilde{g}(a; n) = \lim_{n\to\infty} \frac{i}{\alpha(n) + i} \left(1 - \tilde{G}(a; n)\right) - \lim_{n\to\infty} \frac{i}{\alpha(n) + i}. \tag{210}$$

Now, $\tilde{G}(a; n) \to 0$ for any $a < \bar{a}$, so we have

$$\lim_{n\to\infty} \phi(a; n)\tilde{g}(a; n) = \lim_{n\to\infty} \frac{i}{\alpha(n) + i} - \lim_{n\to\infty} \frac{i}{\alpha(n) + i} = 0. \tag{211}$$

So, in the limit as $n \to \infty$, we have $au'(q_a) = c'(q_a)$ for all $a \in A$ such that $a \leq a_c$. ■

Lemma 21. *In the limit as $n \to \infty$, we have $a_c \to \bar{a}/(1+i)$.*

Proof. First, we know from Proposition 3 that a_c solves

$$\int_{a_c}^{\bar{a}} (x - a_c)\tilde{g}(x;n)dx = (1-\delta)(\bar{a} - a_c)(1 - \tilde{G}(a_c;n)) + \frac{i\bar{a}}{\alpha(n)}. \tag{212}$$

Writing $\delta(n)$, and taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \int_{a_c}^{\bar{a}} (x - a_c)\tilde{g}(x;n)dx = \lim_{n\to\infty} (1-\delta(n))(\bar{a} - a_c)(1 - \tilde{G}(a_c;n)) + \lim_{n\to\infty} \frac{i\bar{a}}{\alpha(n)}, \tag{213}$$

if all limits exist and are finite. For any $a_c < \bar{a}$, we have $\lim_{n\to\infty} \tilde{G}(a;n) \to 0$ and thus $\lim_{n\to\infty} \tilde{G}(a_c;n) \to 0$. Also, we have $\lim_{n\to\infty} \alpha(n) = 1$. Therefore,

$$\lim_{n\to\infty} \int_{a_c}^{\bar{a}} (x - a_c)\tilde{g}(x;n)dx = (\bar{a} - a_c)(1 - \lim_{n\to\infty} \delta(n)) + i\bar{a}. \tag{214}$$

Next, by Lemma 19 we have $\lim_{n\to\infty} \delta(n) = 1 + i$. So, we obtain

$$\lim_{n\to\infty} \int_{a_c}^{\bar{a}} (x - a_c)\tilde{g}(x;n)dx = i\bar{a}. \tag{215}$$

Finally, $\lim_{n\to\infty} \int_{a_c}^{\bar{a}} (x - a_c)\tilde{g}(x;n)dx = \bar{a} - a_c$ because $\tilde{G}$ converges to a degenerate distribution with support $A = \{\bar{a}\}$. So, $a_c \to \bar{a}/(1+i)$ in the limit as $n \to \infty$. ■

Online Appendix A: Informational distortion

In this Appendix, we consider the informational distortion at a "typical" chosen good and see how this varies when the seller-buyer ratio changes.

Recall that an increase in the seller-buyer ratio n leads to a first-order stochastic dominance shift in the distribution $\tilde{G}$, which shifts greater density towards higher values of a. Given that the informational distortion $I_{\tilde{G}}(a;n)$ is lower for higher values of a, we expect that the "typical" value of the informational distortion might decrease due to this indirect effect. In what follows, we provide some conditions under which the "typical" informational distortion is *decreasing* in the seller-buyer ratio n.

Consider any function $a : \mathbb{R}_+ \to A$ where $a' > 0$ and $a(n)$ represents the utility of a "typical" chosen good. For example, we could have $a(n) = \tilde{a}(n)$, the average utility of a chosen good. Proposition 8 provides a necessary and sufficient condition for the absolute value of the the informational distortion at $a(n)$ to be decreasing in n.

Proposition 8. *Suppose that $n > 0$ and let $a : \mathbb{R}_+ \to A$. The "typical" informational distortion, i.e. $I_{\tilde{G}}(a;n)$ at $a = a(n)$, is decreasing in n if and only if*

$$\frac{a'(n)n}{a(n)}\left(\frac{1}{\eta_{\eta_\alpha}(x)}\frac{h'_G(a)a}{h_G(a)} + \frac{g(a)a}{1-G(a)}\right) > 1 \tag{216}$$

where $a = a(n)$, $x = n(1-G(a))$, and $\eta_{\eta_\alpha}(x) \equiv \frac{-\eta'_\alpha(x)x}{\eta_\alpha(x)}$.

The curvature of the function $a(n)$ is crucial for determining whether the "typical" informational distortion is decreasing in n. This curvature depends on properties of both the distribution G and the meeting technology $\mathbb{P}_j$. Condition (216) holds if the function $a(n)$ increases sufficiently sharply with n, the hazard rate $h_G(a)$ increases sufficiently sharply with a, and $1 - G(a)$ decreases sufficiently sharply with a.

In order to apply this general condition, we first need to define a function $a(n)$ which is intended to represent the "typical" informational distortion. It is useful to define $a(n) = \hat{a}(n)$, the unique value of a at which $\frac{\partial}{\partial n}\tilde{g}(a;n) = 0$. In this case, we can prove that $\hat{a}(n)$ must be *over-represented* with choice, i.e. it is more likely to be traded when there is consumer choice. While this is a somewhat arbitrary definition of the "typical" informational distortion, Proposition 8 applies more generally.

We obtain the following sufficient condition for the "typical" information distortion to be increasing in the seller-buyer ratio n.[31]

[31] A necessary and sufficient condition can be found in the proof of Proposition 9.

Proposition 9. *The "typical" informational distortion, i.e.* $I_{\tilde{G}}(a; n)$ *at* $a = \hat{a}(n)$*, is decreasing in* $n > 0$ *if*

$$\left(1 - \frac{\eta_\alpha(n)}{\eta_\sigma(\hat{x})(1 - \eta_\alpha(n))}\right)\left(\left(\frac{g'(\hat{a})(1 - G(\hat{a}))}{g(\hat{a})^2} + 1\right) + 1\right) > 1 \tag{217}$$

where $\hat{a} = \hat{a}(n)$*,* $\hat{x} = n(1 - G(\hat{a}))$*,* $\eta_\sigma(x) \equiv \frac{\sigma'(x)x}{\sigma(x)}$*, and* $\sigma(x) \equiv \frac{-\mathbb{P}_0''(x)x}{\mathbb{P}_0'(x)}$*.*

Observe that if we had $\eta_\alpha(n) = 0$ (i.e. in the limit as $n \to \infty$), the increasing hazard rate condition on G would be sufficient.[32] In general, however, $\eta_\alpha(n) > 0$ and this condition is more restrictive than the increasing hazard rate condition.

Example. Suppose the distribution G is uniform on $[0, 1]$ and the meeting technology $\mathbb{P}_j$ is Poisson. In this case, we have $g'(a) = 0$, $\sigma(x) = x$, and $\eta_\sigma(x) = 1$. A sufficient condition for (217) is $\eta_\alpha(n) < 1/3$. This implies that a seller-buyer ratio $n \geq 2$ (i.e. an entry cost κ sufficiently high) would suffice for Proposition 9 to hold.

Proofs for Online Appendix A

Proof of Proposition 8

Let $a : \mathbb{R}_+ \to A$. Given that $I_{\tilde{G}}(a; n) = 1/h_{\tilde{G}}(a; n)$, to derive a necessary and sufficient condition for $\frac{d}{dn} I_{\tilde{G}}(a(n); n) < 0$, we derive a condition for $\frac{d}{dn} h_{\tilde{G}}(a(n); n) > 0$.

From Lemma 4, we know that

$$h_{\tilde{G}}(a(n); n) = \eta_\alpha(x(n)) h_G(a(n)) \tag{218}$$

where $x(n) = n(1 - G(a(n)))$. Differentiating with respect to n, we have

$$\frac{d}{dn} h_{\tilde{G}}(a(n); n) = \eta_\alpha'(x(n)) x'(n) h_G(a(n)) + \eta_\alpha(x(n)) h_G'(a(n)) a'(n). \tag{219}$$

Now, we have

$$x'(n) = 1 - G(a(n)) - n g(a(n)) a'(n). \tag{220}$$

[32] The increasing hazard rate condition is equivalent to $\frac{g'(a)(1-G(a))}{g(a)^2} + 1 > 0$ for all $a \in A$.

Therefore, we obtain

$$\begin{aligned}\frac{d}{dn}h_{\tilde{G}}(a(n);n) &= \eta_\alpha'(x)h_G(a)[1-G(a)-ng(a)a'(n)]+\eta_\alpha(x)h_G'(a)a'(n)\\ &= \eta_\alpha'(x)g(a)+a'(n)[\eta_\alpha(x)h_G'(a)-\eta_\alpha'(x)h_G(a)ng(a)].\end{aligned}$$

So, we have $\frac{d}{dn}h_{\tilde{G}}(a(n);n)>0$ if and only if

$$a'(n) > \frac{-\eta_\alpha'(x)g(a)}{\eta_\alpha(x)h_G'(a)-\eta_\alpha'(x)h_G(a)ng(a)}, \tag{221}$$

which is equivalent to

$$a'(n) > \frac{1}{\frac{\eta_\alpha(x)h_G'(a)}{-\eta_\alpha'(x)g(a)}+nh_G(a)}. \tag{222}$$

Using $x=n(1-G(a))$ and the definition of $h_G(a)$, this is equivalent to

$$a'(n) > \frac{1}{\frac{1}{\eta_{\eta_\alpha}(x)}\frac{nh_G'(a)}{h_G(a)}+nh_G(a)} \tag{223}$$

where $\eta_{\eta_\alpha}(x)\equiv\frac{-\eta_\alpha'(x)x}{\eta_\alpha(x)}$. In terms of elasticities, this is equivalent to

$$\frac{a'(n)n}{a(n)} > \frac{1}{\frac{1}{\eta_{\eta_\alpha}(\hat{x})}\frac{h_G'(a)a}{h_G(a)}+h_G(a)a}, \tag{224}$$

which yields Proposition 8. ■

Proof of Proposition 9

We first prove the following three lemmas before proving Proposition 9.

Lemma 22 states that, for any given seller-buyer ratio $n>0$, there is a unique utility shock $a_f(n)$ such that the two densities cross, i.e. $\tilde{g}(a_f(n);n)=g(a)$. For higher utilities $a>a_f(n)$, the density of chosen goods $\tilde{g}(a;n)$ is higher than the density of utility shocks $g(a)$. So, the range of utilities $[a_f(n),\bar{a}]$ is *over-represented* with choice, i.e. this range is more likely to be traded when there is consumer choice.

Lemma 22. *For any $n>0$, there exists a unique $a_f(n)$ such that $\tilde{g}(a_f(n);n)=g(a)$ and $\tilde{g}(a;n)>g(a)$ (i.e. a is over-represented) if $a>a_f(n)$ and $\tilde{g}(a;n)<g(a)$ (i.e. a is under-represented) if $a<a_f(n)$.*

Proof. By definition, $a_f(n)$ solves $\tilde{g}(a;n) = g(a)$. Using (74), this implies that

$$\frac{ng(a_f)\alpha'(x_f)}{\alpha(n)} = g(a_f), \tag{225}$$

or, equivalently, $\alpha'(x_f) = \frac{\alpha(n)}{n}$ where $x_f = n(1-G(a_f))$. For any given $n > 0$, the left-hand side is decreasing in x because $\alpha'' < 0$ by Lemma 1 and is therefore increasing in a because $x = n(1 - G(a))$ and $dx/da = -ng(a)$. For $a = a_0$, we have $x = n$ and thus $\alpha'(x) = \alpha'(n) < \frac{\alpha(n)}{n}$ because $\eta_\alpha(n) < 1$ by Lemma 1. For $a = \bar{a}$, we have $x = 0$ and thus $\alpha'(x) = \lim_{x\to 0} \alpha'(x) = 1 > \frac{\alpha(n)}{n}$ by Lemma 1. Therefore, there exists a unique solution $a_f(n)$. Also, $\tilde{g}(a;n) > g(a)$ if and only if $\alpha'(x) > \frac{\alpha(n)}{n} = \alpha'(x_f)$ where $x_f = n(1 - G(a_f))$, which holds if and only if $x < x_f$ because $\alpha'' < 0$ by Lemma 1. So, $\tilde{g}(a;n) > g(a)$ if and only if $n(1 - G(a)) < n(1 - G(a_f))$ or $a > a_f$. ■

Lemma 23. *There exists a unique $\hat{a}(n)$ such that $\frac{\partial}{\partial n}\tilde{g}(a;n) = 0$ and $\frac{\partial}{\partial n}\tilde{g}(a;n) > 0$ if and only if $a > \hat{a}(n)$. We have $\hat{a}(n) > a_f(n)$, i.e. it is an over-represented good.*

Proof. It is shown in the proof of Lemma 2 that there exists a unique $\hat{a}(n)$ such that $\frac{\partial}{\partial n}\tilde{g}(a;n) = 0$ and we have $\frac{\partial}{\partial n}\tilde{g}(a;n) > 0$ if and only if $a > \hat{a}(n)$. It is clear that $\frac{\partial}{\partial n}\tilde{g}(a_f;n) < 0$ and therefore we have $\hat{a} > a_f$. ■

Lemma 24. *For any $n > 0$, we have*

$$\frac{\hat{a}'(n)n}{\hat{a}(n)} = \frac{1 - G(\hat{a})}{g(\hat{a})\hat{a}} \left(1 - \eta_{\eta_\alpha}(n)\frac{\eta_\alpha(n)}{\eta_\sigma(\hat{x})(1 - \eta_\alpha(n))}\right) \tag{226}$$

where $\hat{a} = \hat{a}(n)$, $\eta_\sigma(x) \equiv \frac{\sigma'(x)x}{\sigma(x)}$, $\sigma(x) \equiv \frac{-\mathbb{P}_0''(x)x}{\mathbb{P}_0'(x)}$, and $\hat{x} = n(1 - G(\hat{a}))$.

Proof. From the proof of Lemma 2, for any given $n > 0$ the value $\hat{a}(n)$ satisfies

$$\frac{-\mathbb{P}_0''(\hat{x})\hat{x}}{\mathbb{P}_0'(\hat{x})} = 1 - \eta_\alpha(n) \tag{227}$$

where $\hat{x} = n(1 - G(\hat{a}))$. Now define $\sigma(x) \equiv \frac{-\mathbb{P}_0''(x)x}{\mathbb{P}_0'(x)}$ so $\sigma(\hat{x}) = 1 - \eta_\alpha(n)$. Letting $F(\hat{a}, n) = \sigma(n(1 - G(\hat{a}))) - (1 - \eta_\alpha(n))$ and differentiating, we obtain

$$\hat{a}'(n) = \frac{-\partial F/\partial n}{\partial F/\partial a} = \frac{-\left(\sigma'(\hat{x})(1 - G(\hat{a})) + \eta_\alpha'(n)\right)}{-\sigma'(\hat{x})ng(\hat{a})} \tag{228}$$

and therefore

$$\hat{a}'(n) = \frac{1-G(\hat{a})}{ng(\hat{a})}\left(1 + \frac{\eta'_\alpha(n)\hat{x}}{\frac{\sigma'(\hat{x})\hat{x}}{\sigma(\hat{x})}(1-G(\hat{a}))\sigma(\hat{x})}\right) \tag{229}$$

and therefore, using the fact that $\hat{x} = n(1-G(\hat{a}))$ and $\sigma(\hat{x}) = 1-\eta_\alpha(n)$,

$$\hat{a}'(n) = \frac{1-G(\hat{a})}{ng(\hat{a})}\left(1 + \frac{\eta'_\alpha(n)n}{\frac{\sigma'(\hat{x})\hat{x}}{\sigma(\hat{x})}(1-\eta_\alpha(n))}\right). \tag{230}$$

Letting $\eta_{\eta_\alpha}(x) = \frac{-\eta'_\alpha(x)x}{\eta_\alpha(x)}$ and $\eta_\sigma(x) = \frac{\sigma'(x)x}{\sigma(x)}$, this is equivalent to

$$\hat{a}'(n) = \frac{1-G(\hat{a})}{ng(\hat{a})}\left(1 - \eta_{\eta_\alpha}(n)\frac{\eta_\alpha(n)}{\eta_\sigma(x)(1-\eta_\alpha(n))}\right), \tag{231}$$

which yields equation (226). ■

Substituting (226) into (216) from Proposition 8, $\frac{d}{dn}I_{\tilde{G}}(\hat{a}(n);n) < 0$ if and only if

$$\frac{1-G(\hat{a})}{g(\hat{a})\hat{a}}\left(1 - \eta_{\eta_\alpha}(n)\frac{\eta_\alpha(n)}{\eta_\sigma(\hat{x})(1-\eta_\alpha(n))}\right)\left(\frac{1}{\eta_{\eta_\alpha}(x)}\frac{h'_G(a)a}{h_G(a)} + \frac{g(a)a}{1-G(a)}\right) > 1. \tag{232}$$

This is equivalent to

$$\frac{1}{h_G(a)a}\left(1 - \eta_{\eta_\alpha}(n)\frac{\eta_\alpha(n)}{\eta_\sigma(\hat{x})(1-\eta_\alpha(n))}\right)\left(\frac{1}{\eta_{\eta_\alpha}(x)}\frac{h'_G(a)a}{h_G(a)} + \frac{g(a)a}{1-G(a)}\right) > 1, \tag{233}$$

and, using the fact that

$$\frac{h'_G(a)a}{h_G(a)} = h_G(a)a\left(\frac{g'(a)(1-G(a))}{g(a)^2} + 1\right), \tag{234}$$

this is equivalent to

$$\left(1 - \eta_{\eta_\alpha}(n)\frac{\eta_\alpha(n)}{\eta_\sigma(\hat{x})(1-\eta_\alpha(n))}\right)\left(\frac{1}{\eta_{\eta_\alpha}(x)}\left(\frac{g'(a)(1-G(a))}{g(a)^2} + 1\right) + 1\right) > 1. \tag{235}$$

Finally, $\eta_{\eta_\alpha}(n) \in (0,1)$ implies a sufficient condition is given by (217). ■

Online Appendix B: Model without choice

In this Appendix, we shut down consumer choice. We do this by assuming that buyers are randomly assigned to sellers within meetings. This is effectively equivalent to assuming the meeting technology is bilateral or one-to-one.

First-best allocation

We first solve for the first-best allocation. As in Section 5, this is essentially the solution to a planner's problem where the planner has complete information about buyers' utility shocks and is therefore not constrained by informational frictions. We assume the planner is, however, constrained by the search frictions.

The planner knows the meeting technology $\mathbb{P}_j$, the distribution of utility shocks G, and the cost of entry κ. The planner chooses a seller-buyer ratio n^*, a function $q^* : A \to \mathbb{R}_+$, to solve the following problem:

$$\max_{n\in\mathbb{R}_+,\{q_a\}_{a\in A}} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} [au(q_a) - c(q_a)]\, dG(a) - n\kappa \right\}. \tag{236}$$

It follows from Assumption 3 and Proposition 1 that there exists a unique first-best allocation with $n^* > 0$. Without choice, the only differences are that the distribution of types $\tilde{G}$ is just equal to G and the planner's first order condition for n is

$$\alpha'(n^*)\tilde{s}(\{q_a^*\}_{a\in A}) = \kappa. \tag{237}$$

Without choice, the average trade surplus $\tilde{s}(\{q_a\}_{a\in A})$ no longer depends on the seller-buyer ratio n and condition (237) is equivalent to the standard Hosios condition:

$$\underbrace{\eta_\alpha(n)}_{\text{meeting elasticity}} = \underbrace{\frac{n\kappa}{\alpha(n)\tilde{s}(\{q_a^*\}_{a\in A})}}_{\text{sellers' surplus share}}. \tag{238}$$

Full information benchmark

For completeness, we present the full information benchmark without choice.

Proposition 10. *Without choice, for any $i > 0$ there exists a unique full-information competitive search equilibrium and it satisfies Proposition 2 except that the distribution* *of choices is $\tilde{G} = G$ and (25) is replaced by*

$$\frac{\alpha'(n)\tilde{s}(\{q_a\}_{a\in A})}{1+(1-\eta_\alpha(n))\frac{i}{\alpha(n)}} = \kappa. \tag{239}$$

Without choice, the first main difference is that the distribution of choices $\tilde{G}$ is just equal to the exogenous distribution of utility shocks G. The second main difference is that the Hosios-like condition (239) depends directly on the nominal interest rate i. As in RW, it is equal to the standard Hosios condition only at the Friedman rule, i.e. $i \to 0$. We can recover the results in RW as a limiting case of Proposition 10 where there are no utility shocks and the distribution G is degenerate with support $\{1\}$.

Without consumer choice, including in the special case of RW, there is under-consumption and under-entry relative to the first-best outside the Friedman rule. Interestingly, over-entry is possible with consumer choice but it is not possible without choice. This is likely because there is no longer any "choice externality".

Proposition 11. *Without choice, in any full-information equilibrium where $i > 0$, there is under-entry of sellers and there is underconsumption for all $a \in (a_0, \bar{a}]$.*

At the Friedman rule, competitive search delivers the first-best allocation in terms of both quantities traded and seller entry, i.e. $q_a = q_a^*$ for all $a \in A$ and $n = n^*$.

Private information equilibrium

We now turn to equilibrium with private information but without choice.

Proposition 12. *Without consumer choice, for any $i > 0$ there exists a unique competitive search equilibrium and it satisfies:*

1. *No-trade range. For any $a \in [a_0, a_b]$, $q_a = 0$ and $d_a = 0$.*
2. *Unconstrained range. For any $a \in (a_b, a_c]$, the quantity $q_a > 0$ solves:*

$$(a - \phi(a;n))u'(q_a) = c'(q_a) \tag{240}$$

where

$$\phi(a;n) = \left(1-\frac{1}{\delta}\right)\frac{1-G(a)}{g(a)} - \left(\frac{1}{\delta}\right)\frac{i}{\alpha(n)g(a)} \tag{241}$$

and

$$\delta = \frac{1}{1-\varepsilon_\rho(a_b)}\left(1+\frac{i}{\alpha(n)\rho(a_b)}\right) \tag{242}$$

and the payment d_a is given by (36).

3. *Parts 3-5 of Proposition 3 hold.*

4. *The seller-buyer ratio $n > 0$ is strictly decreasing in κ and satisfies*

$$\frac{\alpha'(n)\tilde{s}(\{q_a\}_{a\in A})}{\delta + (1-\delta)\eta_\alpha(n)} = \kappa. \tag{243}$$

5. *The zero profit condition (39) holds.*

6. *The distribution of choices is $\tilde{G} = G$.*

The main differences between the equilibrium with and without choice are the following. First, the distribution of types $\tilde{G}$ is just the distribution of utility shocks G and it no longer depends on the equilibrium seller-buyer ratio n. Second, the Hosios-like condition (243) depends directly on the nominal interest rate i, as in RW.

Similarly to the model with consumer choice, the planner's solution gives the same allocation as competitive search equilibrium at the Friedman rule if we assume the planner is constrained by the same informational frictions as sellers. Therefore, competitive search equilibrium decentralizes the efficient allocation both with and without consumer choice, but it cannot deliver the first-best allocation at the Friedman rule.

Corollary 5. *Without choice, at the Friedman rule, equilibrium satisfies:*

1. *No-trade range. For any $a \in [a_0, a_b]$, $q_a = 0$, and $d_a = 0$.*

2. *Unconstrained range. For all $a \in (a_b, \bar{a}]$, the quantity q_a satisfies*

$$\left(a - \varepsilon_\rho(a_b)\frac{1-G(a)}{g(a)}\right)u'(q_a) = c'(q_a) \tag{244}$$

and the payment d_a is given by (36).

3. *No meetings are liquidity constrained: $a_c = \bar{a}$.*

4. *Parts 4-6 from Proposition 12 hold.*

At the Friedman rule, there is always underconsumption relative to the first-best quantity, as is the case with consumer choice. As in the full information benchmark, over-entry relative to the first-best is possible only with consumer choice.

Proposition 13. *Without choice, at the Friedman rule, there is underconsumption for all $a \in (a_0, \bar{a})$ and under-entry of sellers relative to the first best.*

Without choice, the informational distortion is just

$$I_G(a) = \frac{1 - G(a)}{g(a)}, \tag{245}$$

which does not depend on the seller-buyer ratio n. The effect of competitive search on the quantity distortion, reflected in the term $\varepsilon_\rho(a_b(n))$, depends on n only through the trading cut-off $a_b(n)$. Overall, the quantity distortion $D(a;n)$ is given by

$$D(a;n) = \underbrace{\varepsilon_\rho(a_b(n))}_{\text{effect of competitive search}} \underbrace{\frac{I_G(a)}{a}}_{\text{relative informational distortion}}. \tag{246}$$

We know that $a_b'(n) \leq 0$, as in the model with choice, and $\varepsilon_\rho(a)$ is increasing in a because G has an increasing hazard rate by Assumption 4, so the term $\varepsilon_\rho(a_b(n))$ is decreasing in the seller-buyer ratio n. Therefore, the quantity distortion $D(a;n)$ is decreasing in n. By similar reasoning to the model with choice, the average quantity distortion $\tilde{D}(n)$ is also decreasing in n. Without choice, greater seller entry can thus reduce the quantity distortion due to private information, but only by reducing the trading cut-off a_b, not by directly affecting the distribution of buyers' choices $\tilde{G}$.

As Corollary 6 shows, in the absence of consumer choice we do not obtain the first-best allocation at the Friedman rule – even in the competitive limit.

Corollary 6. *Without consumer choice, in the competitive limit as $\kappa \to 0$, at the Friedman rule, competitive search equilibrium satisfies:*

1. *The seller-buyer ratio $n \to \infty$.*
2. *No-trade range. For any $a \in [a_0, a_b]$, $q_a = 0$ and $d_a = 0$.*
3. *Unconstrained range. For any $a \in (a_b, \bar{a}]$, the quantity $q_a > 0$ solves:*

$$(a - \phi(a;n))u'(q_a) = c'(q_a) \tag{247}$$

where

$$\phi(a;n) = \varepsilon_\rho(a_b)\frac{1-G(a)}{g(a)} \tag{248}$$

and the payment d_a *is given by*

$$\frac{d_a}{\gamma} = au(q_a) - \int_{a_0}^{a} u(q_x)dx. \tag{249}$$

Proofs for Online Appendix B

Proof of Proposition 10

Without choice, the distribution of choices is equal to G. Other than changing $\tilde{G}$ to G, the results are identical to those for Proposition 2 except for Part 2.

Part 2. Replacing $\tilde{G}(a;n)$ in (127) with $G(a)$, the first-order condition for n is

$$\alpha'(n)\int_{a_0}^{\bar{a}}(1-\delta)v_a + \delta s_a dG(a) + \alpha(n)\frac{\partial}{\partial n}\int_{a_0}^{\bar{a}}(1-\delta)v_a + \delta s_a dG(a) = \delta\kappa. \tag{250}$$

Since $\tilde{G}(a;n) = G(a)$, the second term on the left is zero and we have

$$\alpha'(n)\int_{a_0}^{\bar{a}}(1-\delta)v_a + \delta s_a dG(a) = \delta\kappa. \tag{251}$$

Rearranging, this is equivalent to

$$\alpha'(n)\int_{a_0}^{\bar{a}}\frac{1-\delta}{\delta}\left(au(q_a) - \frac{d_a}{\gamma}\right) + [au(q_a) - c(q_a)]dG(a) = \kappa. \tag{252}$$

Since (80) implies $\int_{a_0}^{\bar{a}}\frac{d_a}{\gamma}dG(a) = \int_{a_0}^{\bar{a}}c(q_a)dG(a) + \frac{n\kappa}{\alpha(n)}$, we obtain

$$\alpha'(n)\int_{a_0}^{\bar{a}}\frac{1-\delta}{\delta}\left(au(q_a) - c(q_a) - \frac{n\kappa}{\alpha(n)}\right) + [au(q_a) - c(q_a)]dG(a) = \kappa. \tag{253}$$

Simplifying further yields

$$\alpha'(n)\int_{a_0}^{\bar{a}}[au(q_a) - c(q_a)]dG(a) = \kappa[\delta + (1-\delta)\eta_\alpha(n)] \tag{254}$$

where $\eta_\alpha(n) \equiv \frac{\alpha'(n)n}{\alpha(n)}$. This is equivalent to

$$\frac{\alpha'(n)\tilde{s}(\{q_a\}_{a\in A})}{\delta + (1-\delta)\eta_\alpha(n)} = \kappa \tag{255}$$

where $\tilde{s}(\{q_a\}_{a\in A}) \equiv \int_{a_0}^{\bar{a}} s_a dG(a)$. With full information, we have $\delta = 1 + \frac{i}{\alpha(n)}$. Substituting into (255), we obtain (239). ■

Proof of Proposition 11

For any $a \in (a_0, \bar{a}]$, we have $(a - \phi^m(a;n))u'(q_a) = c'(q_a)$ where $\phi^m(a;n) \equiv \frac{ai}{i+\alpha(n)} > 0$, so there is underconsumption, i.e. $q_a < q_a^*$. The equilibrium n satisfies

$$\frac{\alpha'(n)\tilde{s}(\{q_a\}_{a\in A})}{1 + (1-\eta_\alpha(n))\frac{i}{\alpha(n)}} = \kappa \tag{256}$$

and the first-best n^* satisfies

$$\alpha'(n^*)\tilde{s}(\{q_a^*\}_{a\in A}) = \kappa. \tag{257}$$

We know that $q_a^* > q_a$ for any $a \in (a_0, \bar{a}]$. Thus $\tilde{s}(\{q_a^*\}_{a\in A}) > \tilde{s}(\{q_a\}_{a\in A})$ and therefore, since κ is constant, we have

$$\alpha'(n^*) < \frac{\alpha'(n)}{1 + (1-\eta_\alpha(n))\frac{i}{\alpha(n)}}. \tag{258}$$

Now, $1 + (1-\eta_\alpha(n))\frac{i}{\alpha(n)} \geq 1$ since $\eta_\alpha(n) < 1$, so $\alpha'(n^*) < \alpha'(n)$. Since $\alpha'' < 0$ by Lemma 1, we have $n^* > n$ and there is always under-entry of sellers. ■

Proof of Proposition 12

Starting with Proposition 3, we can directly replace the distribution of choices with the exogenous distribution, G. We can also replace $\rho(a_b;n)$ with $\rho(a_b)$ and $\varepsilon_\rho(a_b;n)$ with $\varepsilon_\rho(a_b)$ because these no longer depend directly on n. Aside from these changes, the only part that is different from Proposition 3 is Part 6.

Starting with (127) and replacing $\tilde{G}$ with G, the first-order condition for n is

$$\alpha'(n)\int_{a_0}^{\bar{a}}(1-\delta)v_a + \delta s_a dG(a) + \alpha(n)\frac{\partial}{\partial n}\int_{a_0}^{\bar{a}}(1-\delta)v_a + \delta s_a dG(a) = \delta\kappa. \tag{259}$$

Now, the second term on the left is zero because it does not depend directly on n, so

$$\alpha'(n)\int_{a_0}^{\bar{a}}(1-\delta)v_a+\delta s_a dG(a)=\delta\kappa. \tag{260}$$

Rearranging, this is equivalent to

$$\alpha'(n)\int_{a_0}^{\bar{a}}\frac{1-\delta}{\delta}\left(au(q_a)-\frac{d_a}{\gamma}\right)+[au(q_a)-c(q_a)]dG(a)=\kappa. \tag{261}$$

Since the zero profit condition implies $\int_{a_0}^{\bar{a}}\frac{d_a}{\gamma}dG(a)=\int_{a_0}^{\bar{a}}c(q_a)dG(a)+\frac{n\kappa}{\alpha(n)}$, we obtain

$$\alpha'(n)\int_{a_0}^{\bar{a}}\frac{1-\delta}{\delta}\left(au(q_a)-c(q_a)-\frac{n\kappa}{\alpha(n)}\right)+[au(q_a)-c(q_a)]dG(a)=\kappa \tag{262}$$

and simplifying further yields

$$\alpha'(n)\int_{a_0}^{\bar{a}}[au(q_a)-c(q_a)]dG(a)=\kappa[\delta+(1-\delta)\eta_\alpha(n)] \tag{263}$$

where $\eta_\alpha(n)\equiv\frac{\alpha'(n)n}{\alpha(n)}$. This is equivalent to (243) where

$$\delta=\frac{1}{1-\varepsilon_\rho(a_b)}\left(1+\frac{i}{\alpha(n)\rho(a_b)}\right) \tag{264}$$

and $\rho(a)\equiv 1-G(a)$ and $\varepsilon_\rho(a;n)\equiv -a\rho'(a)/\rho(a)$. ■

Proof of Corollary 5

Part 1. Follows from Part 1 of Proposition 12.

Part 2. Setting $i=0$ in Proposition 12, we obtain (244).

Part 3. Setting $i=0$ in (37) implies $a_c=\bar{a}$.

Part 4. Parts 5-8 from Proposition 12 also hold. ■

Proof of Proposition 13

Without consumer choice the equilibrium n satisfies

$$\frac{\alpha'(n)\tilde{s}(\{q_a\}_{a\in A})}{\delta+(1-\delta)\eta_\alpha(n)}=\kappa \tag{265}$$

and the first-best n^* satisfies

$$\alpha'(n^*)\tilde{s}(\{q_a^*\}_{a\in A}) = \kappa. \tag{266}$$

At the Friedman rule, $q_a^* > q_a$ for any $a \in (a_0, \bar{a})$. Thus $\tilde{s}(\{q_a^*\}_{a\in A}) > \tilde{s}(\{q_a\}_{a\in A})$ and therefore, since κ is constant, we have

$$\alpha'(n^*) < \frac{\alpha'(n)}{\delta + (1-\delta)\eta_\alpha(n)}. \tag{267}$$

Now, $\delta + (1-\delta)\eta_\alpha(n) = \delta(1-\eta_\alpha(n)) + \eta_\alpha(n) \geq 1$ since $\delta \geq 1$, so $\alpha'(n^*) < \alpha'(n)$. Since $\alpha'' < 0$, we have $n^* > n$ and there is under-entry of sellers. ■