Consumer Choice and the Cost of Inflation*†

Sephorah Mangin[‡] ⓡ Ayushi Bajaj[§]

March 24, 2022

Abstract

Is inflation more or less costly in economies where consumers have a greater degree of informed choice about their purchases? To answer this question, we introduce consumer choice into a competitive search model of monetary exchange. Consumers can meet multiple sellers and *choose* a seller with whom to trade. Consumers' preferences are given by private utility shocks. When consumers can observe these shocks prior to seller choice, we call this *informed choice*. We calibrate the model to U.S. data and find that a greater degree of informed choice amplifies the negative welfare effects of inflation, making it significantly more costly. *JEL codes*: D82, E31, E40, E50, E52

Keywords: Money; Inflation; Choice; Directed search; Private information

---

*We are grateful to Guillaume Rocheteau and Randy Wright for especially useful feedback. We also thank Michael Choi, Lucas Herrenbrueck, Janet Jiang, and Mario Silva for useful comments, as well as seminar participants at U.C. Irvine, Simon Fraser University, the National University of Singapore, and the Bank of Canada Summer Workshop on Money, Banking, and Payments.

†Author names are listed in random order. Declarations of interest: none.

‡Corresponding author. Research School of Economics, The Australian National University, HW Arndt Building 25A, Kingsley Place, Acton ACT 2601 Australia. Email: sephorah.mangin@anu.edu.au.

§Department of Economics, Monash University. Email: ayushi.bajaj@monash.edu.

1 Introduction

In January 2022, core inflation in the U.S. reached 6.0%, its highest level since 1982.[1] Since the early 1980s, the nature of retail trade has changed radically as a result of various factors, including the rise of the internet. Less than 1% of the U.S. population used the internet in 1990 compared to almost 90% in 2019.[2] Not only are more purchases made online today but consumers have a greater ability to make *informed choices* about their purchases – both online and in store – due to the increased availability of online information about brands and product features. This raises the following question: Does a greater degree of informed choice by consumers make economies more or less vulnerable to the negative effects of inflation?

To answer this question, we build a model that features both consumer choice and monetary exchange. The model is rich enough to allow us to vary the extent to which consumers can make informed choices about which goods to purchase. This enables us to ask a precise question: How does the welfare cost of inflation vary with changes in the extent of consumers' informed choices? We find that a greater degree of consumer choice significantly increases the cost of inflation. As a result, economies in which buyers are more likely to be able to make informed choices – for example, as a result of rising internet availability – may be more sensitive to the effects of lower levels of inflation. This means that the same inflation rate may be more costly today – in terms of its negative welfare effects – than in earlier decades.

When a consumer seeks to purchase a good, he or she generally chooses from a number of goods that are available simultaneously from a range of competing sellers. Discrete choice models with random utility shocks have been used extensively to study this type of choice in the large literature following Anderson, De Palma, and Thisse (1992), but these models do not feature monetary exchange. Search-theoretic models have become the standard way of modelling the microfoundations of monetary exchange, as surveyed in Lagos, Rocheteau, and Wright (2017), but these models do not feature what we call *consumer choice*, i.e. buyers' choice of seller. Typically, buyer-seller meetings are bilateral: each buyer meets at most one seller during a single period of time and chooses to either trade or wait.

[1] U.S. Bureau of Labor Statistics, Consumer Price Index for All Urban Consumers: All Items in U.S. City Average [CPIAUCSL], retrieved from FRED.

[2] World Bank, Internet Users for the United States [ITNETUSERP2USA], retrieved from FRED.

To develop a model that features both consumer choice and monetary exchange, we introduce the possibility of consumer choice into the monetary framework of Rocheteau and Wright (2005), hereafter denoted RW. This framework shares the convenience of the Lagos and Wright (2005) alternating structure and it also features endogenous seller entry. We focus on *competitive search equilibrium.* Buyers and sellers choose to enter submarkets in which terms of trade, or contracts, are posted by market makers. After entering a submarket, buyers and sellers commit to trading at the terms posted in that submarket. Within each submarket, there are search frictions that govern how buyers and sellers meet.

Directed or competitive search is a natural alternative to bargaining in the environment we consider because buyers can meet multiple sellers within a single meeting. At the same time, it is a natural benchmark for welfare analysis since directed or competitive search is often used to decentralize the constrained efficient allocation in search-theoretic environments, as discussed in Wright, Kircher, Julien, and Guerrieri (2021). Moreover, since the cost of inflation is generally much lower when prices are determined by competitive search instead of bargaining, our estimates of the cost of inflation are conservative and can be interpreted as lower bounds.

Our model has two main features that are necessary for consumer choice.

First, search frictions within submarkets are modelled using a meeting technology that features *many-on-one meetings* (sometimes called *multilateral*). During any given period of time, each seller meets exactly *one* buyer, but a buyer may meet *many* sellers. In particular, a buyer can meet either no sellers, one seller, or more than one sellers, but they can trade with only one seller in each period. A *meeting* is an opportunity for buyers to choose one seller from a subset of sellers.

Second, after a meeting takes place, nature draws an i.i.d. preference or utility shock specific to each seller in the meeting. The buyer then chooses to purchase from the seller that maximizes their net utility. The pair consisting of a buyer and their chosen seller is called a *match.* Sellers cannot observe buyers' utility shocks; they are private information for the buyer. We sometimes refer to the realization of a shock as the good's *quality*, but it is really perceived quality (or "suitability") since it is an idiosyncratic preference or "taste" shock.

A buyer's choice of seller is influenced by the information available to the buyer at the time this choice is made. We allow for two possibilities. With probability $\pi \in (0, 1]$, buyers observe the seller-specific utility shocks *before* choosing a seller. In such cases, we say that buyers make an *informed choice* of seller. With probability $1 - \pi$, buyers observe the shock *after* choosing a seller but prior to trade. In such cases, buyers simply randomize across sellers. This flexibility allows us to examine the effect of a change in the extent of consumer choice, i.e. the *degree of choice* π.

An important consequence of consumer choice is that the distribution of utility shocks of *chosen* goods is endogenous and depends on the seller-buyer ratio. More sellers per buyer means that each buyer can choose from a greater number of sellers (on average), which increases the average quality of the goods that are actually chosen by buyers in equilibrium. As a result, both the average quality of a chosen good and the average surplus depends directly on the seller-buyer ratio.

After choosing a seller with whom to trade, buyers choose the quantity of the good to purchase and make the corresponding payment. We focus on incentive-compatible direct revelation mechanisms that induce buyers to reveal their private information to their chosen seller. We establish the existence and uniqueness of equilibrium for any degree of consumer choice π. In equilibrium, there is only one active submarket and sellers offer the same non-linear price schedule that specifies both the quantity traded and the payment in real dollars for any given realization of the buyer's utility shock. Within any meeting, trades may or may not be liquidity constrained. Buyers may spend all of their money, some of their money, or none.

After presenting our key analytic results, we quantify the effect of consumer choice on the welfare cost of inflation. We calibrate the model to match data from Lucas and Nicolini (2015) on money demand in the U.S. from 1915-2008. For our baseline calibration, we target a retail markup of 30% as in Berentsen, Menzio, and Wright (2011), which implies a degree of choice $\pi = 0.54$. That is, 54% of all meetings are ones in which consumers make an informed choice of seller.

We estimate that the welfare cost of going from 0% to 10% inflation is equivalent to 0.93% of consumption at our baseline calibration. To determine the effect of consumer choice on the welfare cost of inflation, we vary the degree of choice π and recalibrate the model using the same calibration strategy for the other parameters. In particular, we compare results for the full choice calibration (i.e. $\pi = 1$) and the random choice calibration (i.e. the limit as $\pi \to 0$). We estimate that the cost of increasing inflation from 0% to 10% is more than twice as high with full choice: 1.45% of consumption compared to 0.61% with random choice. Moreover, we find that the cost of inflation is strictly increasing in the degree of choice π.

An alternative way to measure the welfare cost of inflation is to ask: What level of inflation leads to a welfare cost of 1% (compared to 0% inflation)? At our baseline calibration, this inflation rate is 11%. With full choice, this inflation rate is 7%, and with random choice, this inflation rate is 28%. This suggests that while consumers are better off in economies that feature a greater degree of informed choice, they are significantly more vulnerable to experiencing the negative welfare effects of inflation.

In our model, consumer choice makes inflation more costly because it *amplifies* the negative effects of inflation. With random choice, inflation is costly because buyers hold less money when inflation is higher, which leads to lower quantities traded and lower entry of sellers, which reduces the number of trades. When there is consumer choice, all of these effects continue to hold. However, there is an additional effect of inflation: lower seller entry directly reduces the average quality of chosen goods, which affects welfare by reducing the average match surplus directly (as well as indirectly through quantities). This is because the distribution of chosen goods is endogenous and depends on the seller-buyer ratio when there is choice. In turn, the effect of inflation on the distribution of chosen goods amplifies the negative effects of inflation on money holdings, quantities traded, and seller entry.

Outline. Section 2 discusses the related literature. Section 3 describes the model. Section 4 solves the planner's problem. Section 5 describes competitive search equilibrium and establishes existence and uniqueness of equilibrium. Section 6 presents our key analytic results. Section 7 presents our baseline calibration and some comparative statics. Section 8 provides our estimates of the cost of inflation. Section 9 describes the results of some robustness exercises. Section 10 concludes. The Appendix contains our random choice and full choice calibrations. All proofs are in the Online Appendix, which also contains the comparative statics figures.

2 Related literature

As discussed, our model builds on the environment in RW, which shares the alternating centralized and decentralized markets of Lagos and Wright (2005) but features endogenous seller entry. In RW, the focus is on comparing different market structures (e.g. bargaining and competitive search) that feature bilateral meetings, while our paper examines the effect of consumer choice on the cost of inflation.

There is a large literature on the welfare cost of inflation. Rocheteau and Nosal (2017) provides a summary of estimates of the welfare cost of 10% inflation, which vary from 0.2% to 7.2% of consumption. Cooley and Hansen (1989) estimates the cost of 10% inflation is less than 0.5% of consumption, while Lucas (2000) estimates that it is less than 1%. Lagos and Wright (2005) finds that the cost of 10% inflation is between 3% and 5% of consumption in a monetary model with search and bargaining. In competitive search equilibrium, the cost of inflation is typically much lower than under bargaining, e.g. Rocheteau and Wright (2009) estimates the cost of 10% inflation is between 0.67% and 1.1% of consumption.[3] Recently, Bethune, Choi, and Wright (2020) obtains a relatively low estimate for the cost of inflation – around 1% or less – by identifying a positive market-composition effect of inflation.

Our paper is related to the wide literature on directed and competitive search surveyed in Wright et al. (2021). In particular, we contribute to the literature on directed or competitive search and private information, including Faig and Jerez (2005), Menzio (2007), Guerrieri (2008), Guerrieri, Shimer, and Wright (2010), Moen and Rosen (2011), and Davoodalhosseini (2019). In our environment, both buyers and sellers are ex ante identical and buyers' private utility shocks are realized *after* meetings take place. Importantly, meetings are many-on-one in our environment, allowing buyers to *choose* sellers within meetings. The sequential nature of search in our model, in which buyers first choose a submarket using directed or competitive search and then face a "noisy" process of choosing or matching among the random subset of sellers they meet, shares some similarities with the model of sequentially mixed search developed in Shi (2020). However, in our model buyers' choice of seller within meetings is driven by private utility shocks rather than prices.

In our paper, buyers may or may not observe their utility shocks prior to their choice of seller, allowing us to nest both informed choice and random choice within meetings. The distinction between informed and random choice by buyers is reminiscent of the distinction between informed and uninformed buyers in Lester (2011). However, the meaning of the term "informed" is different here. In our model, *all* buyers observe price schedules and engage in directed or competitive search when choosing submarkets, but *within* meetings buyers can either make an informed choice of seller (i.e. observe utility shocks prior to choosing a seller) or not.

[3] Rocheteau and Wright (2009) use a slightly different formulation to calibrate the model in RW. Instead of seller entry, agents can decide whether to be buyers or sellers.

Related papers that feature many-on-one or multilateral meetings in monetary environments include Julien, Kennes, and King (2008) and Galenianos and Kircher (2008). Julien et al. (2008) introduces multilateral meetings and directed search into the framework of Shi (1995) and Trejos and Wright (1995) with divisible goods and indivisible money. Galenianos and Kircher (2008) develops a model featuring ex ante heterogeneity, private information, and multilateral meetings in which indivisible goods are allocated according to auctions in money holdings. In both Julien et al. (2008) and Galenianos and Kircher (2008), sellers can meet multiple buyers and either money or goods are indivisible. In our paper, by contrast, buyers can meet multiple sellers and both money and goods are divisible.[4]

While we study the effects of consumer choice, some related papers consider monetary environments featuring buyer preference shocks that are private information. Ennis (2008) incorporates private, match-specific buyer preference shocks into the monetary framework of Lagos and Wright (2005). Faig and Jerez (2006) and Dong and Jiang (2014) examine the effect of inflation on the extent of quantity discounts when buyers' valuations are private information, thus extending the theory of non-linear pricing in Maskin and Riley (1984). Faig and Jerez (2006), which builds on Faig and Jerez (2005), is effectively a special case of our model in which there is no seller entry or consumer choice, no individual rationality (IR) constraint, and the distribution of utility shocks is uniform. Dong and Jiang (2014) considers a similar environment that features an IR constraint and price posting with undirected search. More recently, Choi and Rocheteau (2021) develops a search model of retail banking in which consumers' liquidity needs are private information. All of these papers feature bilateral meetings without consumer choice.[5]

[4] An alternative approach is Head and Kumar (2005), which combines the monetary search framework of Shi (1997, 1999) with the price-posting mechanism of Burdett and Judd (1983), which allows buyers to observe a random sample of prices posted by sellers and choose the lowest price. See also Herrenbrueck (2017), which extends the framework of Head and Kumar (2005).

[5] In a monetary search model without private information, Dong (2010) considers the effect of product variety on the welfare cost of inflation when firms can invest to expand product variety. Greater product variety increases welfare by increasing the *probability* of trade in bilateral meetings. Dong (2010) finds the effect of endogenous product variety on the cost of inflation is negligible with competitive search. In a related paper, Silva (2017) incorporates endogenous product variety into a monetary search model featuring monopolistic competition.

3 Model

Time is discrete and continues forever. Each period $t \in \{0, 1, 2, ...\}$ is divided into two subperiods as in Lagos and Wright (2005). During the day, there is a frictionless, centralized market and at night there is a frictional, decentralized market. As in RW, there is a continuum of agents divided into two types: *buyers* and *sellers*. Buyers are ex ante identical and sellers are ex ante identical. The sets of buyers and sellers are denoted B and S respectively. While all agents both produce and consume during the day, buyers and sellers differ at night: buyers wish to consume (but cannot produce) and sellers do not wish to consume (but can produce).

There is a fixed measure of buyers and we normalize $|B| = 1$. All buyers participate in the night market at zero cost, but there is an entry decision by sellers. Only a subset $\bar{S}_t \subseteq S$ of sellers of measure n_t enter the night market. Sellers may or may not choose to enter the night market at cost $k > 0$ and thus $n_t \in \mathbb{R}_+$ is endogenous.[6] Since $|B| = 1$, the measure of sellers who enter, n_t, is also the seller-buyer ratio.

Money is perfectly divisible. The aggregate money supply at date t is $M_t \in \mathbb{R}_+$, which grows at a constant rate $\gamma \in \mathbb{R}_+$, i.e. $M_{t+1} = \gamma M_t$. Money is either injected into the economy ($\gamma > 1$) or withdrawn ($\gamma < 1$) by lump sum transfers during the day. We assume these transfers are to buyers only, and we restrict attention to policies where $\gamma \geq \beta$, where β is the discount factor. When $\gamma = \beta$ (the Friedman rule), we only consider equilibria obtained by taking the limit as $\gamma \to \beta$ from above.

In the day market, the price of goods is normalized to one and the relative price of money is denoted by ϕ_t. The real value of a quantity of money m_t held by an agent at date t is defined as $z_t \equiv \phi_t m_t$ and the aggregate real money supply is $Z_t \equiv \phi_t M_t$. We will focus on steady-state equilibria where all of the aggregate real variables are constant. Since $M_{t+1}/M_t = \gamma$, this implies that in steady state $\phi_{t+1}/\phi_t = 1/\gamma$.

In the night market, prices are determined in competitive search equilibrium, which we discuss in Section 5. The night market has some novel features that enable the possibility of consumer choice.

Many-on-one meetings. A *meeting* is an opportunity for a buyer to choose from among a subset of sellers. While all sellers meet exactly one buyer, a buyer can meet possibly *many* sellers. In particular, each buyer can meet either no sellers, one

[6] We assume the set S is sufficiently large that $n_t \leq |S|$ always.

seller, or more than one seller. The probability that a buyer meets $N \in \{0, 1, 2, ...\}$ sellers is given by $P_N(n) = \Pr(N_i = N)$ where $P_N(n) \in [0, 1]$ and $\sum_{N=0}^{\infty} P_N(n) = 1$. The endogenous probability $\alpha(n)$ that a buyer has the opportunity to trade equals the probability that the buyer meets at least one seller, i.e. $\alpha(n) = 1 - P_N(0)$. Since all sellers meet exactly one buyer, the probability that a seller has the opportunity to trade equals $\alpha(n)/n$, the probability that the seller's good is *chosen* by a buyer.

Throughout the paper, we assume that $P_N(n)$ is Poisson, i.e. $P_N(n) = \frac{n^N e^{-n}}{N!}$ for all $N \in \{0, 1, 2, ...\}$. Given the Poisson assumption, we have $\alpha(n) = 1 - e^{-n}$.

Buyer's choice of seller. After a meeting takes place, nature draws a seller-specific random utility shock a for each seller the buyer meets. The buyer then chooses a single seller with whom to trade in that subperiod.[7] The pair consisting of a buyer and their chosen seller is called a *match*.

There are two different possibilities with respect to buyers' information. With probability $\pi \in (0, 1]$, the buyer observes their utility shocks *before* choosing a seller. With probability $1 - \pi$, the buyer observes their shock *after* choosing a seller but before trade occurs. In the first case, we say that the buyer makes an *informed choice.* In the second case, buyers simply randomize across sellers. We refer to π as the *degree of choice.* We sometimes refer to the case where $\pi = 1$ as *full choice* and the case where $\pi \in (0, 1)$ as *partial choice.* We refer to the limiting case where $\pi \to 0$ as *random choice* since it is effectively equivalent to a model with bilateral meetings and random matching within submarkets.

Distribution of utility shocks. The random utility shocks a are drawn from a bounded, continuous distribution with cdf G that is known to all agents. Importantly, the realizations of the utility shocks are not observed by sellers; they are private information for the buyer. For simplicity, we sometimes refer to a as (perceived) *quality*, but it is intended to capture suitability or fit.

We assume that the distribution G is not degenerate and Assumption 1 is maintained throughout the paper. We later make some additional restrictions on the distribution G in order to prove the existence of equilibrium.

Assumption 1. *The distribution of utility shocks has a twice-differentiable cdf* $G : A \to [0, 1]$*, pdf* $g = G'$ *where* $G' > 0$*, and bounded support* $A = [a_0, \bar{a}] \subseteq \mathbb{R}_+$*.*

[7] Similarly to standard discrete choice models, we assume that consumers choose to purchase from a single firm in each meeting.

The distribution of utility shocks of the goods actually *chosen* by buyers, denoted $\tilde{G}$, is endogenous. Taking $\pi \in (0, 1]$ as given, for any $n \in \mathbb{R}_+$ this distribution has cdf $\tilde{G} : A \to [0, 1]$, which depends on both the equilibrium seller-buyer ratio n and the equilibrium choices made by buyers. For brevity, we refer to G simply as the *distribution of available goods* and $\tilde{G}$ as the *distribution of chosen goods.*

Buyer and seller utility. Sellers can produce on demand any quantity $q \in \mathbb{R}_+$ of a divisible good and the cost of production is $c(q)$, where $c : \mathbb{R}_+ \to \mathbb{R}_+$ and we assume that $c(0) = 0$, $c'(q) > 0$, and $c''(q) \geq 0$ for all $q > 0$. A buyer who consumes quantity q of a good with quality a receives utility $au(q)$, where $u : \mathbb{R}_+ \to \mathbb{R}_+$ and we assume that $u(0) = 0$, $u'(0) = \infty$, $u'(q) > 0$, and $u''(q) < 0$ for all $q > 0$.

The instantaneous utility of a buyer who meets a seller at night at date t is

$$U_t^b = \nu(x_t) - y_t + \beta E_{\tilde{G}_t}(au(q_{a,t})), \tag{1}$$

and the instantaneous utility of a seller who is chosen by a buyer at night at date t is

$$U_t^s = \nu(x_t) - y_t - \beta E_{\tilde{G}_t}(c(q_{a,t})), \tag{2}$$

where x_t is the quantity consumed and y_t is the quantity produced during the day, $q_{a,t}$ is the quantity consumed at night, a is the *quality* of the good consumed, and $\tilde{G}_t$ is the distribution of *chosen goods* at time t.

We assume $\nu'(x) > 0$ and $\nu''(x) < 0$ for all x, and that there exists x^* such that $\nu'(x^*) = 1$. For now, we normalize $\nu(x^*) - x^* = 0$.[8]

4 Planner's problem

Before we consider competitive search equilibrium, we solve the planner's problem. We assume the planner is constrained by the same search frictions and meeting technology as the decentralized market. We also assume that the planner faces the same information about utility shocks as buyers, i.e. the same probability $\pi \in (0, 1]$ of observing these shocks prior to choosing a seller with whom the buyer will trade.

[8] Later, when we calibrate the model in Section 7, we will reverse this normalization.

We say that the planner's solution achieves the constrained efficient allocation. For brevity, we refer to this simply as the *efficient allocation*.

Given the cost of seller entry $k > 0$, the planner chooses a seller-buyer ratio n^*, a function $q^* : A \to \mathbb{R}_+$, and a distribution of chosen goods $\tilde{G} : A \to [0,1]$ to maximize the total surplus created minus the total cost of seller entry, subject to the constraints he faces. That is, the planner solves the following problem:

$$\max_{n\in\mathbb{R}_+,\{q_a\}_{a\in A}} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} [au(q_a) - c(q_a)]\, d\tilde{G}(a;n) - nk \right\} \tag{3}$$

where $\tilde{G}$ represents the planner's optimal choice of seller for each buyer.[9] The planner must take into account the fact that buyers' expected utility from consumption in the night market depends not only on the meeting probability and the quantity of goods traded, but also on the expected quality of the good purchased.

Define $s_a \equiv au(q_a) - c(q_a)$, the trade surplus (or match surplus) for a good of quality a. Let q_a^* denote the efficient quantity of good a and define $s_a^* \equiv au(q_a^*) - c(q_a^*)$. Assume that $s_0^* \geq 0$ where $s_0^* \equiv a_0 u(q_0) - c(q_0)$ and $q_0 = q(a_0)$, so there is (weakly) positive trade surplus for all goods. Define the *expected trade surplus* by

$$\tilde{s}(n;\{q_a\}_{a\in A}) \equiv \int_{a_0}^{\bar{a}} [au(q_a) - c(q_a)] d\tilde{G}(a;n). \tag{4}$$

For simplicity of notation, throughout the paper we sometimes suppress the dependence of the expected trade surplus $\tilde{s}(n;\{q_a\}_{a\in A})$ on the function $q : A \to \mathbb{R}_+$ and let $\tilde{s}(n)$ denote $\tilde{s}(n;\{q_a\}_{a\in A})$ and $\tilde{s}'(n)$ denote $\partial\tilde{s}(n)/\partial n$.

The following assumption ensures the existence of a social optimum where $n^* > 0$. Intuitively, this condition says that the expected trade surplus in the limit as $n \to 0$, i.e. $\lim_{n\to 0} \tilde{s}(n)$, must be greater than k.[10] It follows from our assumptions that, for all $a \in A$, there exists a unique $q_a^* \in \mathbb{R}_+$ such that $au'(q_a^*) = c'(q_a^*)$.

Assumption 2. *The cost of entry is not too high:* $E_G[au(q_a^*) - c(q_a^*)] > k$.

Before presenting the planner's solution, we derive the endogenous distribution of chosen goods. Lemma 1 provides some useful properties.

[9] The planner's distribution of chosen goods will turn out to be equal to the buyers' distribution of chosen goods so we use the same notation, $\tilde{G}$, for simplicity.

[10] Since $\tilde{G} \to G$ as $n \to 0$, as verified in Lemma 1, we have $\lim_{n\to 0} \tilde{s}(n) = E_G[au(q_a^*) - c(q_a^*)]$

The *average quality* of a chosen good is defined by $\tilde{a}_G(n) \equiv E_{\tilde{G}}(a)$, i.e. $\tilde{a}_G(n) = \int_{a_0}^{\bar{a}} a d\tilde{G}(a;n)$.[11] Lemma 1 states that the average quality of a *chosen* good $\tilde{a}(n)$ is greater than the average quality of an *available* good, $E_G(a)$. Moreover, Part 6 of Lemma 1 implies that $\tilde{a}(n)$ is strictly increasing in n, i.e. $\tilde{a}'(n) > 0$. Intuitively, average quality is increasing in the seller-buyer ratio because more sellers per buyer means greater choice of seller and a higher expected quality of the chosen good.

Lemma 1. *Suppose that the seller-buyer ratio $n > 0$.*

1. *For any $\pi \in (0,1]$, the distribution of chosen goods is given by*

$$\tilde{G}(a;n) = \pi \left(\frac{e^{-n(1-G(a))} - e^{-n}}{1 - e^{-n}} \right) + (1-\pi)G(a). \tag{5}$$

2. *In the limit as $n \to 0$, we have $\tilde{G}(a;n) \to G(a)$ and $\tilde{a}(n) \to E_G(a)$.*

3. *In the limit as $n \to \infty$, we have $\tilde{G}(a;n) \to (1-\pi)G(a)$ for all $a \in [a_0, \bar{a})$ and $\tilde{a}(n) \to \pi\bar{a} + (1-\pi)E_G(a)$.*

4. *The distribution of chosen goods $\tilde{G}(a;n)$ first-order stochastically dominates the distribution of available goods $G(a)$ and $\tilde{a}(n) > E_G(a)$.*

5. *If $n_1 > n_2$, the distribution $\tilde{G}(a;n_1)$ first-order stochastically dominates the distribution $\tilde{G}(a;n_2)$ and $\tilde{a}(n_1) > \tilde{a}(n_2)$.*

6. *For any $f : A \to \mathbb{R}_+$ where $f' > 0$, $\tilde{f}'(n) > 0$ where $\tilde{f}(n) \equiv \int_{a_0}^{\bar{a}} f(a) d\tilde{G}(a;n)$.*

We are now ready to describe the planner's solution. Proposition 1 states that there exists a unique social optimum $(n^*, \{q_a^*\}_{a\in A})$ with $n^* > 0$ and provides the necessary conditions for an efficient allocation.

Proposition 1. *There exists a unique social optimum $(n^*, \{q_a^*\}_{a\in A})$ and it satisfies:*

1. *For any $a \in A$, the quantity $q_a^* > 0$ solves*

$$au'(q_a^*) = c'(q_a^*). \tag{6}$$

[11] For simplicity, we generally drop the subscript G and denote $\tilde{a}_G(n)$ simply by $\tilde{a}(n)$.

2. The seller-buyer ratio $n^* > 0$ satisfies

$$\alpha'(n^*)\tilde{s}(n^*;\{q_a^*\}_{a\in A}) + \alpha(n^*)\tilde{s}'(n^*;\{q_a^*\}_{a\in A}) = k. \tag{7}$$

3. For any $\pi \in (0,1]$, the distribution of chosen goods is given by (5).

Equation (7) can be rewritten as a version of the *generalized Hosios condition* derived in Mangin and Julien (2021). This condition generalizes the well-known Hosios (1990) condition, which states that entry is constrained efficient only if sellers' surplus share equals the elasticity of the matching probability for buyers, $\alpha(n)$, with respect to the seller-buyer ratio. Defining the *matching elasticity* by $\eta_\alpha(n) \equiv \alpha'(n)n/\alpha(n)$ and the *surplus elasticity* by $\eta_s(n) \equiv \tilde{s}'(n)n/\tilde{s}(n)$, condition (7) says

$$\underbrace{\eta_\alpha(n)}_{\text{matching elasticity}} + \underbrace{\eta_s(n;\{q_a\}_{a\in A})}_{\text{surplus elasticity}} = \underbrace{\frac{nk}{\alpha(n)\tilde{s}(n;\{q_a\}_{a\in A})}}_{\text{sellers' surplus share}}. \tag{8}$$

We have not yet discussed equilibrium, but it is useful to refer to the term on the right as the sellers' surplus share. Given that our equilibrium features free entry of sellers at cost k, sellers' total expected payoff will be equal to the total cost of seller entry, nk, and the total surplus created is $\alpha(n)\tilde{s}(n)$. Therefore, the term on the right will be sellers' surplus share in equilibrium. The generalized Hosios condition (8) says that constrained efficiency requires sellers' surplus share to be equal to the matching elasticity plus the surplus elasticity.

Since s_a^* is increasing in a, Lemma 1 implies that the expected trade surplus $\tilde{s}(n)$ is increasing in the seller-buyer ratio.[12] Therefore, the surplus elasticity $\eta_s(n)$ is positive. Intuitively, more sellers per buyer means greater choice for buyers, which increases both the average quality of chosen goods and the quantities traded (since q_a^* is increasing in a), thus increasing the average trade surplus. Equivalently, there is a positive externality arising from the effect of seller entry on the average surplus when there is consumer choice. When the generalized Hosios condition (8) holds, both the search externalities and this "choice externality" are internalized.

[12]It is established in the proof of Proposition 1 that both q_a^* and s_a^* are increasing in a.

5 Competitive search equilibrium

Competitive search is an equilibrium concept developed in Moen (1997) and Shimer (1996). A large literature on directed or competitive search has followed. The basic idea is that either buyers or sellers, or market makers, can post prices or contracts that specify the terms of trade offered. Search is directed in the sense that buyers and sellers choose which *submarket* to enter, where each submarket corresponds to a particular specification of the terms of trade. Commitment is key: buyers and sellers who enter a submarket *commit* to trade at the terms specified within that submarket. Within each submarket, there are search frictions.

As in Rocheteau and Wright (2005), we assume there are agents called "market makers" who can open submarkets by posting terms of trade or contracts.[13] Market makers take into account the expected relationship between the posted terms of trade or contracts and the seller-buyer ratio n. In our environment, market makers post contracts $\{(q_a, d_a)\}_{a \in A}$ which specify the quantity of the good q_a and the payment in real dollars d_a *contingent on the buyer's utility shock for their chosen seller.*

Within each submarket, meetings take place, buyers choose sellers, and trade occurs as described in Section 3.

Within meetings – and also within matches between each buyer and their chosen seller – buyers' utility shocks are private information and they cannot be observed directly by sellers. However, buyers may choose to reveal their private information within matches through their choice of contract (q_a, d_a) offered by the chosen seller. By the revelation principle, it is without loss of generality to focus on incentive-compatible direct mechanisms $\{(q_a, d_a)\}_{a \in A}$ that induce buyers to truthfully reveal their private information to their chosen sellers.

Within each period, the timing is as follows. At the start of each day, the market makers announce the submarkets $\{(q_a, d_a)\}_{a \in A}$ that will be open that night, implying an expected n for each submarket. During the day, agents trade in the centralized market and readjust their real balances, and then choose a submarket in which to trade at night, in a manner consistent with expectations. During the night, agents trade goods and money in the decentralized market in their chosen submarket, where they are bound by the posted contracts $\{(q_a, d_a)\}_{a \in A}$ in that submarket.

[13] While these "market makers" are not able to clear the market, we use this term in order to be consistent with the terminology in Rocheteau and Wright (2005).

Let Ω denote the set of open submarkets, where each submarket $\omega \in \Omega$ is characterized by $(\{(q_a, d_a)\}_{a\in A}, n)_\omega$. Let W^b and W^s denote the value functions for buyers and sellers respectively in the day market, and let V^b and V^s denote the value functions for buyers and sellers respectively in the night market.

Centralized market. In the CM, a buyer with real balance z solves:

$$W^b(z) = \max_{\hat{z},x,y\in\mathbb{R}_+} \{\nu(x) - y + \beta V^b(\hat{z})\}, \tag{9}$$

subject to $\hat{z} + x = z + T + y$, where T is her real transfer and $\hat{z}$ is the real balances carried forward into that period's decentralized market. Substituting into (9) yields

$$W^b(z) = z + T + \max_{\hat{z},x\in\mathbb{R}_+} \{\nu(x) - x - \hat{z} + \beta V^b(\hat{z})\}. \tag{10}$$

Thus, the buyer's $\hat{z}$ is independent of z, and $W^b(z) = z + W^b(0)$, which is linear.

Similarly, a seller with real balance z_s in the centralized market solves:

$$W^s(z_s) = \max_{\hat{z},x,y\in\mathbb{R}_+} \left\{\nu(x) - y + \beta \max\left[V^s(\hat{z}), W^s\left(\frac{\hat{z}}{\gamma}\right)\right]\right\}, \tag{11}$$

subject to $\hat{z} + x = z_s + y$. Substituting into (11), we obtain

$$W^s(z_s) = z_s + \max_{\hat{z},x\in\mathbb{R}_+} \left\{\nu(x) - x - \hat{z} + \beta \max\left[V^s(\hat{z}), W^s\left(\frac{\hat{z}}{\gamma}\right)\right]\right\}. \tag{12}$$

Thus, the seller's $\hat{z}$ is independent of z_s, and $W^s(z_s) = z_s + W^s(0)$.

Decentralized market. The equilibrium distribution of chosen goods $\tilde{G}$ is given by buyers' optimal choices of sellers. In any meeting, the buyer chooses the seller that maximizes $v_a \equiv au(q_a) - d_a/\gamma$, the buyer's ex post trade surplus.

For a seller in the decentralized night market,

$$V^s(z_s) = \max_{\omega\in\Omega} \left\{\frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[-c(q_a) + W^s\left(\frac{z_s + d_a}{\gamma}\right)\right] d\tilde{G}(a; n) + \left[1 - \frac{\alpha(n)}{n}\right] W^s\left(\frac{z_s}{\gamma}\right)\right\} - k \tag{13}$$

where each submarket $\omega \in \Omega$ is characterized by $(\{(q_a, d_a)\}_{a\in A}, n)$. A seller chooses ω among the set of open submarkets and has the opportunity to trade only if chosen.

While all sellers meet exactly one buyer, the probability a seller is *chosen* is $\alpha(n)/n$. It is straightforward to verify that the seller's choice of real balances is $\hat{z} = 0$.[14]

For a buyer in the decentralized night market,

(14)

$$V^b(z) = \max_{\omega \in \Omega} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \mathbf{1}_a \left[au(q_a) + W^b \left(\frac{z - d_a}{\gamma} \right) \right] d\tilde{G}(a;n) + \left[1 - \alpha(n) \int_{a_0}^{\bar{a}} \mathbf{1}_a \, d\tilde{G}(a;n) \right] W^b \left(\frac{z}{\gamma} \right) \right\}$$

where $\mathbf{1}_a$ is an indicator function that is equal to one if $z \geq d_a$ and zero otherwise. A buyer chooses ω among the set of open submarkets and gets the opportunity to trade if she meets at least one seller and has sufficient money z to pay the posted d_a for her chosen good. If she either fails to meet a seller or does not have sufficient money, she does not trade. Using $W^b(z) = z + W^b(0)$ we obtain

$$V^b(z) = \max_{\omega \in \Omega} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \mathbf{1}_a \left[au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a;n) + \frac{z}{\gamma} + W^b(0) \right\}. \tag{15}$$

Thus, the buyer's choice of z from (10) is given by

$$\max_{z \in \mathbb{R}_+} \left\{ -z + \beta \max_{\omega \in \Omega} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a;n) + \frac{z}{\gamma} \right\} \right\} \tag{16}$$

subject to the liquidity constraint, $d_a \leq z$ for all $a \in A$.

Defining $i \equiv \frac{\gamma - \beta}{\beta}$, the nominal interest rate, the above problem is equivalent to

$$\max_{z \in \mathbb{R}_+,\, \omega \in \Omega} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a;n) - i \frac{z}{\gamma} \right\}, \tag{17}$$

subject to $d_a \leq z$ for all $a \in A$ plus the constraint that a submarket with posted contracts $\{(q_a, d_a)\}_{a \in A}$ will attract measure n of sellers per buyer, where n satisfies

$$\frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[-c(q_a) + \frac{d_a}{\gamma} \right] d\tilde{G}(a;n) \leq k \tag{18}$$

and $n \geq 0$ with complementary slackness. Due to buyers' private information, we impose some additional constraints on problem (17), which we discuss next.

[14] Using $W^s(z_s) = z_s + W^s(0)$, (13) simplifies to $V^s(z_s) = z_s/\gamma + V^s(0)$. Substituting into (12), the choice of $\hat{z}$ is given by the first order condition $-1 + \beta/\gamma \leq 0$, where $-1 + \beta/\gamma \; = 0$ if $\hat{z} > 0$. Since we only consider the case $\gamma = \beta$ by taking the limit as $\gamma \to \beta$ from above, $\hat{z} = 0$.

5.1 Existence, uniqueness, and characterization

We focus on incentive-compatible direct mechanisms that induce buyers to reveal their private information to their chosen sellers. Given this, we need to impose on problem (17) two additional constraints: an incentive compatibility (IC) constraint and an individual rationality (IR) constraint. The IR constraint for buyers is

$$au(q_a) - \frac{d_a}{\gamma} \geq 0 \tag{19}$$

for all $a \in A$. This condition states that buyers must receive a (weakly) positive ex post trade surplus, otherwise they will not trade. The IC constraint is given by

$$au(q_a) - \frac{d_a}{\gamma} \geq au(q_{a'}) - \frac{d_{a'}}{\gamma} \tag{20}$$

for all $a, a' \in A$. Intuitively, this condition states that a buyer with utility shock a cannot do better by choosing a contract $(q_{a'}, d_{a'})$ instead of (q_a, d_a).

We restrict attention to steady-state monetary equilibria where $z > 0$ and $n > 0$. We will later prove that there is a unique solution to the market makers' problem and thus there is only one active submarket in equilibrium. Anticipating this result, we simply denote equilibrium by $(\{(q_a, d_a)\}_{a \in A}, z, n)$ and define it as follows.

Definition 1. *A* competitive search equilibrium *is a list* $(\{(q_a, d_a)\}_{a \in A}, z, n)$ *and a distribution of chosen goods* $\{\tilde{G}(a; n)\}_{a \in A}$ *where* $(q_a, d_a) \in \mathbb{R}_+^2$ *for all* $a \in A$, $\tilde{G}(a; n) \in [0, 1]$ *for all* $a \in A$, *and* $z, n \in \mathbb{R}_+ \backslash \{0\}$, *such that* $(\{(q_a, d_a)\}_{a \in A}, z, n)$ *maximizes (17) subject to constraint (18), the liquidity constraint* $d_a \leq z$ *for all* $a \in A$, *plus the IR constraint (19) and the IC constraint (20), and* $\{\tilde{G}(a; n)\}_{a \in A}$ *represents buyers' optimal choices of sellers.*

Lemma 2 tells us that there may exist a non-empty range of utility shocks a such that trade does not occur in equilibrium, i.e. $q_a = 0$. When the good chosen by a buyer within a meeting falls within this range, we call such meetings *no-trade meetings*. There may also exist a non-empty range of utility shocks such that buyers' purchases are constrained by their money holdings, i.e. $d_a = z$. When the good chosen by a buyer within a meeting falls within this range, we call such meetings *liquidity constrained*. Outside of these ranges, we refer to meetings as *unconstrained*.

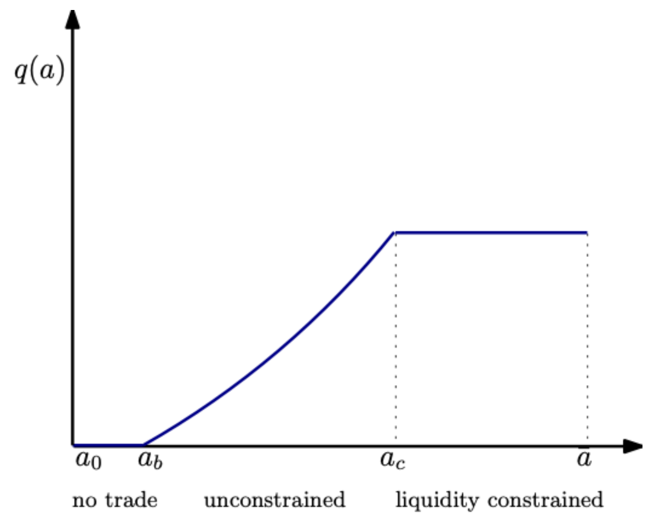

Figure 1: Example of no-trade, unconstrained, and liquidity constrained ranges

Lemma 2. *In any equilibrium where $i > 0$, there exist a_b, $a_c \in A$ such that*

1. *No-trade range: $q_a = 0$ and $d_a = 0$ for all $a \in [a_0, a_b]$.*
2. *Unconstrained range: $q_a > 0$ and $d_a < z$ for all $a \in (a_b, a_c)$.*
3. *Liquidity constrained range: $q_a = q_{a_c} > 0$ and $d_a = z$ for all $a \in [a_c, \bar{a}]$.*

Before presenting Proposition 2, it will be useful to define $\rho(a;n) \equiv 1 - \tilde{G}(a;n)$, the probability that a chosen good has quality greater than a. Applying Lemma 2, the probability that a meeting results in trade is given by $\rho(a_b;n) = 1 - \tilde{G}(a_b;n)$. We also define $\varepsilon_\rho(a;n) \equiv -a\rho'(a;n)/\rho(a;n)$, the elasticity of $\rho(a;n)$ with respect to a, where $\rho'(a;n) \equiv \frac{\partial \rho(a;n)}{\partial a}$. This elasticity can be calculated as follows:

$$\varepsilon_\rho(a;n) = \frac{a\tilde{g}(a;n)}{1 - \tilde{G}(a;n)}. \tag{21}$$

For simplicity, we assume $a_0 = 0$ throughout the rest of the paper. We also make the following assumptions, which ensure the existence of equilibrium.

Definition 2. *The* virtual valuation function $\psi_G : A \to \mathbb{R}$ *is given by*

$$\psi_G(a) \equiv a - \frac{1 - G(a)}{g(a)}. \tag{22}$$

Assumption 3. *The distribution is regular, i.e.* $\psi'_G(a) > 0$.

The requirement that the virtual valuation function is strictly increasing, i.e. $\psi'_G(a) > 0$, is known as *regularity* in the mechanism design literature.[15]

Assumption 4. *The cost of entry is not too high:* $E_G[au(q_a^0) - c(q_a^0)] > k$.

Assumption 4 says the expected trade surplus in the limit as $n \to 0$ must be greater than k, otherwise no sellers enter. Since $\tilde{G} \to G$ as $n \to 0$ by Lemma 1, $\lim_{n\to 0} \tilde{s}(n) = E_G[au(q_a^0) - c(q_a^0)]$ where $q_a^0 \equiv \lim_{n\to 0} q_a(n)$ is given by Lemma 3.[16]

Lemma 3. *For all* $a \in [a_0, a_b]$, $q_a^0 = 0$ *and, for all* $a \in (a_b, \bar{a}]$, q_a^0 *satisfies*

$$\left(a - \frac{1 - G(a)}{g(a)}\right) u'(q_a) = c'(q_a) \tag{23}$$

where $a_b^0 \in [a_0, \bar{a})$ *is the unique solution to* $\psi_G(a) = 0$.

We can now present our main result, which establishes the existence and uniqueness of equilibrium and provides a characterization. The requirement that $G''(a) \leq 0$ is a sufficient but not a necessary condition for existence.

Proposition 2. *Suppose that* $G''(a) \leq 0$ *for all* $a \in A$. *For any* $\pi \in (0, 1]$ *and* $i > 0$, *there exists a unique competitive search equilibrium and it satisfies:*

1. *No-trade range. For any* $a \in [a_0, a_b]$, $q_a = 0$ *and* $d_a = 0$.

2. *Unconstrained range. For any* $a \in (a_b, a_c]$, *the quantity* $q_a > 0$ *solves:*

$$(a - \phi(a; n))u'(q_a) = c'(q_a) \tag{24}$$

where

$$\phi(a; n) = \left(1 - \frac{1}{\delta}\right)\left(\frac{1 - \tilde{G}(a; n)}{\tilde{g}(a; n)}\right) - \left(\frac{1}{\delta}\right) \frac{i}{\alpha(n)\tilde{g}(a; n)} \tag{25}$$

[15] Regularity does not entail much loss of generality as this condition is weaker than both the increasing hazard rate condition and log-concavity.

[16] Assumption 4 is more complicated than Assumption 2 because q_a depends on n in equilibrium, but the planner's solution q_a^* is independent of n.

and

$$\delta = \frac{1}{1-\varepsilon_\rho(a_b;n)}\left(1+\frac{i}{\alpha(n)\rho(a_b;n)}\right). \tag{26}$$

Also, $d_a/\gamma = au(q_a) - \int_{a_0}^{a} u(q_x)dx$.

3. *Liquidity constrained range. For any* $a \in [a_c, \bar{a}]$, $q_a = q_{a_c}$ *and* $d_a = d_{a_c}$.

4. *The value of* a_c *satisfies*

$$\int_{a_c}^{\bar{a}} (a-a_c)\tilde{g}(a;n)da = -\left(1-\frac{1}{\delta}\right)\int_{a_c}^{\bar{a}}[\tilde{G}(a;n)-\tilde{G}(a_c;n)]da+\left(\frac{1}{\delta}\right)\frac{i\bar{a}}{\alpha(n)}. \tag{27}$$

5. *Real money holdings* $z > 0$ *is given by* $z = d_{a_c}$.

6. *The seller-buyer ratio* $n > 0$ *is strictly decreasing in* k *and satisfies*

$$\alpha'(n)\tilde{s}(n;\{q_a\}_{a\in A}) + \alpha(n)\tilde{s}'(n;\{q_a\}_{a\in A}) = k. \tag{28}$$

7. *The zero profit condition is satisfied:*

$$\frac{\alpha(n)}{n}\int_{a_0}^{\bar{a}}\left[-c(q_a)+\frac{d_a}{\gamma}\right]d\tilde{G}(a;n) = k. \tag{29}$$

8. *For any* $\pi \in (0,1]$, *the distribution of chosen goods is given by*

$$\tilde{G}(a;n) = \pi\left(\frac{e^{-n(1-G(a))}-e^{-n}}{1-e^{-n}}\right) + (1-\pi)G(a). \tag{30}$$

The equilibrium distribution of chosen goods $\tilde{G}$ is the same as the planner's. With probability π the buyer can observe the utility shocks a prior to choosing a seller. In this case, buyers always choose the highest quality seller they meet. The distribution of chosen goods therefore equals the distribution across buyers of the highest quality a among the sellers a buyer meets, conditional on meeting at least one seller. With probability $1-\pi$, the buyer observes the shock only after choosing a seller. In this case, buyers randomize across the sellers they meet. The distribution of chosen goods is therefore equal to the distribution of available goods,

G. In general, for any $\pi \in (0,1]$, the cdf of the equilibrium distribution $\tilde{G}$ is a weighted average of these two possibilities. In the limit as $\pi \to 0$, we have $\tilde{G} \to G$.

A version of the generalized Hosios condition holds *endogenously* in our environment featuring competitive search since the equilibrium condition (28) is equivalent in form to the planner's condition (7). The only difference between the equilibrium condition (28) and the planner's condition (7) is that the quantities q_a traded in equilibrium may be different than the efficient quantities q_a^*. Since the expected trade surplus $\tilde{s}(n; \{q_a\}_{a \in A})$ depends not only on the seller-buyer ratio n but also on the quantities q_a, seller entry is not necessarily efficient. However, seller entry is efficient *provided that the quantity traded is efficient*, i.e. $q_a = q_a^*$ for all $a \in A$. Note that in the limit as $\pi \to 0$, the standard Hosios condition applies.

In equilibrium, the endogenous value of a_b may or may not be equal to a_0. If $a_b = a_0$, we refer to the equilibrium as *full trade* because all meetings result in trade.[17] Alternatively, if $a_b > a_0$, we refer to the equilibrium as *partial trade*.

6 Results

We first present some results regarding consumption and seller entry. Next, we consider whether the Friedman rule restores efficiency.

6.1 Consumption and seller entry

There are two margins for efficiency: the *intensive margin* (related to quantity traded or consumption) and the *extensive margin* (related to seller entry). To fix terminology, we say that there is *underconsumption* of any good of quality a whenever the quantity traded in equilibrium is less than the efficient quantity, i.e. $q_a < q_a^*$, and there is *overconsumption* whenever $q_a > q_a^*$. We say that there is *under-entry* of sellers whenever the equilibrium seller-buyer ratio is less than the efficient ratio, i.e. $n < n^*$, and there is *over-entry* whenever $n > n^*$.

Consumption. Consider expression (25), which gives us the equilibrium quantities for the trading range that is unconstrained, $a \in (a_b, a_c]$. Given that the efficient

[17] While $q_0 = 0$ since we assume $a_0 = 0$, the distribution G is assumed to have no mass points and therefore the probability that a_0 is the quality of a chosen good is zero.

quantity q_a^* satisfies $au'(q_a^*) = c'(q_a^*)$, it is clear that we have *underconsumption* if $\phi(a;n) > 0$, *overconsumption* if $\phi(a;n) < 0$, and *efficient* consumption if $\phi(a;n) = 0$.

To better understand expression (25), we can interpret it as a weighted average of two terms, where the endogenous weights are $1/\delta \in (0, 1]$ and $1 - 1/\delta \in [0, 1)$.

$$\phi(a;n) = \left(1 - \frac{1}{\delta}\right) \underbrace{\left(\frac{1 - \tilde{G}(a;n)}{\tilde{g}(a;n)}\right)}_{\text{weakly positive, } \geq 0} + \left(\frac{1}{\delta}\right) \underbrace{\frac{-i}{\alpha(n)\tilde{g}(a;n)}}_{\text{negative, } <0} \tag{31}$$

Whether or not we have equilibrium overconsumption or underconsumption for a good of quality a depends on the relative weights given to each of these two terms, as well as their values at a. If the positive term dominates, we have underconsumption, while if the negative term dominates we have overconsumption. If the two terms exactly offset each other, we have efficient consumption at quality a.

Proposition 3 describes the three possible equilibrium outcomes in terms of underconsumption or overconsumption ranges for $i > 0$ (as depicted in Figure 2).

Proposition 3. *Let $a_u \equiv \max\{a_c, a_d\}$ where $a_d \equiv a_c - \phi(a_c)$, and let a_p solve $\tilde{G}(a_p;n) = 1 + \frac{i}{\alpha(n)(1-\delta)}$. For any $i > 0$, there are three possible equilibrium outcomes:*

1. *If $a_p \leq a_c$, there is underconsumption on (a_0, a_p), overconsumption on (a_p, a_u), and underconsumption on $(a_u, \bar{a}]$.*
2. *If $a_p \geq a_c$, there is underconsumption on $(a_0, \bar{a}]$.*
3. *If $a_b = a_0$, there is overconsumption on (a_0, a_d) and underconsumption on $(a_d, \bar{a}]$.*

Seller entry. Given that the generalized Hosios condition holds endogenously under competitive search, we know that the equilibrium seller-buyer ratio n is efficient *provided that the quantities traded q_a are efficient*. However, the quantities traded are not efficient whenever $i > 0$ and therefore seller entry is not necessarily efficient. Proposition 4 states that there can be over-entry, under-entry, or efficient entry of sellers outside the Friedman rule. We can find examples of each possibility.

Proposition 4. *In any equilibrium where $i > 0$, there may be either under-entry, over-entry, or efficient entry of sellers.*

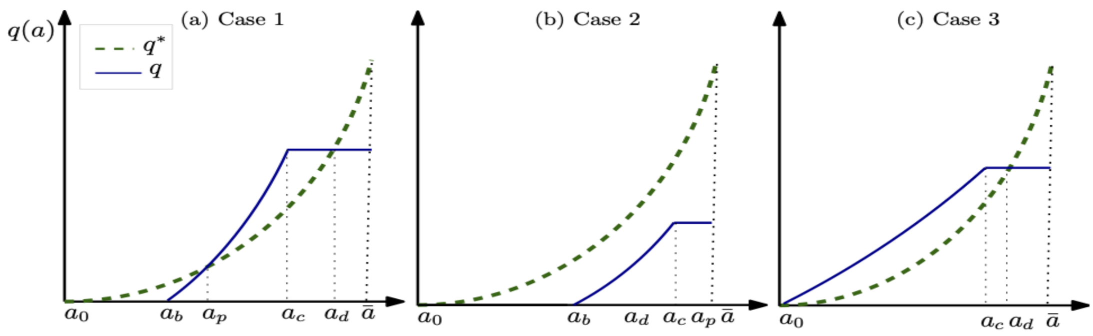

Figure 2: Examples of the three cases of under/over consumption in Proposition 3

While we know that entry must be efficient if the quantity traded is efficient, the converse is not true. There are examples where entry is efficient but the quantities traded are not. When this occurs, the efficiency of entry is really just “coincidental”.

6.2 Does the Friedman rule deliver efficiency?

In RW, there is efficiency along both the intensive and extensive margins when the Friedman rule is imposed. That is, both the quantity traded and the level of entry of sellers are efficient. In our environment, there can be inefficiencies along *both* margins at the Friedman rule. Importantly, these inefficiencies are due to buyers' private information, not the presence of consumer choice.

Corollary 1. *At the Friedman rule ($i \to 0$), for any $\pi \in (0,1]$ equilibrium satisfies:*

1. *No-trade range. For any $a \in [a_0, a_b]$, $q_a = 0$, and $d_a = 0$.*

2. *Unconstrained range. For all $a \in (a_b, \bar{a}]$, the quantity q_a satisfies*

$$\left(a - \varepsilon_\rho(a_b; n)\frac{1 - \tilde{G}(a;n)}{\tilde{g}(a;n)}\right) u'(q_a) = c'(q_a). \tag{32}$$

   Also, $d_a/\gamma = au(q_a) - \int_{a_0}^{a} u(q_x)dx$.

3. *No meetings are liquidity constrained: $a_c = \bar{a}$.*

4. *Parts 5-8 from Proposition 2 hold.*

Proposition 5 says the Friedman rule delivers efficiency if and only if the equilibrium is full trade (i.e. $a_b = a_0$). First, it is clear from (32) that the efficient quantities are traded at the Friedman rule if and only if $\varepsilon_\rho(a_b; n) = 0$, which is true if and only if $a_b = a_0 = 0$. Second, the equilibrium condition (28) is equivalent to the planner's condition given the same function q_a, i.e. *given the quantities traded are efficient.*

Proposition 5. *At the Friedman rule, we have efficiency, i.e.* $n = n^*$ *and* $q_a = q_a^*$ *for all* a*, if and only if the equilibrium is full-trade* ($a_b = a_0$)*.*

Proposition 6 tells us that, in any partial-trade equilibrium where $a_b > a_0$, the Friedman rule results in *underconsumption,* i.e. $q_a < q_a^*$ for all $a \in (a_0, \bar{a})$. Only for two specific qualities of chosen goods, a_0 and $\bar{a}$, are the efficient quantities traded.

Proposition 6. *At the Friedman rule, there is underconsumption for all* $a \in (a_0, \bar{a})$ *if the equilibrium is partial-trade* ($a_b > a_0$)*.*

The reason why the Friedman rule does not yield efficiency along the intensive margin is not only because there is underconsumption in no-trade meetings. Even if we consider meetings that *do* result in trade, there is underconsumption. Intuitively, in any partial trade equilibrium, sellers need to compensate for the fact that there is a range of meetings in which no trade occurs. Sellers compensate for the no-trade meetings by charging higher prices over the trading range, which implies that less than the efficient quantity is consumed even within the trading range.

Given that there is underconsumption at the Friedman rule (unless $a_b = a_0$), seller entry is not necessarily efficient. There may be either under-entry or over-entry of sellers. Therefore, the Friedman rule does not generally deliver efficiency along either the intensive or extensive margin.

Proposition 7. *At the Friedman rule, there can be either under-entry, over-entry, or efficient entry of sellers.*

In our baseline calibration in Section 7, there is *over-entry* of sellers, i.e. $n > n^*$, at the Friedman rule. We can also find examples of under-entry and efficient entry.

7 Calibration

We calibrate the model to match the data from Lucas and Nicolini (2015) on money demand in the U.S. from 1915-2008.[18] The period length is set to one year. We set $\beta = 1/(1+r)$ to match a real interest rate of $r = 0.03$ as in Bethune et al. (2020). We use the 3-month U.S. T-bill rate as a measure of the nominal interest rate i. The average nominal interest rate i for the period 1915-2008 is $i = 0.0383$. Money demand $L(i)$ is defined as $M1/GDP$.

In the model, money demand is $L(i) = z/Y$ where z is real money holdings and Y is real GDP given by $Y = x^* + \alpha(n)\tilde{d}(n)$, where x^* is the quantity consumed in the CM, $\tilde{d}(n) \equiv \int_{a_0}^{\bar{a}} \frac{d_a}{\gamma} d\tilde{G}(a;n)$, the average payment for a chosen good, and $\alpha(n) = 1 - e^{-n}$, the probability a buyer has the opportunity to trade.

We assume that $c(q) = q$ and $u(q) = \frac{(q+\epsilon)^{1-\sigma} - \epsilon^{1-\sigma}}{1-\sigma}$ where $\sigma \in (0,1)$ and $\epsilon \approx 0$. The CM utility function is $\nu(x) = A \log x$. Since $\nu'(x^*) = 1$, we have $x^* = A$. We assume the distribution of utility shocks G is uniform on $[0,1]$.

Parameter		*Target*	
DM utility curvature, $1-\sigma$	0.719	elasticity of money demand, η_L	-0.16
CM utility parameter, A	1.99	average money demand, $L(i)$	0.272
cost of entry, k	0.0184	buyers' surplus share, $\theta(n)$	0.50
degree of choice, π	0.54	decentralized market markup, μ_{DM}	1.30

Table 1: Baseline calibration

Baseline calibration. For our baseline calibration, we calibrate four parameters (A, σ, k, π) to match four targets. The target for the steady state level of money demand in the model, $L(i)$ where $i = 0.0383$, is equal to 0.272, the average money demand in the data for 1915-2008. The target for the elasticity of money demand $L(i)$ with respect to i, denoted by η_L, is equal to -0.16, the elasticity in the data for 1915-2008. Our third target is *buyers' surplus share*, defined by $\theta(n) \equiv \tilde{v}(n)/\tilde{s}(n)$, where $\tilde{v}(n) \equiv \int_{a_0}^{\bar{a}} v_a d\tilde{G}(a;n)$ and $v_a \equiv au(q_a) - \frac{d_a}{\gamma}$. We treat $\theta(n)$ as a proxy for buyers' bargaining power and target $\theta(n) = 0.5$.[19] Our fourth target is the *markup* in the decentralized market. Defining average quantity by $\tilde{q}(n) \equiv \int_{a_0}^{\bar{a}} q_a d\tilde{G}(a;n)$,

[18] Lucas and Nicolini (2015) adjust the measure of $M1$ to generate a stable money demand curve.

[19] Note that $\theta(n) = \theta$ is the value of buyer's surplus share that would deliver the same buyer/seller shares as the Kalai (proportional) bargaining solution with parameter θ.

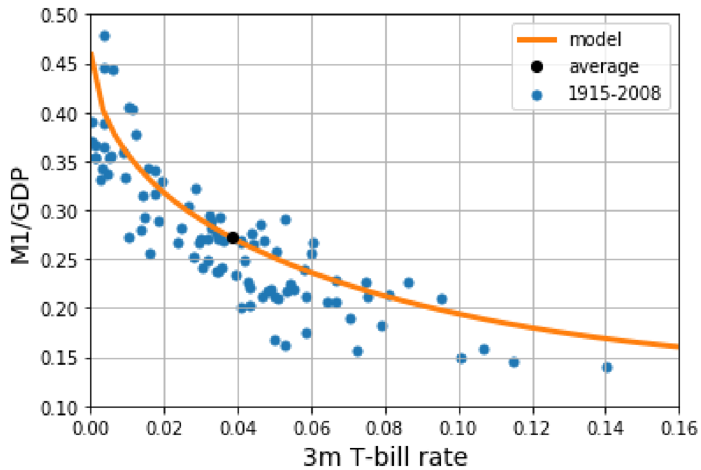

Figure 3: Data vs model predictions for money demand (by nominal interest rate i)

the average unit price is $\tilde{p}(n) \equiv \tilde{d}(n)/\tilde{q}(n)$. The DM markup is defined by $\mu_{DM} \equiv \tilde{p}(n)/c'(q)$, which is equal to the average price $\tilde{p}(n)$ since we assume $c(q) = q$ for our calibration. We follow Berentsen et al. (2011) in targeting a DM markup of $\mu_{DM} = 1.3$ to reflect a retail markup of 30%.

Discussion of calibration strategy. In the monetary search literature featuring bargaining, buyers' bargaining power is a parameter and it is generally calibrated to match either the DM markup or the aggregate markup. Since prices are determined in competitive search equilibrium in our model, we cannot do this because buyers' surplus share is endogenous. However, it is important to ensure that we fix buyers' surplus share when we compare calibrations for different values of π in Section 8 because the cost of inflation depends strongly on buyers' surplus share, as discussed in Craig and Rocheteau (2008). Our strategy is to ensure that we "fix" buyers' surplus share at steady state through our choice of k and match the DM markup through our choice of π. Given that we can match the DM markup as a separate target, the choice of target for buyers' surplus share is somewhat arbitrary, hence we simply set $\theta(n) = 0.5$. In Section 9, we provide a robustness exercise where we vary this target and show that our main result is preserved.

Table 2 provides a summary of the equilibrium outcomes for our baseline calibration. The equilibrium features underconsumption of goods of *all* qualities (i.e. there is no overconsumption). This is an example of the second case of equilibrium

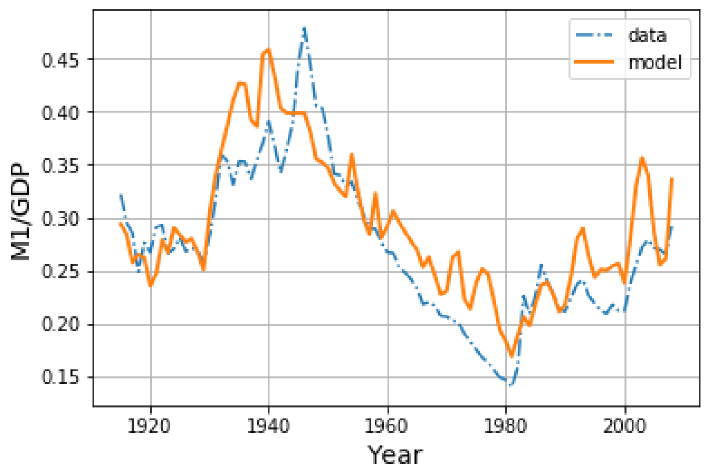

Figure 4: Data vs model predictions for money demand (by year)

described in Proposition 3 and depicted in the middle panel of Figure 2. The equilibrium is also *partial trade*: around 23% of meetings do not result in any trade. Around 33% of meetings and 26% of trades are liquidity constrained. Buyers spend around 41% of their money holdings on average.

We do not target the output share of the decentralized market, but it is around 9%.[20] We also do not target price dispersion, but it is close to the empirical estimates in Kaplan and Menzio (2015). Defining unit prices by $p_a \equiv \frac{d_a/\gamma}{q_a}$ for all traded goods (i.e. DM markup since $c'(q) = 1$), *price dispersion* is defined as the standard deviation of normalized prices across all trades.[21] Price dispersion is 25% at our baseline calibration, which fits well within the range of empirical estimates, 19% to 36%, found in Table 2 of Kaplan and Menzio (2015) and is equal to their estimate of 25% for the broader definition of goods which aggregates brands (but not sizes).[22]

Comparative statics. We provide some comparative statics results for the cost of entry k, the inflation rate $\tau \equiv \gamma - 1$, and the degree of choice π. Table 2 summarizes the effects of a 1% increase in the parameters k, $\gamma \equiv 1 + \tau$, and π from

[20] In the literature, values of the DM output share vary from less than 10% in Lagos and Wright (2005) to 25% in Bethune et al. (2020) and 42% in Berentsen et al. (2011).

[21] Standard deviations are expressed as a percentage of the mean throughout the paper.

[22] We believe the "brand aggregation" measure in Kaplan and Menzio (2015) is the most relevant since goods are not strictly identical in our environment where consumers experience idiosyncratic utility or preference shocks that differ across goods.

our baseline calibration. In Table 2, we can see that greater informed choice among buyers increases seller entry, increases the average quality of a chosen good, and increases the average quantity of goods purchased. Greater choice also increases money holdings and the average payment, as well as increasing the average size of the trade surplus. Buyers' surplus share does not change by much when we increase the degree of choice by 1%, but it decreases slightly at the baseline calibration. The average price or DM markup also decreases slightly at baseline. Total real output or GDP and welfare (defined in Section 8) are both increasing in the degree of choice at baseline. The Online Appendix contains some figures to illustrate the comparative statics over a wider range of parameter values.

	Baseline	$1+\tau$ ($\uparrow$ inflation)	k ($\uparrow$ cost)	π ($\uparrow$ choice)
seller-buyer ratio, n	3.08	-2.3%	-1.1%	0.9%
meeting prob, $\alpha(n)$	0.95	-0.4%	-0.2%	0.1%
average quality, $\tilde{a}(n)$	0.62	-0.4%	-0.2%	0.3%
average quantity, $\tilde{q}(n)$	0.20	-7.3%	-0.7%	0.9%
average payment, $\tilde{d}(n)$	0.26	-6.1%	-0.5%	0.9%
money holdings, z/γ	0.60	-8.5%	0.1%	0.3%
average surplus, $\tilde{s}(n)$	0.12	-2.6%	-0.5%	0.7%
buyer share, $\theta(n)$	0.50	-0.6%	-0.6%	-0.0%
price or markup, $\tilde{p}(n)$	1.30	1.3%	0.2%	-0.0%
price dispersion	0.25	1.1%	0.6%	-0.1%
total real output, Y	2.23	-0.7%	-0.1%	0.1%
total welfare, W	0.43	-0.5%	-0.2%	0.1%

Table 2: Equilibrium outcomes and comparative statics at baseline calibration

8 Welfare cost of inflation

In this section, we present our estimates of the welfare cost of inflation and show how it varies with the degree of informed choice by consumers. We start by defining total welfare in economy E by[23]

$$W(E) = \alpha(n) \int_{a_0}^{\bar{a}} [au(q_a) - c(q_a)] d\tilde{G}(a; n) - nk + \nu(x^*) - x^* + 1. \tag{33}$$

[23] Note that adding one is a normalization that ensures $W(E)$ is positive for all calibrations we consider. It does not affect our estimates of the welfare cost of inflation.

Since consumers' utility depends on both quality and quantity, in order to calculate the consumption sacrifice in terms of quantity alone we first convert to a welfare-equivalent "representative" economy in which the quantity of goods traded is constant and quality is normalized to one. That is, we find quantity q such that

$$W(E) = \alpha(n)[u(q) - c(q)] - nk + \nu(x^*) - x^* + 1. \tag{34}$$

If total consumption is multiplied by a factor of $\Delta \in [0, 1]$, then welfare is given by

$$W(E, \Delta) = \alpha(n)[u(\Delta q) - c(q)] - nk + \nu(\Delta x^*) - x^* + 1. \tag{35}$$

We measure the welfare cost of moving from economy E to E' by the share of total consumption that consumers are willing to give up in order to go from economy E' to E. That is, the cost is $1 - \Delta$ where $\Delta \in [0, 1]$ satisfies $W(E, \Delta) = W(E')$.

We compute the welfare cost of 10% inflation relative to both 0% inflation and the Friedman rule. In particular, we find $\Delta_0 \in [0, 1]$ such that $W(\gamma = 1, \Delta_0)$ is equal to $W(\gamma = 1.1, \Delta = 1)$. The value $1 - \Delta_0$ is the percentage of total consumption that consumers are willing to give up in order to go from 10% inflation to 0% inflation. We also find $\Delta_F \in [0, 1]$ such that $W(\gamma = \beta, \Delta_F)$ is equal to $W(\gamma = 1.1, \Delta = 1)$. The value $1 - \Delta_F$ is the percentage of total consumption that consumers are willing to give up in order to go from 10% inflation to the Friedman rule.

8.1 How does consumer choice affect the cost of inflation?

As we would expect, consumer choice increases the level of welfare. Starting at our random choice calibration, ($\pi = 0$), we estimate that increasing the degree of choice to our baseline level ($\pi = 0.54$) delivers a welfare gain worth 1.58% of total consumption. Similarly, starting at our baseline degree of choice ($\pi = 0.54$), an increase in the degree of choice to $\pi = 1$ delivers a welfare gain worth 2.75% of total consumption. The positive effect of greater choice on welfare is intuitive. A greater degree of informed choice by consumers increases both the average quality and the average quantity traded, as well as increasing seller entry. However, the effect of choice on the welfare cost of inflation is not clear.

Figure 5 illustrates how the welfare cost of inflation varies with the degree of choice π. For any given value of π, we recalibrate the parameters (A, σ, k) to match

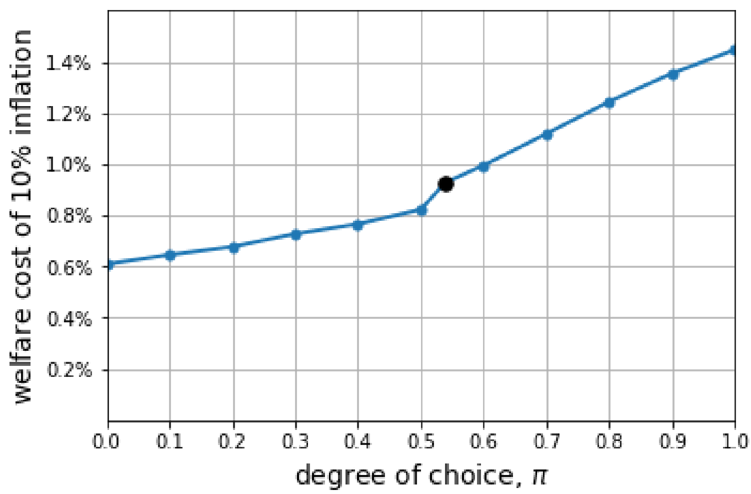

Figure 5: Welfare cost of 0% to 10% inflation for different values of π (recalibrated)

the first three targets of our baseline calibration. Figure 5 shows that the cost of inflation is strictly increasing with the degree of consumer choice π.

Table 3 provides our estimates of the welfare cost of inflation. Recall that $1-\Delta_0$ (or $1-\Delta_F$) denotes the welfare cost of moving from 0% (or the Friedman rule) to 10% inflation. We focus on comparing our baseline calibration ($\pi = 0.54$), full choice calibration ($\pi = 1$), and random choice calibration ($\pi = 0$). Details of the full choice and random choice calibrations can be found in Appendix A.

	$1-\Delta_0$	$1-\Delta_F$
Random ($\pi = 0$)	0.61%	0.79%
Baseline ($\pi = 0.54$)	0.93%	1.11%
Full choice ($\pi = 1$)	1.45%	1.64%

Table 3: Welfare cost of inflation (baseline, full choice, and random choice)

At our baseline calibration ($\pi = 0.54$), the cost of increasing inflation from 0% to 10% is 0.93% of consumption, while the cost of moving from the Friedman rule to 10% inflation is 1.11% of consumption. When we recalibrate the model after imposing random choice ($\pi = 0$), the welfare cost of increasing inflation from 0% to 10% is 0.61% of consumption and the cost of moving from the Friedman rule to 10% inflation is 0.79% of consumption. On the other hand, when we recalibrate the model with full choice ($\pi = 1$), the cost of increasing inflation from 0% to 10% is

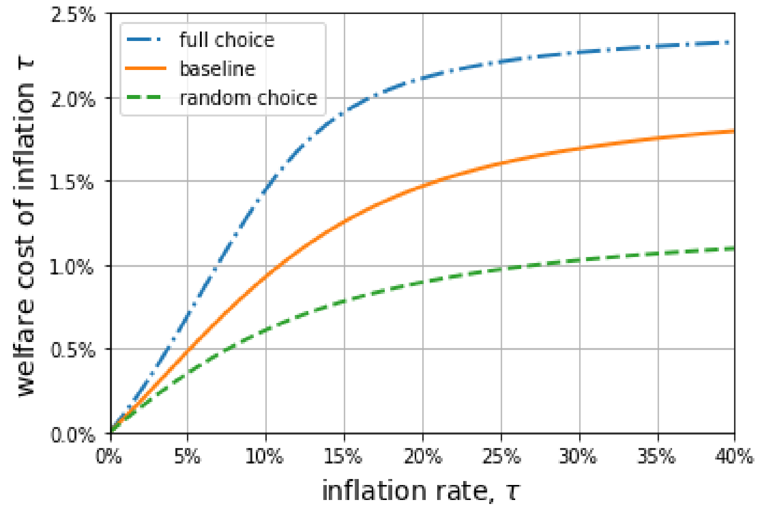

Figure 6: Welfare cost of 0% to τ inflation for different inflation rates τ

more than twice as high at 1.45% of consumption, while the cost of moving from the Friedman rule to 10% inflation is 1.64% of consumption.[24]

Figure 6 depicts the welfare cost of increasing inflation from 0% to τ for various inflation rates τ at our baseline, random choice, and full choice calibrations. We can see that a welfare cost of 1% of consumption requires an inflation rate of around 11% in our baseline calibration. With random choice, a very high inflation rate of around 28% is required for the same welfare cost. With full choice, a relatively low inflation rate of around 7% delivers the same welfare cost. This suggests that economies featuring a greater degree of informed choice can experience the same extent of negative welfare effects from lower levels of inflation.

8.2 Why is the cost of inflation higher with consumer choice?

To understand better the negative effects of inflation on welfare in our model, Table 4 shows how the equilibrium outcomes change when the economy shifts from either the Friedman rule or 0% inflation to 10% inflation at the baseline calibration.

[24] In Section 9, we show that our main result – greater choice increases the cost of inflation – still holds when we vary π and adjust k to match the target surplus share, while keeping the utility parameters (A, σ) at their baseline levels. This confirms that the difference in the welfare cost is driven by variation in the degree of choice π, not by differences across calibrations in either the utility parameters (A, σ) or the buyer surplus share $\theta(n)$.

We also include the efficient outcomes (given baseline $\pi = 0.54$) for comparison.[25]

	Efficient	Friedman rule	0% inflation	10% inflation
seller-buyer ratio, n	3.26	3.28	3.13	2.46
meeting prob, $\alpha(n)$	0.96	0.96	0.96	0.91
average quality, $\tilde{a}(n)$	0.63	0.63	0.62	0.60
average quantity, $\tilde{q}(n)$	0.35	0.27	0.21	0.11
average payment, $\tilde{d}(n)$	-	0.33	0.27	0.16
money holdings, z/γ	-	1.13	0.65	0.33
average surplus, $\tilde{s}(n)$	0.14	0.13	0.12	0.09
buyer share, $\theta(n)$	-	0.51	0.50	0.46
price or markup, $\tilde{p}(n)$	-	1.23	1.28	1.46
price dispersion	-	0.24	0.25	0.27
total real output, Y	-	2.31	2.25	2.13
total welfare, W	0.45	0.44	0.44	0.42

Table 4: Equilibrium outcomes at different inflation rates (baseline calibration)

When the economy shifts from 0% to 10% inflation, the seller-buyer ratio falls by 21.4%. As a result, the meeting probability for buyers falls and average quality drops by 3.4%. Money holdings fall dramatically by 48.8%, while average quantity traded decreases by 49.2%, average payment falls by 42.3%, and average surplus drops by 24.3%. As inflation jumps from 0% to 10%, buyers' surplus share falls by 8.5%. The average price or DM markup rises by 13.7% and price dispersion rises by 9.6%. Total real output or GDP decreases by 5.2% and welfare falls by 4.5%.

The effects of inflation at our baseline calibration lie somewhere in between the effects at the two extremes of full choice ($\pi = 1$) and random choice ($\pi = 0$). To see how the sensitivity of various equilibrium outcomes to changes in inflation varies with the degree of choice π, Table 5 compares the comparative statics effect of a 1% increase in the parameter $1 + \tau$ (for inflation rate τ) for our three calibrations.

As Table 5 shows, greater choice *amplifies* the sensitivity of the economy to changes in inflation. First of all, it increases the sensitivity of seller entry to inflation. In response to a 1% increase in $1 + \tau$, the seller-buyer ratio falls by 3.0% with full choice compared to just 1.4% with random choice. Consumer choice also results in a higher sensitivity of average quality to changes in inflation, since average quality is unchanged with random choice. At the same time, average quantity, average payments and money holdings are also more sensitive to changes in inflation when

[25] Notice that we have *over-entry* of sellers at the Friedman rule relative to the efficient allocation.

	Random ($\pi = 0$)	Baseline ($\pi = 0.54$)	Full choice ($\pi = 1$)
seller-buyer ratio, n	-1.4%	-2.3%	-3.0%
meeting prob, $\alpha(n)$	-0.7%	-0.4%	-0.0%
average quality, $\tilde{a}(n)$	0.0%	-0.4%	-0.5%
average quantity, $\tilde{q}(n)$	-6.1%	-7.3%	-9.2%
average payment, $\tilde{d}(n)$	-4.4%	-6.1%	-8.3%
money holdings, z/γ	-7.3%	-8.5%	-9.7%
average surplus, $\tilde{s}(n)$	-1.7%	-2.6%	-3.7%
buyer share, $\theta(n)$	-1.1%	-0.6%	-0.8%
price or markup, $\tilde{p}(n)$	1.8%	1.3%	1.0%
price dispersion	1.5%	1.1%	1.3%
total real output, Y	-0.3%	-0.7%	-1.6%
total welfare, W	-0.3%	-0.5%	-0.9%

Table 5: Effect of a 1% increase in $1 + \tau$ (inflation τ) for baseline, full choice, random

there is greater choice. Finally, the sensitivity of average surplus to changes in inflation is amplified by greater choice. In response to a 1% increase in $1 + \tau$, average surplus falls by 3.7% with full choice compared to just 1.7% with random choice. The effects at baseline π fall somewhere in between these two extremes.

9 Robustness

In this section, we establish the robustness of our main result that the welfare cost of inflation is increasing in the degree of choice π. First, we consider how our results change when we vary the target surplus share, which is $\theta(n) = 0.5$ for our baseline calibration. Second, we present the results of two different experiments where we shut down either endogenous seller entry or endogenous surplus shares.

9.1 Effect of surplus share target

Table 6 reports our estimates of the welfare cost of inflation when we vary the target value of buyers' surplus share and recalibrate the model using the same strategy (for baseline, full choice, and random choice). When we vary the target surplus share, baseline π also varies and equals the value π_μ that matches the DM markup.

While our exact estimates of the cost of inflation depend on the target value of buyers' surplus share, it is clear from Table 6 that the cost of inflation is increasing in the degree of choice for every level of the target buyers' surplus share.

	$\theta(n) = 0.4$	$\theta(n) = 0.5$	$\theta(n) = 0.6$
Random ($\pi = 0$)	0.25%	0.61%	0.81%
Baseline ($\pi = \pi_\mu$)	0.55%	0.93%	0.94%
Full choice ($\pi = 1$)	0.90%	1.45%	1.65%

Table 6: Welfare cost of 0% to 10% inflation for different values of target buyer share

Since we focus on competitive search, our estimates of the cost of inflation can be viewed as lower bounds when compared to environments featuring bargaining. In such environments, the cost of inflation is sensitive to changes in the bargaining parameter. In Lagos and Wright (2005), the cost of inflation decreases as buyers' bargaining parameter θ increases because the severity of the hold-up problem decreases as $\theta \to 1$. In environments such as ours that feature competitive search, there is no hold-up problem. Buyers' surplus share is endogenous and depends crucially on the equilibrium seller-buyer ratio. As Table 6 shows, the cost of inflation actually *increases* in our model as we increase the target for buyers' surplus share.

9.2 Results of experiments

Our main result regarding the effect of choice on the welfare cost of inflation does not depend on either of two features of our model: (1) endogenous seller entry; and (2) endogenous surplus shares. To demonstrate this, we conduct two experiments. In the first experiment, we shut down endogenous seller entry by fixing the seller-buyer ratio to $n = \bar{n}$. In the second experiment, we shut down endogenous surplus shares by fixing buyers' surplus share to $\theta(n) = \bar{\theta}$.[26] For both experiments, the equilibrium conditions are the same as Proposition 2 except that entry cost k is replaced by endogenous J (where J is equal to equilibrium expected seller utility before entry cost). To calculate welfare, we use definition (33) and set $k = 0$.

Table 7 compares the cost of inflation for our main model and the experiments. We use the same calibration strategy as our main model except we treat $\bar{\theta}$ or $\bar{n}$ as a calibrated parameter (instead of k). For our main results (a) in Table 7, we

[26] In one sense, Experiment 2 is similar to the fixed surplus shares in a model featuring bargaining. However, it is different because we use competitive search and the equilibrium surplus shares are always the efficient ones (conditional on the quantities traded being efficient). This is why we still have relatively low costs of inflation in Experiment 2 compared to bargaining models.

recalibrate (A, σ) to match the money demand targets when we vary π (as for Table 3). We also include an additional check (b) where we keep (A, σ) at the baseline parameters when we vary π. For both (a) and (b), we match the surplus share target $\theta(n) = 0.5$ by adjusting k (or $\bar{\theta}$ or $\bar{n}$) to ensure comparability of welfare costs.

	Main model	Exper 1 ($n = \bar{n}$)	Exper 2 ($\theta(n) = \bar{\theta}$)
(a) Recalibrated (A, σ)			
Random ($\pi = 0$)	0.61%	0.58%	0.58%
Baseline ($\pi = 0.54$)	0.93%	0.87%	1.16%
Full choice ($\pi = 1$)	1.45%	1.22%	1.87%
(b) Baseline (A, σ)			
Random ($\pi = 0$)	0.52%	0.56%	0.56%
Baseline ($\pi = 0.54$)	0.93%	0.87%	1.16%
Full choice ($\pi = 1$)	1.14%	0.96%	1.53%

Table 7: Welfare cost of 0% to 10% inflation for main model and experiments

With random choice, the cost of inflation is the same for both experiments because fixing $\theta(n) = \bar{\theta}$ and fixing $n = \bar{n}$ are equivalent. This is because the standard Hosios condition applies under random choice, i.e. $\eta_\alpha(n) = 1 - \theta(n)$. However, whenever $\pi > 0$ and there is some degree of consumer choice, the generalized Hosios condition applies, i.e. $\eta_\alpha(n) + \eta_s(n; \{q_a\}_{a \in A}) = 1 - \theta(n)$, and fixing the seller-buyer ratio is not equivalent to fixing the surplus shares. As a result, the welfare cost of inflation differs across these two experiments when there is consumer choice.

For both experiments, Table 7 (a) shows that our main result – greater choice increases the welfare cost of inflation – is confirmed when we vary π and recalibrate (A, σ) using the same strategy as our baseline calibration. For Experiment 1 (exogenous n), the cost of inflation is around twice as high with full choice compared to random choice. For Experiment 2 ($\theta(n)$ exogenous), the cost of inflation is more than three times as high with full choice compared to random choice.

Table 7 (b) shows that, for both our main model and both of our experiments, our result that greater choice increases the cost of inflation also holds when we vary π and adjust k to match the target surplus share, but keep the utility parameters (A, σ) equal to the baseline parameters. This confirms that, in all three cases, the difference in the welfare cost of inflation is not due to changes in the utility parameters (A, σ), or differences in the buyer surplus share $\theta(n)$, across calibrations for different π, but is instead due to variation in the degree of choice π.

10 Conclusion

Inflation in the U.S. has reached its highest level for forty years. During this period, the nature of retail trade has changed radically. As a result of the rise of the internet since the 1990s, consumers today face much greater availability of online information about brands and products prior to making purchases. This paper asks the question: How does consumer choice affect the welfare cost of inflation?

To answer this question, we introduce consumer choice into a search-theoretic model of monetary exchange. We find that a greater degree of informed choice by consumers makes inflation significantly more costly for an economy. This suggests that while consumers benefit greatly from the ability to make more informed choices about their purchases, this feature of the economy also makes consumers more vulnerable to the negative welfare effects of inflation.

In future work, we believe it would be interesting to examine in more detail the complex interactions between private information and consumer choice in environments with monetary exchange. We also believe it would be interesting to use our model of consumer choice and monetary exchange to further explore the implications of changes in the structure of retail trade – for example, the rise of online transactions and various online platforms – for monetary policy.

Appendix

Appendix A: Full choice and random choice calibrations

For the full choice calibration, we set $\pi = 1$ and then calibrate the remaining three parameters (A, σ, k) to match the first three targets of our baseline calibration. Table 8 reports the calibrated parameters and targets.

Parameter		*Target*	
DM utility curvature, $1-\sigma$	0.815	elasticity of money demand, η_L	-0.16
CM utility parameter, A	1.75	average money demand, $L(i)$	0.272
cost of entry, k	0.0081	buyers' surplus share, $\theta(n)$	0.50

Table 8: Full choice calibration ($\pi = 1$)

Table 9 summarizes the equilibrium outcomes and the comparative statics effects of a 1% increase in the parameters k, $\gamma \equiv 1+\tau$, and π for the full choice calibration.[27]

	Baseline	$1+\tau$ ($\uparrow$ inflation)	k ($\uparrow$ cost)	π ($\uparrow$ choice)
seller-buyer ratio, n	7.09	-3.0%	-0.9%	1.0%
meeting prob, $\alpha(n)$	1.00	-0.0%	-0.0%	0.0%
average quality, $\tilde{a}(n)$	0.86	-0.5%	-0.1%	0.6%
average quantity, $\tilde{q}(n)$	0.36	-9.2%	-0.8%	1.8%
average payment, $\tilde{d}(n)$	0.41	-8.3%	-0.7%	1.7%
money holdings, z/γ	0.58	-9.7%	-0.1%	0.5%
average surplus, $\tilde{s}(n)$	0.11	-3.7%	-0.5%	1.4%
buyer share, $\theta(n)$	0.50	-0.8%	-0.7%	0.4%
price or markup, $\tilde{p}(n)$	1.16	1.0%	0.1%	-0.1%
price dispersion	0.12	1.3%	1.0%	-1.0%
total real output, Y	2.16	-1.6%	-0.1%	0.3%
total welfare, W	0.29	-0.9%	-0.2%	0.4%

Table 9: Comparative statics for full choice calibration ($\pi = 1$)

For the random choice calibration, we set $\pi = 0$ and then calibrate the remaining three parameters (A, σ, k) to match the first three targets of our baseline calibration. Table 10 reports the calibrated parameters and targets.

[27] Since $\pi = 1$ with full choice, we instead calculate the effect of a 1% *decrease* in π and then reverse the sign in Table 9.

Parameter		*Target*	
DM utility curvature, $1-\sigma$	0.641	elasticity of money demand, η_L	-0.16
CM utility parameter, A	2.06	average money demand, $L(i)$	0.272
cost of entry, k	0.0363	buyers' surplus share, $\theta(n)$	0.50

Table 10: Random choice calibration ($\pi = 0$)

Table 11 summarizes the equilibrium outcomes and the comparative statics effects of a 1% increase in the parameters k, $\gamma \equiv 1+\tau$, and π for the random choice calibration.

	Baseline	$1+\tau$ ($\uparrow$ inflation)	k ($\uparrow$ cost)	π ($\uparrow$ choice)
seller-buyer ratio, n	1.26	-1.4%	-0.9%	1.0%
meeting prob, $\alpha(n)$	0.72	-0.7%	-0.5%	0.5%
average quality, $\tilde{a}(n)$	0.50	0.0%	0.0%	0.2%
average quantity, $\tilde{q}(n)$	0.14	-6.1%	-0.4%	0.6%
average payment, $\tilde{d}(n)$	0.20	-4.4%	-0.1%	0.5%
money holdings, z/γ	0.60	-7.3%	0.4%	0.4%
average surplus, $\tilde{s}(n)$	0.13	-1.7%	-0.2%	0.5%
buyer share, $\theta(n)$	0.50	-1.1%	-0.7%	-0.1%
price or markup, $\tilde{p}(n)$	1.46	1.8%	0.3%	0.0%
price dispersion	0.37	1.5%	0.5%	0.0%
total real output, Y	2.21	-0.3%	0.0%	0.1%
total welfare, W	0.48	-0.3%	-0.1%	0.1%

Table 11: Comparative statics for random choice calibration ($\pi = 0$)

With random choice, the direction of the effects is generally the same as with full choice, but the magnitude is often significantly lower. The only differences in direction are *(i)* average quality, which does not vary in the absence of choice; and *(ii)* money holdings, which are locally decreasing in entry cost with full choice, but increasing with random choice. At our baseline calibration with partial choice ($\pi = 0.54$), money holdings are locally increasing in entry cost, but non-monotonic (and decreasing over most of the parameter range) as Figure 8 shows.

References

S. P. Anderson, A. De Palma, and J.-F. Thisse. *Discrete choice theory of product differentiation*. MIT press, 1992.

A. Berentsen, G. Menzio, and R. Wright. Inflation and unemployment in the long run. *American Economic Review*, 101(1):371–98, February 2011.

Z. Bethune, M. Choi, and R. Wright. Frictional goods markets: Theory and applications. *The Review of Economic Studies*, 87(2):691–720, 2020.

K. Burdett and K. L. Judd. Equilibrium price dispersion. *Econometrica*, pages 955–969, 1983.

M. Choi and G. Rocheteau. A model of retail banking and the deposits channel of monetary policy. *Working paper*, 2021.

T. F. Cooley and G. D. Hansen. The inflation tax in a real business cycle model. *The American Economic Review*, pages 733–748, 1989.

B. Craig and G. Rocheteau. Inflation and welfare: A search approach. *Journal of Money, Credit and Banking*, 40(1):89–119, 2008.

S. M. Davoodalhosseini. Constrained efficiency with adverse selection and directed search. *Journal of Economic Theory*, 183:568–593, 2019.

M. Dong. Inflation and variety. *International Economic Review*, 51(2):401–420, 2010.

M. Dong and J. H. Jiang. Money and price posting under private information. *Journal of Economic Theory*, 150:740–777, 2014.

H. M. Ennis. Search, money, and inflation under private information. *Journal of Economic Theory*, 138(1):101–131, 2008.

M. Faig and B. Jerez. A theory of commerce. *Journal of Economic Theory*, 122(1): 60–99, 2005.

M. Faig and B. Jerez. Inflation, prices, and information in competitive search. *Advances in Macroeconomics*, 6(1):1–34, 2006.

M. Galenianos and P. Kircher. A model of money with multilateral matching. *Journal of Monetary Economics*, 55:1054–1066, 2008.

V. Guerrieri. Inefficient unemployment dynamics under asymmetric information.

Journal of Political Economy, 116(4):667–708, 2008.

V. Guerrieri, R. Shimer, and R. Wright. Adverse selection in competitive search equilibrium. *Econometrica*, 78(6):1823–1862, 2010.

A. Head and A. Kumar. Price dispersion, inflation, and welfare. *International Economic Review*, 46(2):533–572, 2005.

L. Herrenbrueck. An open-economy model with money, endogenous search, and heterogeneous firms. *Economic Inquiry*, 55(4):1648–1670, 2017.

A. J. Hosios. On the Efficiency of Matching and Related Models of Search and Unemployment. *Review of Economic Studies*, 57(2):279–298, 1990.

B. Julien, J. Kennes, and I. King. Bidding for Money. *Journal of Economic Theory*, 142:196–217, 2008.

G. Kaplan and G. Menzio. The morphology of price dispersion. *International Economic Review*, 56(4):1165–1206, 2015.

R. Lagos and R. Wright. A unified framework for monetary theory and policy analysis. *Journal of Political Economy*, 113(3):463–484, 2005.

R. Lagos, G. Rocheteau, and R. Wright. Liquidity: A new monetarist perspective. *Journal of Economic Literature*, 55(2):371–440, 2017.

B. Lester. Information and prices with capacity constraints. *American Economic Review*, 101(4):1591–1600, 2011.

R. E. Lucas and J. P. Nicolini. On the stability of money demand. *Journal of Monetary Economics*, 73:48–65, 2015.

R. J. Lucas. Inflation and welfare. *Econometrica*, 68(2):247–274, 2000.

S. Mangin and B. Julien. Efficiency in search and matching models: A generalized hosios condition. *Journal of Economic Theory*, 193:105208, 2021.

E. Maskin and J. Riley. Monopoly with incomplete information. *The RAND Journal of Economics*, 15(2):171–196, 1984.

G. Menzio. A theory of partially directed search. *Journal of Political Economy*, 115 (5):748–769, 2007.

E. Moen. Competitive Search Equilibrium. *Journal of Political Economy*, 105(2): 385–411, 1997.

E. R. Moen and A. Rosen. Incentives in competitive search equilibrium. *The Review* *of Economic Studies*, 78(2):733–761, 2011.

G. Rocheteau and E. Nosal. *Money, Payments, and Liquidity.* MIT Press, 2017.

G. Rocheteau and R. Wright. Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium. *Econometrica*, 73(1):175–202, 2005.

G. Rocheteau and R. Wright. Inflation and welfare with trading frictions. *Monetary Policy in Low Inflation Economies, ed. Ed Nosal and D. Altig*, 2009.

S. Shi. Sequentially mixed search and equilibrium price dispersion. *Working paper*, 2020.

S. Shi. Money and prices: A model of search and bargaining. *Journal of Economic Theory*, 67(2):467–496, 1995.

S. Shi. A divisible search model of fiat money. *Econometrica*, pages 75–102, 1997.

S. Shi. Search, inflation and capital accumulation. *Journal of Monetary Economics*, 44(1):81–103, 1999.

R. Shimer. PhD Thesis. *M.I.T.*, 1996.

M. Silva. New monetarism with endogenous product variety and monopolistic competition. *Journal of Economic Dynamics and Control*, 75:158–181, 2017.

N. Stokey, R. Lucas, and E. C. Prescott. *Recursive Methods in Economic Dynamics.* Harvard University Press, 1989.

A. Trejos and R. Wright. Search, bargaining, money, and prices. *Journal of Political Economy*, 103(1):118–141, 1995.

R. Wright, P. Kircher, B. Julien, and V. Guerrieri. Directed search and competitive search equilibrium: A guided tour. *Journal of Economic Literature*, 59(1):90–148, 2021.

Online Appendix A: Proofs

Proofs for Section 4

Proof of Lemma 1

Part 1. Since we assume the planner faces the same information as the buyer, with probability $\pi \in (0, 1]$ the planner can observe the utility shocks a prior to choosing a seller. We verify in the proof of Proposition 1 that in this case the planner always chooses the seller with the highest utility shock among those the buyer meets. With probability $1 - \pi$, the planner cannot observe the utility shocks prior to choosing a seller and they simply choose a seller at random.

Using the fact that the distribution of the maximum of $N \geq 1$ draws is $(G(a))^N$, and weighting by the probability $P_N(n)$ that exactly N sellers meet a buyer, conditional on $N \geq 1$, we obtain

$$\tilde{G}(a; n) = \frac{\pi \sum_{N=1}^{\infty} P_N(n)(G(a))^N}{\alpha(n)} + (1 - \pi)G(a). \tag{36}$$

Given that we assume a Poisson distribution, substituting $P_N(n) = \frac{n^N e^{-n}}{N!}$ and $\alpha(n) = 1 - e^{-n}$ into the above yields

$$\tilde{G}(a; n) = \frac{\pi \left(e^{-n} \sum_{N=0}^{\infty} \frac{(nG(a))^N}{N!} - e^{-n} \right)}{1 - e^{-n}} + (1 - \pi)G(a) \tag{37}$$

which, using the fact that $\sum_{N=0}^{\infty} \frac{(nG(a))^N}{n!} = e^{-n(G(a))}$, simplifies to (5).

Part 2. Taking the limit as $n \to 0$, we have

$$\lim_{n \to 0} \tilde{G}(a; n) = \pi \lim_{n \to 0} \left(\frac{e^{-n(1-G(a))} - e^{-n}}{1 - e^{-n}} \right) + (1 - \pi)G(a) = G(a) \tag{38}$$

using L'Hopital's rule. Therefore, $\tilde{a}(n) \to E_G(a)$.

Part 3. Taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \tilde{G}(a;n) = \pi \lim_{n\to\infty} \left(\frac{e^{-n(1-G(a))} - e^{-n}}{1 - e^{-n}} \right) + (1-\pi)G(a) = (1-\pi)G(a) \tag{39}$$

for any $a \in [a_0, \bar{a})$ and $\lim_{n\to\infty} \tilde{G}(\bar{a};n) = 1$. Therefore, $\tilde{a}(n) \to \pi\bar{a} + (1-\pi)E_G(a)$.

Part 4. For $n > 0$, we have $\tilde{G}(a;n) < G(a)$ for $a \in A$. To see this, let $w_N(n) = P_N(n)/\alpha(n)$. Using (36), $\tilde{G}(a;n) = \sum_{N=1}^{\infty} w_N(n)[\pi(G(a))^N + (1-\pi)G(a)]$. Since $\tilde{G}(a;n)$ is a weighted average of the term $\pi(G(a))^N + (1-\pi)G(a)$ for all $N > 1$, and $(G(a))^N < G(a)$ for all $N > 1$ and $a \in (a_0, \bar{a})$, and $G(a)^N = G(a)$ for $a = a_0$ or $a = \bar{a}$, we have $\tilde{G}(a;n) < G(a)$ for all $\pi \in (0, 1]$. Therefore, $\tilde{G}(a;n)$ first order stochastically dominates $G(a)$ and $\tilde{a}(n) > E_G(a)$.

Part 5. Consider any $f : A \to \mathbb{R}_+$ such that $f' > 0$. For any n_1 and n_2 such that $n_1 > n_2$, Part 6 implies $\tilde{f}(n_1) > \tilde{f}(n_2)$, i.e. $\int_{a_0}^{\bar{a}} f(a)d\tilde{G}(a;n_1) > \int_{a_0}^{\bar{a}} f(a)d\tilde{G}(a;n_2)$. Thus $\tilde{G}(a;n_1) \leq \tilde{G}(a;n_2)$ and $\tilde{G}(a;n_1)$ first order stochastically dominates $\tilde{G}(a;n_2)$.

Part 6. Applying Leibniz' integral rule gives us

$$\tilde{f}'(n) = \int_{a_0}^{\bar{a}} f(a) \frac{\partial \tilde{g}(a;n)}{\partial n} da. \tag{40}$$

First, we show that there exists a unique cutoff $\hat{a} \in A$ such that $\frac{\partial \tilde{g}(a;n)}{\partial n} > 0$ for $a > \hat{a}$ and $\frac{\partial \tilde{g}(a;n)}{\partial n} < 0$ for $a < \hat{a}$. To start with, we have

$$\tilde{g}(a;n) = \pi \left(\frac{ng(a)e^{-n(1-G(a))}}{1 - e^{-n}} \right) + (1-\pi)g(a). \tag{41}$$

Differentiating (41) with respect to n, we obtain

$$\frac{\partial \tilde{g}(a;n)}{\partial n} = \pi g(a) \left[\frac{e^{-n(1-G(a))}[(1 - n(1-G(a)))(1-e^{-n}) - ne^{-n}]}{(1-e^{-n})^2} \right] \tag{42}$$

and therefore $\frac{\partial \tilde{g}(a;n)}{\partial n} > 0$ if and only if

$$(1 - n(1-G(a)))(1-e^{-n}) - ne^{-n} > 0, \tag{43}$$

or, equivalently,

$$G(a) > \frac{1}{1-e^{-n}} - \frac{1}{n}. \tag{44}$$

Defining $\hat{a} = G^{-1}\left(\frac{1}{1-e^{-n}} - \frac{1}{n}\right)$, we have $\frac{\partial \tilde{g}(a;n)}{\partial n} > 0$ if and only if $a > \hat{a}$.

We can use the cutoff $\hat{a}$ to rewrite $\tilde{f}'(n)$ as follows:

$$\tilde{f}'(n) \equiv \int_{a_0}^{\hat{a}} f(a)\frac{\partial \tilde{g}(a;n)}{\partial n}da + \int_{\hat{a}}^{\bar{a}} f(a)\frac{\partial \tilde{g}(a;n)}{\partial n}da. \tag{45}$$

We therefore have $\tilde{f}'(n) > 0$ if and only if

$$\int_{\hat{a}}^{\bar{a}} f(a)\frac{\partial \tilde{g}(a;n)}{\partial n}da > -\int_{a_0}^{\hat{a}} f(a)\frac{\partial \tilde{g}(a;n)}{\partial n}da > 0. \tag{46}$$

Given that $f'(a) > 0$, and both sides of (46) are positive, by definition of $\hat{a}$, a sufficient condition for $\tilde{f}'(n) > 0$ is

$$\int_{\hat{a}}^{\bar{a}} f(\hat{a})\frac{\partial \tilde{g}(a;n)}{\partial n}da \geq -\int_{a_0}^{\hat{a}} f(\hat{a})\frac{\partial \tilde{g}(a;n)}{\partial n}da, \tag{47}$$

which is true iff $\int_{\hat{a}}^{\bar{a}} \frac{\partial \tilde{g}(a;n)}{\partial n}da \geq -\int_{a_0}^{\hat{a}} \frac{\partial \tilde{g}(a;n)}{\partial n}da$, or equivalently $\int_{a_0}^{\bar{a}} \frac{\partial \tilde{g}(a;n)}{\partial n}da \geq 0$. Applying Leibniz' integral rule again, $\int_{a_0}^{\bar{a}} \frac{\partial \tilde{g}(a;n)}{\partial n}da = \frac{\partial}{\partial n}\int_{a_0}^{\bar{a}} \tilde{g}(a;n)da = 0$, since $\int_{a_0}^{\bar{a}} \tilde{g}(a;n)da = 1$. Therefore, $\tilde{f}'(n) > 0$. ■

Proof of Proposition 1

The first-order condition with respect to q_a is

$$\alpha(n)[au'(q_a) - c'(q_a)]\tilde{g}(a;n) = 0 \tag{48}$$

and the first order-condition with respect to n is

$$\alpha'(n)\tilde{s}(n;\{q_a\}_{a\in A}) + \alpha(n)\tilde{s}'(n;\{q_a\}_{a\in A}) = k. \tag{49}$$

We can verify that $s_a^* = au(q_a^*) - c(q_a^*)$ is strictly increasing in a. Differentiating s_a^*,

$$\frac{ds_a^*}{da} = u(q_a^*) + [au'(q_a^*) - c'(q_a^*)] \frac{dq^*}{da}. \tag{50}$$

Since $au'(q_a^*) - c'(q_a^*) = 0$ by (48) if $n^* > 0$, we have $\frac{ds_a^*}{da} = u(q_a^*) > 0$ for all $a \in (a_0, \bar{a}]$. Given that s_a^* is strictly increasing in a and $s_0^* \geq 0$ where $s_0^* \equiv a_0 u(q_0) - c(q_0)$, we have $s_a^* \geq 0$ for all $a \in A$. Therefore, all chosen goods $a \in A$ are traded if $a_0 > 0$, and q_a satisfies $au'(q_a) = c'(q_a)$. If $a_0 = 0$, we have $q_a = 0$ since $\lim_{q \to 0} c'(q)/u'(q) = 0$.

Since s_a^* is strictly increasing in a, the planner chooses the seller with the highest utility shock a whenever possible, i.e. with probability π, and randomizes across sellers otherwise, i.e. with probability $1 - \pi$. The distribution of chosen goods, $\tilde{G}(a; n)$, is therefore equal to (5).

Existence and uniqueness of the solution to the planner's problem follows from Proposition 2, which is proven below. For the planner's problem, we know that $s_a^* \geq 0$ for all $a \in A$ and thus all chosen goods are traded. Setting $i = 0$ in Proposition 2 results in equilibrium conditions that are equivalent to the planner's FOCs. It follows that there exists a unique solution to the planner's problem with $n^* > 0$ provided that Assumption 4 holds, except that $q_a^0 = q_a^*$ since q_a^* does not depend directly on n. That is, Assumption 2 suffices. ■

Proofs for Section 5 (except Proposition 2)

Proof of Lemma 2

Define $v_a \equiv au(q_a) - d_a/\gamma$, the buyer's ex post trading surplus, and $\dot{v}_a \equiv v'(a)$. First, it follows from the facts that $v_a \geq 0$ for all a and $\dot{v}_a = u(q_a) \geq 0$ that there exists a unique $a_b \in A$ such that $q_a = 0$ and $d_a = 0$ if and only if $a \leq a_b$.

Next, let $f(a) = \frac{z}{\gamma} - au(q_a) + v_a$. Constraint (59) binds if and only if $f(a) = 0$. Differentiating, we have $f'(a) = -(u(q_a) + au'(q_a)q'(a)) + \dot{v}_a$. Using $\dot{v}_a = u(q_a)$, this implies that $f'(a) = -au'(q_a)q'(a)$. Since $u'(q_a) > 0$ and $q'(a) \geq 0$ is a constraint, we have $f'(a) \leq 0$. Therefore, there exists a unique $a_c \in A$ such that $f(a) = 0$ and constraint (59) binds if and only if $a \in [a_c, \bar{a}]$, so $d_a = z$ and thus $\frac{z}{\gamma} = au(q_a) - v_a$. Differentiating, we have $au'(q_a)q'(a) = 0$ for all $a \in [a_c, \bar{a}]$. Since $u'(q_a) > 0$ and $q'(a) \geq 0$ is a constraint, this requires $q'(a) = 0$ and thus $q_a = q_{a_c}$ on $[a_c, \bar{a}]$.

Finally, it is clear that $a_0 \leq a_b$ and $a_c \leq \bar{a}$. It remains only to show that $a_b \leq a_c$. We have $q_{a_b} = 0$ while $q_{a_c} > 0$, so $q_{a_b} \leq q_{a_c}$ and thus $a_b \leq a_c$ because $q'(a) \geq 0$. ■

Proof of Lemma 3

In the limit as $n \to 0$, we have $\tilde{G}(a; n) \to G(a)$ by Lemma 1. As $n \to 0$, we have $i/\alpha(n) \to \infty$ so $\delta \to \infty$. Also, $1/\delta \to 0$ implies that $\phi(a; n) \to \frac{1-G(a)}{g(a)}$ on $(a_b, a_c]$. From Lemma 7, we have $[1 - \frac{\phi(a_b;n)}{a_b}]a_b = 0$, which is equivalent to $\psi_G(a_b) = 0$ where $\psi_G(a) \equiv a - \frac{1-G(a)}{g(a)}$. By Assumption 3, we have $\psi'_G(a) > 0$. Also, we have $\psi_G(a_0) = \psi_G(0) \leq 0$ and $\psi_G(\bar{a}) = \bar{a} > 0$. Therefore, there exists a unique solution $a_b \in [a_0, \bar{a})$ to $\psi_G(a) = 0$. Finally, as $n \to 0$, the condition for a_c reduces to

$$(\bar{a} - a_c)[1 - G(a_c)] = \bar{a} \left[\frac{-\psi_G(a_b)}{a_b - \psi_G(a_b)} \right] (1 - G(a_b)). \tag{51}$$

Given that $\psi_G(a_b) = 0$, if $a_b > 0$ then the right-hand side is zero, which implies $a_c = \bar{a}$. In the limit as $a_b \to 0$, the right-hand side is also zero, so $a_c = \bar{a}$. ■

Proof of Proposition 2

Our strategy is to solve for the equilibrium in two stages. First, we take z and n as given and solve for $\{(q_a, d_a)\}_{a \in A}$ (inner maximization problem). Second, we solve for z and n (outer maximization problem) given the solutions for $\{(q_a, d_a)\}_{a \in A}$.

We first solve the inner and outer maximization problems. Next, we use the results to prove Parts 1 to 8 of Proposition 2. Finally, we prove existence and uniqueness of equilibrium. Proofs for all lemmas used in this section to prove Proposition 2 are found at the end of this section (unless included earlier).

Stage 1. Inner maximization problem

In the first stage, taking $z > 0$ and $n > 0$ as given (we later prove this), the market makers' problem is to maximize (17) subject to (18) at equality, plus a liquidity constraint $d_a \leq z$ for all $a \in A$, the IC constraint (20), and the IR constraint (19). Ignoring constants, the market maker's inner maximization problem is:

$$\max_{\{(q_a, d_a)\}_{a \in A}} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a; n) - i\frac{z}{\gamma} \right\}, \tag{52}$$

subject to

$$\frac{\alpha(n)}{n}\int_{a_0}^{\bar{a}}\left[-c(q_a)+\frac{d_a}{\gamma}\right]d\tilde{G}(a;n)=k, \tag{53}$$

and, for all $a, a' \in A$,

$$\begin{aligned} d_a &\leq z, & (54)\\ au(q_a)-\frac{d_a}{\gamma} &\geq au(q_{a'})-\frac{d_{a'}}{\gamma}, & (55)\\ au(q_a)-\frac{d_a}{\gamma} &\geq 0, & (56)\\ d_a, q_a &\geq 0. & (57)\end{aligned}$$

To solve the inner maximization problem (52), we transform the above problem as follows. Defining $v_a \equiv au(q_a) - d_a/\gamma$, the buyer's ex post trading surplus, and $\dot{v}_a \equiv v'(a)$, the following lemma simplifies the (IC) constraint. This is a standard result and the proof is omitted.

Lemma 4. *The incentive compatibility (IC) constraint holds if and only if (i) $q'(a) \geq 0$, and (ii) $\dot{v}_a = u(q_a)$.*

We can now use $v_a \equiv au(q_a) - d_a/\gamma$ and Lemma 4 to re-write the problem as an optimal control problem where q_a is the control variable, v_a is the state variable, and δ is the Lagrange multiplier associated with the seller entry constraint (53). For simplicity, we assume that $a_0 = 0$.

In the first stage, we take z, n, δ as given and later solve for these. Given that $a_0 = 0$, we have $v_0 = 0$. Using $v_a \equiv au(q_a) - d_a/\gamma$ to eliminate d_a in the above, and substituting in the constraint (53), the inner maximization problem becomes

$$\max_{\{(q_a,v_a)\}_{a\in A}}\left\{\alpha(n)\int_{a_0}^{\bar{a}}\{(1-\delta)v_a+\delta\left[au(q_a)-c(q_a)\right]\}\tilde{g}(a;n)da-\delta nk-i\frac{z}{\gamma}\right\}, \tag{58}$$

subject to $v_0 = 0$ and, for all $a \in A$,

$$au(q_a) - v_a \leq \frac{z}{\gamma}, \tag{59}$$

$$\dot{v}_a = u(q_a), \tag{60}$$

$$q'(a) \geq 0, \tag{61}$$

$$q_a, v_a \geq 0. \tag{62}$$

The inner maximization problem is a standard optimal control problem with q_a as the control variable and v_a as the state variable. We can therefore apply the Maximum Principle to find the necessary conditions for the optimal path of the control and state variables. To solve the inner maximization problem, we ignore the condition $q'(a) \geq 0$ and later verify that it holds in Lemma 8. Ignoring the constants, the current value Hamiltonian for the optimal control problem is:

$$H = \alpha(n)\{(1-\delta)v_a + \delta\left[au(q_a) - c(q_a)\right]\}\tilde{g}(a;n) + \lambda_a u(q_a) \tag{63}$$

where λ_a is the costate variable, and the Lagrangian is:

$$L = \alpha(n)\{(1-\delta)v_a + \delta\left[au(q_a) - c(q_a)\right]\}\tilde{g}(a;n) + \lambda_a u(q_a) + \mu_a\left[\frac{z}{\gamma} - au(q_a) + v_a\right] + \theta_a q_a + \eta_a v_a \tag{64}$$

where μ_a, θ_a and η_a are the Lagrangian multipliers associated with the liquidity constraint, non-negativity constraint, and IR constraint respectively.

The FOCs and the transversality condition are as follows:

$$\frac{\partial L}{\partial q_a} = \alpha(n)\delta\left[au'(q_a) - c'(q_a)\right]\tilde{g}(a;n) + (\lambda_a - \mu_a a)\, u'(q_a) + \theta_a = 0, \tag{65}$$

$$\frac{\partial L}{\partial v_a} = (1-\delta)\alpha(n)\tilde{g}(a;n) + \mu_a + \eta_a = -\dot{\lambda}_a, \tag{66}$$

$$\frac{\partial L}{\partial \lambda_a} = \dot{v}_a = u(q_a), \tag{67}$$

$$(68) \qquad \lambda_{\bar{a}} v_{\bar{a}} = 0.$$

For the inequality constraints, the conditions are:

$$
\begin{aligned}
(69) \qquad \mu_a &\geq 0,\ \mu_a\left(\frac{z}{\gamma} - au(q_a) + v_a\right) = 0, \\
(70) \qquad \theta_a &\geq 0,\ \theta_a q_a = 0, \\
(71) \qquad \eta_a &\geq 0,\ \eta_a v_a = 0.
\end{aligned}
$$

The following lemma provides expressions for λ_a and Σ_{a_c}, where $\Sigma_a \equiv \int_a^{\bar{a}} \mu_x dx$.

Lemma 5. *For all $a \in [a_0, a_c]$, we have the following:*

$$(72) \qquad \lambda_a = \alpha(n)(1-\delta)[1 - \tilde{G}(a;n)] + \Sigma_{a_c} + \int_a^{\bar{a}} \eta_x dx$$

and

$$(73) \qquad \Sigma_{a_c} = \frac{\alpha(n)}{\bar{a}} \int_{a_c}^{\bar{a}} [\delta(x - a_c)\tilde{g}(x;n) + (1-\delta)(\tilde{G}(a_c;n) - \tilde{G}(x;n)]dx.$$

The next lemma uses our assumption that $a_0 = 0$.

Lemma 6. *If $a_0 = 0$, we obtain the following:*

$$(74) \qquad \delta = 1 + \frac{\Sigma_{a_c} + \int_{a_0}^{\bar{a}} \eta_x dx}{\alpha(n)}.$$

To determine q_a for all $a \in A$, it remains only to determine δ, a_b, and a_c.

By Lemma 2, there are three intervals to consider.
Case 1. For any $a \in [a_0, a_b]$, $v_a = 0$ for all a and therefore $q_a = 0$.
Case 2. For any $a \in [a_b, a_c]$, we have $\theta_a = 0$ and $\mu_a = 0$, so q_a solves

$$(75) \qquad \alpha(n)\delta\left[au'(q_a) - c'(q_a)\right]\tilde{g}(a;n) \ = -\lambda_a u'(q_a).$$

Using the above two lemmas, plus the fact that $\int_a^{\bar{a}} \eta_x dx = \int_a^{a_b} \eta_x dx$ for all a since

$\eta_a = 0$ for $a > a_b$, and therefore $\int_a^{\bar{a}} \eta_x dx = \int_a^{a_b} \eta_x dx = 0$ for $a \geq a_b$, we can write

$$(a - \phi(a;n))u'(q_a) = c'(q_a) \tag{76}$$

where

$$\phi(a;n) = -\left(\frac{1-\delta}{\delta}\right)\left(\frac{1-\tilde{G}(a;n)}{\tilde{g}(a;n)}\right) - \frac{\Sigma_{a_c}}{\alpha(n)\delta\tilde{g}(a;n)}. \tag{77}$$

Case 3. For any $a \in [a_c, \bar{a}]$, we have $\theta_a = 0$ and $q_a = q_{a_c}$ by Lemma 2.

The following lemma will prove useful in deriving Proposition 2.

Lemma 7. *We have either $a = \phi(a;n)$ or $a = 0$ for all $a \leq a_b$.*

Proof. For $a = a_b$, we have $q_{a_b} = 0$. Using (76) above, we have

$$\lim_{a \to a_b} (a - \phi(a;n)) = \lim_{a \to a_b} \left[1 - \frac{\phi(a;n)}{a}\right] a = \lim_{q \to 0} \frac{c'(q)}{u'(q)} = 0 \tag{78}$$

since we have $\lim_{q \to 0} \frac{c'(q)}{u'(q)} = 0$ by assumption. Therefore, by continuity of the function q_a, we have either $\frac{\phi(a_b)}{a_b} = 1$, or equivalently $a_b = \phi(a_b)$, or $a_b = 0$. Similarly, we have $a = \phi(a;n)$ or $a = 0$ for all $a < a_b$. ■

Stage 2. Outer maximization problem

The outer maximization problem we solve next is

$$\max_{z,n,\delta} \left\{ J(n,z,\delta) - \delta nk - i\frac{z}{\gamma} \right\}, \tag{79}$$

where we define

$$J(n,z,\delta) \equiv \max_{\{(q_a,v_a)\}_{a \in A}} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \{(1-\delta)v_a + \delta\left[au(q_a) - c(q_a)\right]\} \tilde{g}(a;n)da \right\}, \tag{80}$$

subject to $v_0 = 0$ and, for all $a \in A$, constraints (59), (60), (61), and (62).

To solve the outer maximization problem, the function $J(n,z,\delta)$ is equivalent to

(81)
$$J(n,z,\delta)=\max_{\{(q_a,v_a)\}_{a\in A}}\left\{\begin{array}{c}\int_{a_0}^{\bar{a}}\alpha(n)\{(1-\delta)v_a+\delta\left[au(q_a)-c(q_a)\right]\}\tilde{g}(a;n)da\\+\int_{a_0}^{\bar{a}}\left[\mu_a\left(\frac{z}{\gamma}-au(q_a)+v_a\right)+\eta_a v_a+\lambda_a u(q_a)+\theta_a q_a\right]da\end{array}\right\}.$$

Define $\tilde{s}(n)\equiv\int_{a_0}^{\bar{a}}s_a d\tilde{G}(a;n)$ and $\tilde{v}(n)\equiv\int_{a_0}^{\bar{a}}v_a d\tilde{G}(a;n)$. Returning to our original formulation to eliminate δ, the above problem is equivalent to

(82)
$$\max_{z,n}\left\{\hat{J}(n,z)-i\frac{z}{\gamma}\right\},$$

where

(83)
$$\hat{J}(n,z)=\max_{\{(q_a,v_a)\}_{a\in A}}\left\{\alpha(n)\tilde{v}(n)+\int_{a_0}^{\bar{a}}\left[\mu_a\left(\frac{z}{\gamma}-au(q_a)+v_a\right)+\eta_a v_a+\lambda_a u(q_a)+\theta_a q_a\right]da\right\}$$

subject to the constraint

(84)
$$\frac{\alpha(n)}{n}[\tilde{s}(n)-\tilde{v}(n)]\leq k$$

and $n\geq 0$ with complementary slackness.

Using the envelope theorem, the first-order conditions for z and n respectively are

(85)
$$\int_{a_0}^{\bar{a}}\mu_a da=i$$

and

(86)
$$\alpha'(n)\tilde{v}(n)+\alpha(n)\tilde{v}'(n)=0.$$

Using the fact that $\mu_a=0$ for all $a<a_c$, by definition of a_c, we have $\int_{a_0}^{\bar{a}}\mu_a da=\Sigma_{a_c}$. The FOC for z given by (85) thus becomes:

(87)
$$\Sigma_{a_c}=i,$$

Substituting $\Sigma_{a_c} = i$ into expression (74) in Lemma 6, the above yields

$$\delta = 1 + \frac{i + \int_{a_0}^{\bar{a}} \eta_x dx}{\alpha(n)}. \tag{88}$$

Finally, we verify that the condition $q'(a) \geq 0$ is indeed satisfied provided that $G''(a) \leq 0$. This is a sufficient but not a necessary condition for $q'(a) \geq 0$.

Lemma 8. *If $G''(a) \leq 0$ for all $a \in A$, then $q(.)$ is weakly increasing for all $a \in A$ and $q'(a) > 0$ for all $a \in (a_b, a_c)$.*

Proof of Parts 1 to 8

Part 1. Follows from the definition of a_b.

Part 2. From above, for any $a \in [a_b, a_c]$, we have

$$(a - \phi(a;n))u'(q_a) = c'(q_a) \tag{89}$$

where, using $\Sigma_{a_c} = i$ plus expression (77) for $\phi(a;n)$, we have

$$\phi(a;n) = -\left(\frac{1-\delta}{\delta}\right)\left(\frac{1-\tilde{G}(a;n)}{\tilde{g}(a;n)}\right) - \frac{i}{\alpha(n)\delta\tilde{g}(a;n)}. \tag{90}$$

The expression for δ can be derived as follows. Using $[1 - \frac{\phi(a_b;n)}{a_b}]a_b = 0$ from Lemma 7 plus expression (90) for $\phi(a;n)$, we have

$$\left[1 + \left(\frac{1-\delta}{\delta}\right)\left(\frac{1-\tilde{G}(a_b;n)}{a_b\tilde{g}(a;n)}\right) + \frac{i}{\alpha(n)\delta a_b\tilde{g}(a_b;n)}\right]a_b = 0 \tag{91}$$

If $a_b = 0$ then $\delta = 1 + \frac{i}{\alpha(n)}$ from (74). If $a_b > 0$, the above implies that

$$i = -\alpha(n)[\delta a_b\tilde{g}(a_b;n) + (1-\delta)(1-\tilde{G}(a_b;n))]. \tag{92}$$

For any $a_b \geq 0$, the value of δ is given by the following expression:

$$\delta = \frac{1 - \tilde{G}(a_b;n) + \frac{i}{\alpha(n)}}{1 - \tilde{G}(a_b;n) - a_b\tilde{g}(a_b;n)} \tag{93}$$

which is equivalent to (26) using expression (21).

Also, $\dot{v}_a = u(q_a)$ implies $v_a - v_0 = \int_{a_0}^{a} u(q_x)dx$, so $v_a = \int_{a_0}^{a} u(q_x)dx$ since $v_0 = 0$. We can derive d_a/γ from v_a using the fact that $v_a \equiv au(q_a) - d_a/\gamma$.

Part 3. Clear from Lemma 2.

Part 4. Using $\Sigma_{a_c} = i$ and expression (73), the value of a_c is given by (27).

Part 5. Clear from the definition of a_c.

Part 6. The first-order condition for $n > 0$ given by (86) can be written as

$$\alpha'(n)\tilde{s}(n) + \alpha(n)\tilde{s}'(n) = k, \tag{94}$$

using the ZPC constraint (84) at equality. More precisely, this is equivalent to

$$\alpha'(n)\tilde{s}(n; \{q_a\}_{a\in A}) + \alpha(n)\tilde{s}'(n; \{q_a\}_{a\in A}) = k. \tag{95}$$

The fact that n is strictly decreasing in k is proven in Lemma 11 below.

Part 7. The zero profit condition is given by (84), using the definition of v_a.

Part 8. Since v_a is increasing in a, the highest draw is always chosen by buyers whenever possible, i.e. with probability π, and buyers randomize otherwise, i.e. with probability $1 - \pi$. Therefore the cdf of chosen goods is given by (5). ■

Proof of existence and uniqueness

We first prove existence and uniqueness of the solution to the inner maximization problem and then prove the same for the outer maximization problem.

Inner maximization. We prove that, given z and n from the outer maximization problem, the solution to the inner maximization problem exists and is unique.

Existence. A solution to the problem exists because the set of admissible paths is non-empty and compact, and there exists an admissible path for which the objective is finite. For example, the path $q_a = 0$ and $v_a = (a-1)u(q_a)$ for all $a \in A$ is admissible (since $v_0 = 0$, $au(q_a) - v_a \leq z/\gamma$, $q_a \geq 0$, $v_a \geq 0$, and $\dot{v}_a = u(q_a) + (a-1)u'(q_a)q'(a) = u(q_a)$, and $q'(a) \geq 0$). Also, the objective is finite under this path. Finally, the set of feasible paths is compact since $q_a \in [0, q^*_{\bar{a}}]$ where $q^*_{\bar{a}}$ solves $\bar{a}u'(q_{\bar{a}}) = c'(q_{\bar{a}})$ and $v_a \in [0, v_{\bar{a}}]$ where $v_{\bar{a}} = u(q^*_{\bar{a}})[\bar{a} - a_0]$ since $v_a = \int_{a_0}^{a} u(q_x)dx$.

Uniqueness. The Hamiltonian $H(q_a, v_a, \lambda_a)$, where λ_a is the co-state variable given by the Maximum Principle, is strictly concave in the control and state variables (q_a, v_a) for all a. Therefore, the solution is an optimum that solves the inner maximization problem and it is unique. To establish strict concavity, differentiating $H(q_a, v_a, \lambda_a)$ with respect to q_a yields

$$\begin{aligned}
\frac{\partial H}{\partial q_a} &= \alpha(n)\delta[u'(q_a) - c'(q_a)]\tilde{g}(a;n) + \lambda_a u'(q_a), \\
\frac{\partial^2 H}{\partial q_a^2} &= \alpha(n)\delta[u''(q_a) - c''(q_a)]\tilde{g}(a;n) + \lambda_a u''(q_a) \equiv -X,
\end{aligned}$$

where $X > 0$, since $u''(q_a) < 0$ and $c''(q_a) > 0$. Differentiating $H(q_a, v_a, \lambda_a)$ with respect to v_a, we obtain $\frac{\partial H}{\partial v_a} = \alpha(n)(1-\delta)\tilde{g}(a;n)$ and $\frac{\partial^2 H}{\partial v_a^2} = 0$. Finally, $\frac{\partial^2 H}{\partial v_a \partial q_a} = 0$, so we get the Hessian matrix, $\mathbb{H} = \begin{bmatrix} -X & 0 \\ 0 & 0 \end{bmatrix}$. Since $\mathbf{x}^T \mathbb{H} \mathbf{x} < 0$ for all $\mathbf{x} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$, the Hessian $\mathbb{H}$ is negative definite and the Hamiltonian is strictly concave in (q_a, v_a).

Outer maximization. We prove that, given $\{(q_a, v_a)\}_{a \in A}$ from the inner maximization problem, the solution (n, z) to the outer maximization problem exists and is unique, and n, z are interior solutions with $n, z > 0$ if Assumption 4 holds. To establish this result, we first prove that there exists a non-empty set of solutions n, denoted by $N(k)$, that solves the problem. We then show that equilibrium is unique if $n > 0$ for all $n \in N(k)$, and finally we prove that $n > 0$ for any $n \in N(k)$.

Taking $\{(q_a, v_a)\}_{a \in A}$ as given by the inner maximization problem, and ignoring constants, the outer maximization problem is equivalent to

$$\max_{z,n} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[a u(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a;n) + (\Sigma_{a_c} - i) \frac{z}{\gamma} \right\}, \tag{96}$$

subject to

$$\frac{\alpha(n)}{n} \int_{a_0}^{\bar{a}} \left[-c(q_a) + \frac{d_a}{\gamma} \right] d\tilde{G}(a;n) \leq k \tag{97}$$

and $n \geq 0$ with complementary slackness, where $\{(q_a, v_a)\}_{a \in A}$ solves the inner maximization problem.

Lemma 9. *The set of solutions $N(k)$ is nonempty and upper hemicontinuous.*

Proof. Since $\alpha(n)$ is a bijection, we can rewrite (96) in terms of α as follows:

$$\max_{z,\alpha}\left\{\alpha\int_{a_0}^{\bar{a}}\left[au(q_a)-\frac{d_a}{\gamma}\right]d\tilde{G}(a;\alpha)+(\Sigma_{a_c}-i)\,\frac{z}{\gamma}\right\}. \tag{98}$$

The objective function is continuous and, without loss of generality, we can restrict (z,α) to the following compact set:

$$\Delta=\{(z,\alpha):\alpha\in[0,1],\ z/\gamma\in[0,\bar{a}u(q_{\bar{a}})]\} \tag{99}$$

since $q\in[0,q_{\bar{a}}^*]$ where $q_{\bar{a}}^*$ solves $\bar{a}u'(q_{\bar{a}})=c'(q_{\bar{a}})$, and we have $z/\gamma<\bar{a}u(q_{\bar{a}})$. The constraint (97) can therefore be written as $(z,\alpha)\in\Gamma(k)$ for all $k\geq 0$, where $\Gamma(k)$ is a continuous and compact-valued correspondence. Applying the Theorem of the Maximum (Theorem 3.6 in Stokey, Lucas, and Prescott, 1989), the correspondence that gives the set of solutions for α is nonempty and upper hemicontinuous, and therefore also $N(k)$ is nonempty and upper hemicontinuous. ■

The following lemma establishes that any strictly positive solution $n\in N(k)$ must be unique. Since we know that $z=d_{a_c}>0$ where $d_a/\gamma=au(q_a)-v_a$, and $\{(q_a,v_a)\}_{a\in A}$ is given by the inner maximization problem, Lemma 10 implies that any solution (n,z) where $n>0$ is unique.

Lemma 10. *If $N^+\subseteq N(k)$ and $N^+\subseteq\mathbb{R}_+\backslash\{0\}$, then $N^+=\{n\}$.*

Proof. Consider any solution $n\in N(k)$ such that $n>0$. Defining $\Phi(n)\equiv\alpha(n)\tilde{v}(n)$, the solutions n satisfy the first-order condition (86), which says $\Phi'(n)=0$. We show that $\Phi''(n)<0$ and thus any solution is unique. Using (41), for any $\pi\in(0,1]$ we have

$$\Phi(n)=\pi\int_{a_0}^{\bar{a}}ne^{-n(1-G(a))}v_a g(a)da+(1-\pi)(1-e^{-n})\int_{a_0}^{\bar{a}}v_a g(a)da. \tag{100}$$

Using Leibniz's integral rule, plus the envelope theorem,

$$\begin{aligned}\Phi'(n) &= \pi\left(\int_{a_0}^{\bar{a}}e^{-n(1-G(a))}v_a g(a)da-\int_{a_0}^{\bar{a}}n(1-G(a))e^{-n(1-G(a))}v_a g(a)da\right)\\ &\quad+(1-\pi)e^{-n}\int_{a_0}^{\bar{a}}v_a g(a)da. \end{aligned}\tag{101}$$

By integration by parts on the second integral in $\Phi'(n)$ above, we obtain

(102)

$$\Phi'(n) = \pi \left(\int_{a_0}^{\bar{a}} e^{-n(1-G(a))}(1-G(a))v'(a)da + e^{-n}v(a_0) \right) + (1-\pi)e^{-n} \int_{a_0}^{\bar{a}} v_a g(a)da > 0.$$

Differentiating (102), we find that

(103)

$$\Phi''(n) = -\left(\pi \int_{a_0}^{\bar{a}} e^{-n(1-G(a))}(1-G(a))^2 v'(a)da + \pi e^{-n}v(a_0) + (1-\pi)e^{-n} \int_{a_0}^{\bar{a}} v_a g(a)da \right) < 0.$$

The fact that $\Phi''(n) < 0$ follows from the fact that $v'(a) = u(q_a) \geq 0$ for all a and $v'(a) > 0$ for some a and also $v(a_0) = 0$. Therefore, any solution $n > 0$ is unique. ■

From Lemma 9, we know that, for any given $k \geq 0$, there exists a non-empty set of solutions $N(k)$ that solves problem (96). We also know that any solution z is interior, since $z/\gamma = \bar{a}u(q_{\bar{a}})$ implies $v_{\bar{a}} = \bar{a}u(q_{\bar{a}}) - \bar{z}/\gamma = 0$ and therefore $v_a = 0$ for all $a \in A$. We now prove that, for any $n \in N(k)$, we have $n \in \mathbb{R}_+ \backslash \{0\}$ provided that Assumption 4 holds. Also, the function $n(k)$ is strictly decreasing in k.

Lemma 11. *Any solution $n \in N(k)$ is interior, i.e. $n \in \mathbb{R}_+ \backslash \{0\}$. The function $n(k)$ is strictly decreasing in k.*

Proof. First, we show there exists an interior solution $n > 0$. Define $\Lambda(n) \equiv \alpha(n)\tilde{s}(n)$. The first-order condition (94) says $\Lambda'(n) = k$. We prove there exists $n > 0$ such that $\Lambda'(n) = k$ if Assumption 4 holds. We have $\lim_{n\to\infty} \Lambda'(n) = 0$, and

$$\lim_{n\to 0} \Lambda'(n) = \int_{a_0}^{\bar{a}} \lim_{n\to 0} s(a; q_a(n))dG(a) \tag{104}$$

where $\lim_{n\to 0} s(a; q_a(n)) = s(a; \lim_{n\to 0} q_a(n))$. If the following condition holds:

$$E_G[au(q_a^0) - c(q_a^0)] > k \tag{105}$$

where $q_a^0 \equiv \lim_{n\to 0} q_a(n)$, there exists $n > 0$ that satisfies $\Lambda'(n) = k$ provided that $\Lambda''(n) < 0$ (which we prove below).

Next, any interior solution $n > 0$ is better than $n = 0$. Define the value function:

$$V(k, \gamma) \equiv \max_{z,n} \left\{ \alpha(n) \int_{a_0}^{\bar{a}} \left[au(q_a) - \frac{d_a}{\gamma} \right] d\tilde{G}(a; n) + (\Sigma_{a_c} - i) \frac{z}{\gamma} \right\}. \tag{106}$$

Since we know that z is interior, we have $V(k,\gamma) \equiv \max_n \{\alpha(n)\tilde{v}(n)\}$ since $\int_{a_0}^{\bar{a}} \mu_a = i$. If $n = 0$ then $V(k,\gamma) = 0$. If $n > 0$, $V(k,\gamma) \equiv \max_n \{\alpha(n)\tilde{s}(n) - nk\}$ using constraint (97) with equality. Letting $\Lambda(n) = \alpha(n)\tilde{s}(n)$, we have $V(k,\gamma) > 0$ if $\Lambda(n) - nk > 0$. Thus the candidate solution $n > 0$ is better than $n = 0$ if $\Lambda(n) > nk$ for $n > 0$. Using the fact that $\Lambda'(n) = k$, it suffices to show that $\Lambda''(n) < 0$ and $\frac{\Lambda'(n)n}{\Lambda(n)} < 1$ for $n > 0$. Similarly to Lemma 10, using (41), for any $\pi \in (0,1]$ we have

$$\Lambda(n) = \pi \int_{a_0}^{\bar{a}} n e^{-n(1-G(a))} s(a)g(a)da + (1-\pi)(1-e^{-n}) \int_{a_0}^{\bar{a}} s(a)g(a)da \tag{107}$$

and using Leibniz's integral rule, plus the envelope theorem, yields

$$\begin{aligned} \Lambda'(n) &= \pi \left(\int_{a_0}^{\bar{a}} e^{-n(1-G(a))} s_a g(a)da - \int_{a_0}^{\bar{a}} n(1-G(a)) e^{-n(1-G(a))} s_a g(a)da \right) \\ &\quad + (1-\pi)e^{-n} \int_{a_0}^{\bar{a}} s_a g(a)da. \end{aligned} \tag{108}$$

Therefore, we have

$$\begin{aligned} \frac{\Lambda'(n)n}{\Lambda(n)} &= \frac{\pi \int_{a_0}^{\bar{a}} n e^{-n(1-G(a))} s_a g(a)da + (1-\pi) n e^{-n} \int_{a_0}^{\bar{a}} s_a g(a)da}{\pi \int_{a_0}^{\bar{a}} n e^{-n(1-G(a))} s_a g(a)da + (1-\pi)(1-e^{-n}) \int_{a_0}^{\bar{a}} s_a g(a)da} \\ &\quad - \frac{\pi \int_{a_0}^{\bar{a}} n^2 (1-G(a)) e^{-n(1-G(a))} s_a g(a)da}{\pi \int_{a_0}^{\bar{a}} n e^{-n(1-G(a))} s_a g(a)da + (1-\pi)(1-e^{-n}) \int_{a_0}^{\bar{a}} s_a g(a)da.} \end{aligned} \tag{109}$$

So, $\frac{\Lambda'(n)n}{\Lambda(n)} < 1$ for $n > 0$ provided that $ne^{-n} \leq 1 - e^{-n}$, which is true (note this is equivalent to $\eta_\alpha(n) \leq 1$).

Finally, $\Phi(n) = \Lambda(n) - nk$ for $n > 0$, so $\Phi'(n) = \Lambda'(n) - k$ and $\Phi''(n) = \Lambda''(n)$. Since $\Phi''(n) < 0$ from the proof of Lemma 10, we have $\Lambda''(n) < 0$. It follows that, for any $n \in N(k)$, we have $n > 0$. Since we assume $k > 0$, this implies $n \in \mathbb{R}_+ \backslash \{0\}$.

Since n is unique by Lemma 10, there is a function $n : \mathbb{R}_+ \backslash \{0\} \to \mathbb{R}_+ \backslash \{0\}$ such that $n(k)$ solves $\Lambda'(n) = k$. Clearly, n is strictly decreasing in k since $\Lambda''(n) < 0$. ■

Proof of Lemma 5

Start with the fact that

$$(1-\delta)\alpha(n)\tilde{g}(a;n) + \mu_a + \eta_a = -\dot{\lambda}_a \tag{110}$$

from the FOC (66) above. Integrating both sides over $[a, \bar{a}]$, we obtain

$$(111) \qquad -\int_a^{\bar{a}} \dot{\lambda}_x dx = \int_a^{\bar{a}} (1-\delta)\alpha(n)\tilde{g}(x;n)dx + \int_a^{\bar{a}} \mu_x dx + \int_a^{\bar{a}} \eta_x dx$$

and therefore

$$(112) \qquad -(\lambda_{\bar{a}} - \lambda_a) = \alpha(n)(1-\delta)\int_a^{\bar{a}} \tilde{g}(x;n)dx + \int_a^{\bar{a}} \mu_x dx + \int_a^{\bar{a}} \eta_x dx.$$

The transversality condition $\lambda_{\bar{a}} v_{\bar{a}} = 0$ implies $\lambda_{\bar{a}} = 0$ since $v_{\bar{a}} > 0$. Substituting $\Sigma_a \equiv \int_a^{\bar{a}} \mu_x dx$ into the above, and setting $\lambda_{\bar{a}} = 0$ yields

$$(113) \qquad \lambda_a = \alpha(n)(1-\delta)\int_a^{\bar{a}} \tilde{g}(x;n)dx + \Sigma_a + \int_a^{\bar{a}} \eta_x dx.$$

Now, $\mu_a = 0$ for all $a \in [a_0, a_c]$, thus $\Sigma_a = \int_a^{\bar{a}} \mu_x dx = \int_{a_c}^{\bar{a}} \mu_x dx = \Sigma_{a_c}$ for all $a \in [a_0, a_c]$. Substituting into (113), and using the fact that $\int_a^{\bar{a}} \tilde{g}(x;n)dx = [\tilde{G}(x;n)]_a^{\bar{a}} = 1 - \tilde{G}(a;n)$, we obtain (72).

For the second part, using (65) and Lemma 2, for all $a \in [a_c, \bar{a}]$ we have

$$(114) \qquad \alpha(n)\delta\left[au'(\bar{q}) - c'(\bar{q})\right]\tilde{g}(a;n) + (\lambda_a - \mu_a a)\, u'(\bar{q}) = 0$$

where $\bar{q} \equiv q_{a_c}$, and, for all $a \in [a_c, \bar{a}]$, we also have

$$(115) \qquad \alpha(n)\delta\left[a_c u'(\bar{q}) - c'(\bar{q})\right]\tilde{g}(a;n) + \lambda_{a_c} u'(\bar{q}) = 0.$$

Using the above two equations, and dividing both sides by $u'(\bar{q})$, we obtain

$$(116) \qquad \alpha(n)\delta(a - a_c)\tilde{g}(a;n) = -\lambda_a + \mu_a a + \lambda_{a_c}.$$

Substituting (113) for both λ_a and λ_{a_c} into the above, and simplifying, yields

$$(117) \qquad \alpha(n)[\delta(a - a_c)\tilde{g}(a;n) + (1-\delta)(\tilde{G}(a_c;n) - \tilde{G}(a;n)] = -\Sigma_a + \mu_a a + \Sigma_{a_c}.$$

Finally, $\Sigma_a = \int_a^{\bar{a}} \mu_x dx$ implies that $\dot{\Sigma}_a = -\mu_a$ and thus we obtain

$$(118) \qquad \alpha(n)[\delta(a - a_c)\tilde{g}(a;n) + (1-\delta)(\tilde{G}(a_c;n) - \tilde{G}(a;n)] = -\Sigma_a - \dot{\Sigma}_a a + \Sigma_{a_c}.$$

Integrating both sides over $[a_c, \bar{a}]$, we have

(119)

$$\alpha(n)\int_{a_c}^{\bar{a}}[\delta(x-a_c)\tilde{g}(x;n)+(1-\delta)(\tilde{G}(a_c;n)-\tilde{G}(x;n)]dx = \int_{a_c}^{\bar{a}}\left(-\Sigma_x - \dot{\Sigma}_x x + \Sigma_{a_c}\right)dx$$

where $\int_{a_c}^{\bar{a}}\left(-\Sigma_x - \dot{\Sigma}_x x + \Sigma_{a_c}\right)dx = -\left(\int_{a_c}^{\bar{a}}\Sigma_x + \dot{\Sigma}_x x \; dx\right) + [\Sigma_{a_c}x]_{a_c}^{\bar{a}}$. Using integration by parts, $\int_{a_c}^{\bar{a}}\Sigma_x + \dot{\Sigma}_x x \; dx = [\Sigma_x x]_{a_c}^{\bar{a}} = \Sigma_{\bar{a}}\bar{a} - \Sigma_{a_c}a_c = -\Sigma_{a_c}a_c$, and $[\Sigma_{a_c}x]_{a_c}^{\bar{a}} = \Sigma_{a_c}\bar{a} - \Sigma_{a_c}a_c$. Substituting $\int_{a_c}^{\bar{a}}\left(-\Sigma_x - \dot{\Sigma}_x x + \Sigma_{a_c}\right)dx = \Sigma_{a_c}\bar{a}$ into the above yields

$$(120) \qquad \alpha(n)\int_{a_c}^{\bar{a}}[\delta(x-a_c)\tilde{g}(x;n)+(1-\delta)(\tilde{G}(a_c;n)-\tilde{G}(x;n)]dx = \Sigma_{a_c}\bar{a}$$

and we therefore obtain (73). ■

Proof of Lemma 6

To start with, we have

$$(121) \qquad \alpha(n)\delta\left[au'(q_a) - c'(q_a)\right]\tilde{g}(a;n) + (\lambda_a - \mu_a a)\,u'(q_a) + \theta_a = 0$$

from the FOC (65) for q_a. Dividing both sides by q_a, we obtain

$$(122) \qquad \alpha(n)\delta\left[a - \frac{c'(q_a)}{u'(q_a)}\right]\tilde{g}(a;n) + (\lambda_a - \mu_a a) = \frac{-\theta_a}{u'(q_a)}.$$

Taking the limit as $q_a \to 0$, and using $\lim_{q\to 0} u'(q) = +\infty$ and $\lim_{q\to 0}\frac{c'(q)}{u'(q)} = 0$ yields

(123)

$$\lim_{q\to 0}\alpha(n)\delta\left[a - \frac{c'(q)}{u'(q)}\right]\tilde{g}(a;n) + (\lambda_a - \mu_a a) + \frac{\theta_a}{u'(q)} = \alpha(n)\delta a\tilde{g}(a;n) + (\lambda_a - \mu_a a) = 0$$

for any $a \leq a_b$ and therefore

$$(124) \qquad \lambda_a = -\alpha(n)\delta a\tilde{g}(a;n) - \mu_a a$$

for any $a \leq a_b$. In particular, we have

$$(125) \qquad \lambda_{a_0} = -\alpha(n)\delta a_0\tilde{g}(a_0;n) - \mu_{a_0}a_0.$$

If $a_0 = 0$, then the above implies that $\lambda_{a_0} = 0$. Next, applying Lemma 5 to the special case $a = a_0$, we have

$$\lambda_{a_0} = \alpha(n)(1-\delta) + \Sigma_{a_c} + \int_{a_0}^{\bar{a}} \eta_x dx. \tag{126}$$

Therefore, if $a_0 = 0$, we have $\lambda_{a_0} = \alpha(n)(1-\delta) + \Sigma_{a_c} + \int_{a_0}^{\bar{a}} \eta_x dx = 0$. ■

Proof of Lemma 8

For all $a \leq a_b$, we have $q_a = 0$ and $q'(a) = 0$. For all a greater than or equal to a_c, q_a is constant and thus $q'(a) = 0$. For $a \in (a_b, a_c)$, implicit differentiation of

$$(a - \phi(a;n))u'(q_a) = c'(q_a) \tag{127}$$

yields

$$q'(a) = \frac{-[1-\phi'(a)]u'(q_a)}{[a-\phi(a;n)]u''(q_a) - c''(q_a)} \tag{128}$$

where $\phi(a;n)$ can be simplified to:

$$\phi(a;n) = -\left(\frac{(1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}}{\delta\tilde{g}(a;n)}\right). \tag{129}$$

Differentiating the above yields

$$\phi'(a) = \frac{1-\delta}{\delta} + \frac{\left[(1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}\right]\tilde{g}'(a;n)}{\delta\tilde{g}(a;n)^2}. \tag{130}$$

Since $u'(q_a) > 0$ and $u''(q_a) < 0$ and $c''(q_a) > 0$ and $a - \phi(a;n) > 0$, we have $q'(a) \geq 0$ provided that $\phi'(a) < 1$. Rearranging, this is true provided that

$$\left(\frac{(1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}}{\tilde{G}(a;n)}\right)\left(\frac{\tilde{g}'(a;n)\tilde{G}(a;n)}{\tilde{g}(a;n)^2}\right) < 2\delta - 1. \tag{131}$$

To prove this, we first show that

$$(132)\qquad \frac{(1-\delta)(1-\tilde{G}(a;n))+\frac{i}{\alpha(n)}}{\tilde{G}(a;n)} < 2\delta - 1.$$

Rearranging the above and simplifying, this is equivalent to

$$(133)\qquad \delta(1+\tilde{G}(a;n)) > 1 + \frac{i}{\alpha(n)}.$$

For any $a \in (a_b, a_c)$, this is true if $\delta \geq 1 + \frac{i}{\alpha(n)}$, which is true since

$$(134)\qquad \delta = \frac{1-\tilde{G}(a_b;n)+\frac{i}{\alpha(n)}}{1-\tilde{G}(a_b;n)-a_b\tilde{g}(a_b;n)} \geq 1 + \frac{i}{\alpha(n)(1-\tilde{G}(a_b;n))}.$$

Next, we prove that $G''(a) \leq 0$ is a sufficient (but not necessary) condition for

$$(135)\qquad \frac{\tilde{G}(a;n)\tilde{g}'(a;n)}{\tilde{g}(a;n)^2} \leq 1.$$

Combining (41) with (5), and differentiating (41), yields

$$(136)\qquad \frac{\tilde{G}(a;n)\tilde{g}'(a;n)}{\tilde{g}(a;n)^2} \leq \frac{1-e^{-nG(a)}}{ng(a)}\left(\frac{g'(a)+ng(a)^2}{g(a)}\right)$$

Rearranging the above, we have

$$(137)\qquad \frac{\tilde{G}(a;n)\tilde{g}'(a;n)}{\tilde{g}(a;n)^2} \leq \left(1-e^{-nG(a)}\right)\left(\frac{G''(a)}{ng(a)^2}+1\right).$$

Since $1 - e^{-nG(a)} \leq 1$, it suffices to show that $\frac{G''(a)}{ng(a)^2} \leq 0$. Therefore, if $G''(a) \leq 0$ then $q'(a) > 0$ for all $a \in (a_b, a_c)$. ■

Proofs for Section 6

We first present a lemma that is used to prove Proposition 3.

Lemma 12. *In any equilibrium where $i > 0$,*

1. *There exists a unique cutoff $a_p \in [a_b, \bar{a}]$ such that (i) if $a_p \leq a_c$, there is underconsumption for all $a \in (a_0, a_p)$ and overconsumption for all $a \in (a_p, a_c)$, and (ii) if $a_p > a_c$, there is underconsumption for all $a \in (a_0, a_p)$.*

2. There exists a unique cutoff $a_d \in [a_b, \bar{a}]$ such that (i) if $a_c \leq a_d$, there is overconsumption for all $a \in [a_c, a_d)$ and underconsumption for all $a \in (a_d, \bar{a}]$, and (ii) if $a_c > a_d$, there is underconsumption for all $a \in [a_c, \bar{a}]$.

Proof. *Part 1.* *(i)* For $a \in (a_0, a_b]$, there is underconsumption, i.e. $q_a < q_a^*$, since $q_a = 0$ but $q_a^* > 0$. For $a \in (a_b, a_c]$, we have $a - \phi(a;n) = c'(q_a)/u'(q_a)$ and $a = c'(q_a^*)/u'(q_a^*)$, where $c'(q)/u'(q)$ is increasing in q, so $q_a < q_a^*$ (i.e. underconsumption) for $a \in (a_b, a_c]$ if and only if $\phi(a;n) > 0$. Rearranging (25) yields

$$\phi(a;n) = -\left(\frac{(1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}}{\delta \tilde{g}(a;n)}\right), \tag{138}$$

and therefore $\phi(a;n) > 0$ if and only if

$$-\left((1-\delta)(1-\tilde{G}(a;n)) + \frac{i}{\alpha(n)}\right) > 0. \tag{139}$$

Rearranging, $\phi(a;n) > 0$ if and only if

$$\tilde{G}(a;n) < 1 + \frac{i}{\alpha(n)(1-\delta)}. \tag{140}$$

Since $\tilde{G}'(a;n) = \tilde{g}(a;n) \geq 0$, and $\tilde{G}(a_0;n) = 0$ and $\tilde{G}(\bar{a};n) = 1$, while $1 + \frac{i}{\alpha(n)(1-\delta)} \in [0,1]$, there exists a unique cut-off $a_p \in (a_b, \bar{a}]$ such that $\phi(a;n) > 0$ and there is underconsumption for all $a \in (a_0, a_p)$ where a_p satisfies

$$\delta = 1 + \frac{i}{\alpha(n)[1 - \tilde{G}(a_p;n)]} \tag{141}$$

provided that $a_p \leq a_c$. If $a \in (a_p, a_c)$ then $\phi(a;n) < 0$ and there is overconsumption. *(ii)* If $a_p > a_c$, the range of overconsumption (a_p, a_c) is empty and we have underconsumption for all $a \in (a_0, a_p)$.

Part 2. *(i)* For $a \in [a_c, \bar{a}]$, $q_a = q_{a_c}$ where $a_c - \phi(a_c) = c'(q_{a_c})/u'(q_{a_c})$ and $a = c'(q_a^*)/u'(q_a^*)$. Since $c'(q)/u'(q)$ is increasing in q, we have $q_a > q_a^*$ (i.e. overconsumption) if and only if $a < a_c - \phi(a_c)$. Defining $a_d \equiv a_c - \phi(a_c)$, we have overconsumption for $a \in [a_c, a_d)$ and underconsumption for $a \in (a_d, \bar{a}]$. *(ii)* If $a_d < a_c$, the interval $[a_c, a_d)$ is empty and we have underconsumption for all $a \in [a_c, \bar{a}]$. ■

Proof of Proposition 3

Part 1. Suppose that $a_d = \max\{a_c, a_d\}$. Follows from combining Parts 1 and 2 of Lemma 12 if $a_p \leq a_c \leq a_d$. Suppose that $a_c = \max\{a_c, a_d\}$. Follows from combining Parts 1 and 2 of Lemma 12 if $a_p \leq a_c$ and $a_d < a_c$.

Part 2. If $a_p > a_c$, then $\phi(a; n) > 0$ for all $a \in (a_0, a_p)$ from Part 1 *(ii)* in Lemma 12. In particular, $\phi(a_c) > 0$, so we get $a_c > a_d$. The rest follows from combining Parts 1 and 2 in Lemma 12. If $a_p = a_c$, the result follows from Part 1.

Part 3. If $a_b = a_0$ then $\delta = 1 + \frac{i}{\alpha(n)}$ and (141) implies $\tilde{G}(a_p; n) = 0$ and thus $a_p = a_b = a_0$. Since $a_p \leq a_c$, Part 1 implies there is overconsumption on (a_0, a_u) and underconsumption on $(a_u, \bar{a}]$ where $a_u = \max\{a_c, a_d\}$. Since $\phi(a_c) < 0$ by (**??**), we have overconsumption at a_c. Therefore, $a_c < a_u$ and $a_u = a_d$. ■

Proof of Corollary 1

Part 1. Follows from Part 1 of Proposition 2.

Part 2. Setting $i = 0$ in expression (26) for δ in Proposition 2, we obtain

$$\delta = \frac{1}{1 - \varepsilon_\rho(a_b; n)}. \tag{142}$$

Setting $i = 0$ in expression (25) for $\phi(a; n)$ in Proposition 2, and substituting (142) into (25), we obtain (32).

Part 3. Starting with equation (27) in Proposition 2, setting $i = 0$ implies $a_c = \bar{a}$.

Part 4. Parts 5-8 from Proposition 2 also hold. ■

Proof of Proposition 5

In any full-trade equilibrium where $a_b = a_0$, letting $i \to 0$ gives same allocation as planner. If $a_b = a_0$, then $\varepsilon_\rho(a_b; n) = \varepsilon_\rho(a_0; n) = 0$ and Corollary 1 implies that q_a satisfies $au'(q_a) = c'(q_a)$ for all $a \in A$, which is equivalent to the planner's FOC (6). Also, we know that $\alpha'(n)\tilde{s}(n) + \alpha(n)\tilde{s}'(n) = k$, which is equivalent to the planner's FOC (7). Finally, buyers always choose the highest quality seller in any meeting and therefore the distribution of chosen goods is equal to the distribution of the maximum, given by (5), which is the same as the distribution of goods chosen by the planner. Therefore, $i \to 0$ gives the same allocation as the planner. Conversely, $q_a = q_a^*$ for all a requires that $a_b = a_0 = 0$. ■

Proof of Proposition 6

For any $a \in (a_0, a_b]$, the efficient quantity is not traded even when $i \to 0$ since $q_a^* > 0$ but $q_a = 0$. The efficient quantity is traded at $a_0 = 0$ since $q_0 = q_0^* = 0$. To get the efficient quantity at $i \to 0$ for any $a \in (a_b, \bar{a}]$, Corollary 1 implies that we require either $\varepsilon_\rho(a_b; n) = 0$ or $a = \bar{a}$. This is true only if $a_b = a_0 = 0$ or $a = \bar{a}$. In general, if the equilibrium is partial trade ($a_b > a_0$), then for any $a \in (a_0, \bar{a})$, we have $q_a \neq q_a^*$. Since $\varepsilon_\rho(a_b; n) > 0$, there is underconsumption for all $a \in (a_0, \bar{a})$. ■

Proof of Proposition 7

At the Friedman rule, entry is efficient if the equilibrium is full trade. In any partial-trade equilibrium, we know from Proposition 6 that $q_a^* > q_a$ for any $a \in (a_0, \bar{a})$ at the Friedman rule. The equilibrium n satisfies

$$\alpha'(n)\tilde{s}(n; \{q_a\}_{a \in A}) + \alpha(n)\tilde{s}'(n; \{q_a\}_{a \in A}) = k \tag{143}$$

and the efficient n^* satisfies

$$\alpha'(n^*)\tilde{s}(n^*; \{q_a^*\}_{a \in A}) + \alpha(n^*)\tilde{s}'(n^*; \{q_a^*\}_{a \in A}) = k. \tag{144}$$

We know $q_a^* > q_a$ for any $a \in (a_0, \bar{a})$, but we cannot infer anything about whether there is under-entry ($n < n^*$), over-entry ($n > n^*$), or efficient entry ($n = n^*$). We can find examples of equilibria for each of these three possibilities. ■

Online Appendix B: Comparative statics

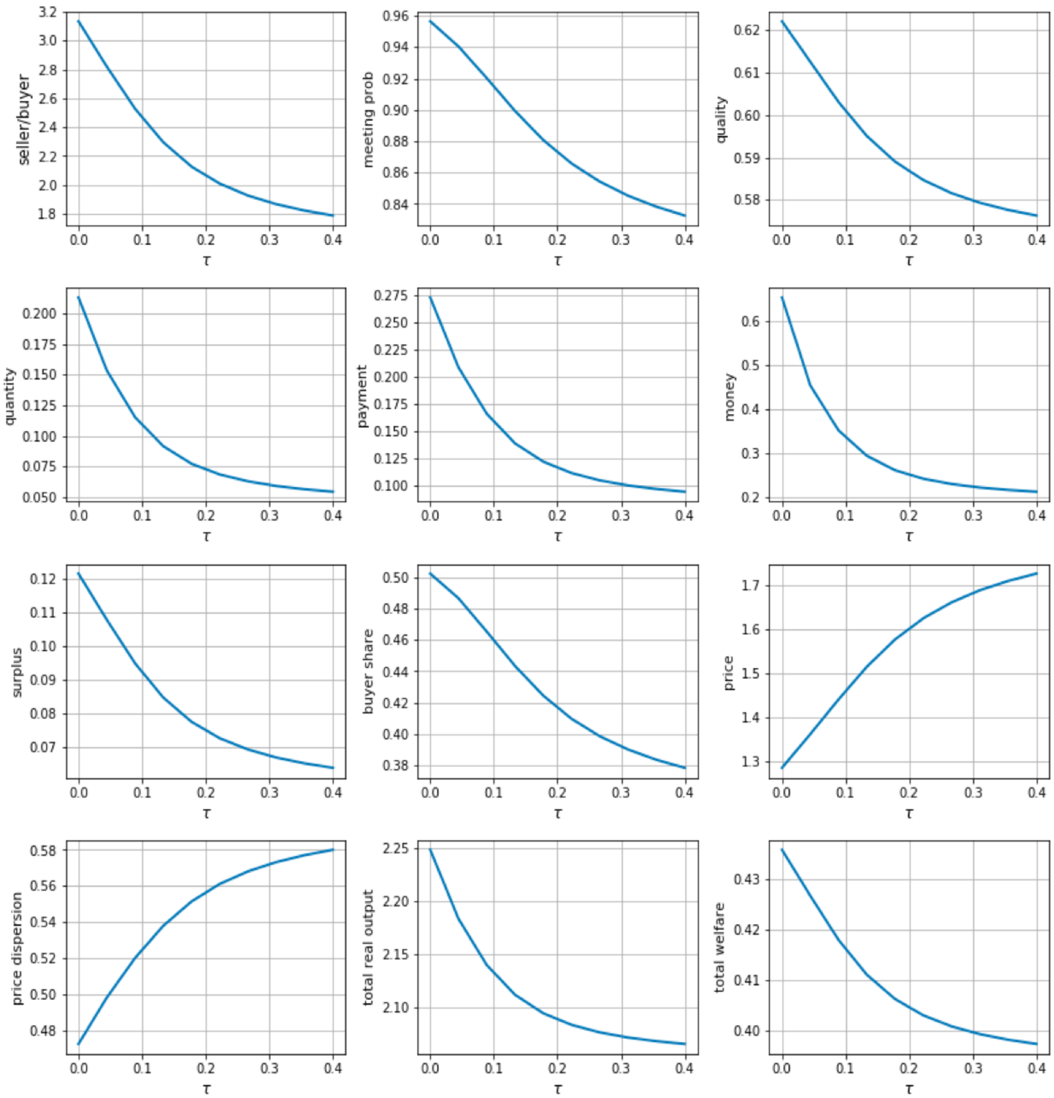

Figure 7: Comparative statics with respect to inflation rate τ

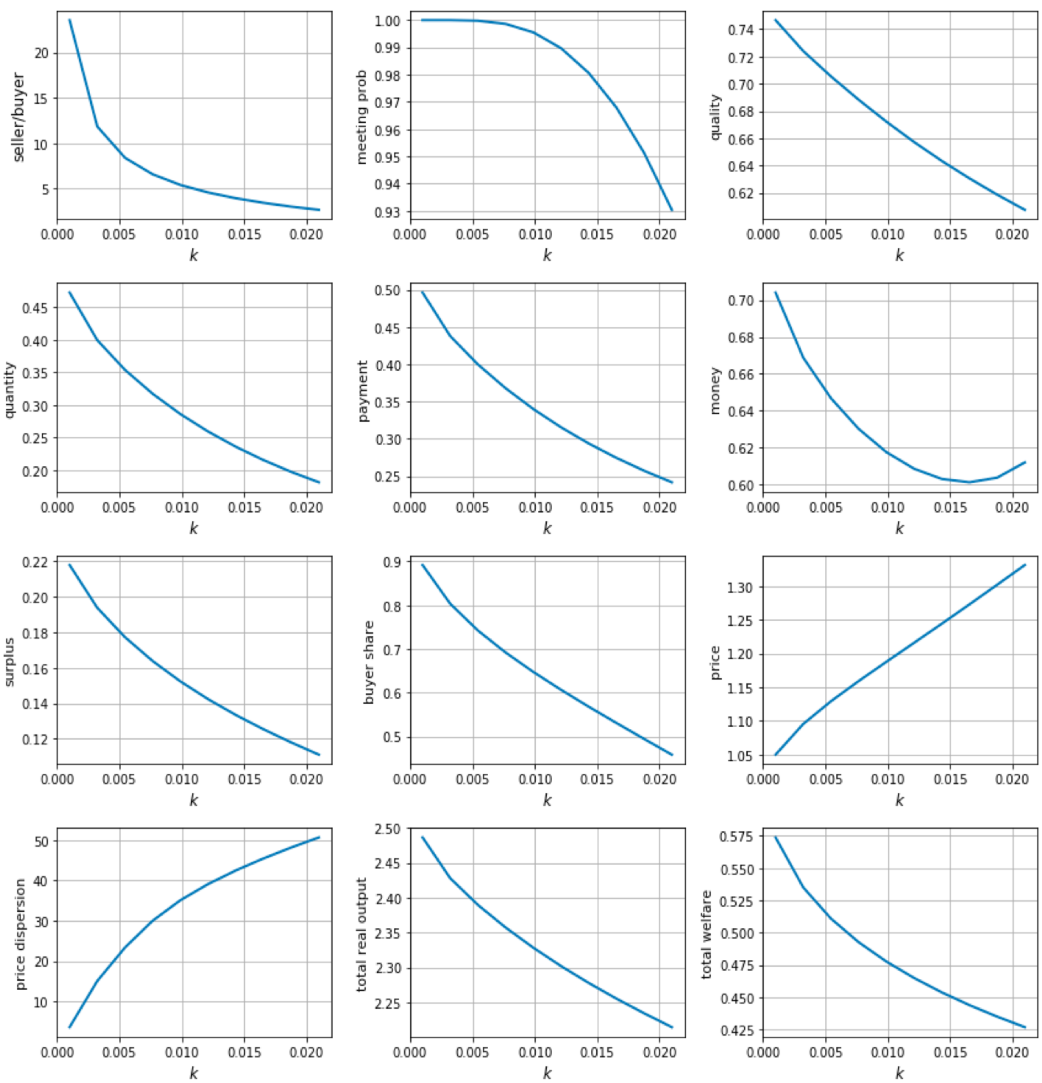

Figure 8: Comparative statics with respect to entry cost k

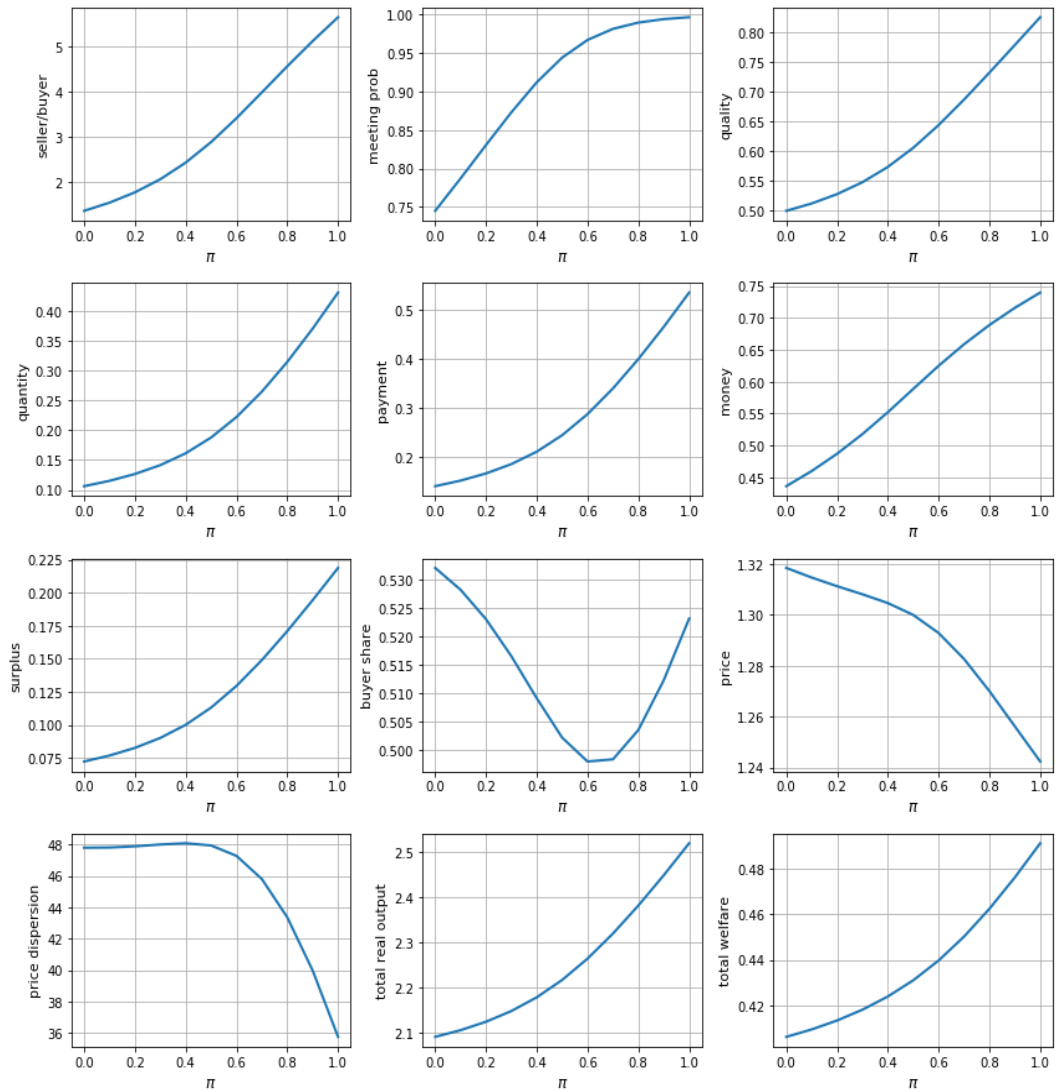

Figure 9: Comparative statics with respect to degree of choice π