Consumer Choice and Nonlinear Pricing in Competitive Search Equilibrium*

Sephorah Mangin†

June 30, 2025

Abstract

We study the effects of competition on nonlinear pricing in a competitive search model where the distribution of buyer types is *endogenous* due to consumer choice. In the first stage (i.e. competitive search), sellers compete by offering price-quantity schedules that attract buyers to their submarket. In the second stage (i.e. imperfect competition), buyers meet a finite number of sellers within a submarket and *choose* a seller from within their multilateral meeting. The *type* of a consumer is their private valuation of their chosen seller's good. The endogenous distribution of types depends on the distribution of valuations, the degree of competition, and properties of the *search technology* (i.e. the distribution of the number of sellers a buyer meets). We find that greater competition decreases the extent of nonlinear pricing and reduces the quantity distortions. In the competitive limit, the effects of private information are eliminated.

JEL codes: D40, D43, D82, D83

Keywords: Competitive search; Private information; Search technology; Multilateral meetings; Nonlinear pricing; Imperfect competition

---

*I would like to thank Ayushi Bajaj, Jonathan Chiu, Michael Choi, Mohammad Davoodalhosseini, Lucas Herrenbrueck, Janet Jiang, Guillaume Rocheteau, Mario Silva, and Yu Zhu for very useful comments. I also thank seminar participants at the Australian National University, Bank of Canada, U.C. Irvine, Simon Fraser University, and the National University of Singapore.

†Research School of Economics, The Australian National University, HW Arndt Building 25A, Kingsley Place, Acton ACT 2601 Australia. Email: sephorah.mangin@anu.edu.au.

1 Introduction

In retail trade, consumer preferences are not directly observed by firms. As a result, firms often elicit these preferences through second-degree price discrimination on the basis of quantity. To do this, firms offer price-quantity schedules that are nonlinear, i.e. the unit price is decreasing with the quantity purchased. There is recent evidence that nonlinear pricing is widespread in retail trade, with Bornstein and Peter (2025) finding that over 90% of retail sales are of multi-sized products featuring quantity discounts. While this form of price discrimination has been classically studied for the case of monopolists in Mussa and Rosen (1978) and Maskin and Riley (1984), it is less clear how *competition* affects the quantity distortions due to price discrimination in oligopolistic environments where firms directly compete for customers.

This paper examines how competition affects price discrimination and the extent of nonlinear pricing. To do so, we develop a search-theoretic model of retail trade. As is standard in retail trade environments, we assume that sellers post price-quantity schedules that they commit to *ex ante* in order to attract buyers. Buyers then choose sellers through two distinct stages. First, they focus their search on the group of sellers with the most attractive price-quantity schedules. Next, buyers contact or "meet" a finite subset of sellers, which we call their *choice set*. Buyers then choose which seller to purchase from and how much to buy, where buyers' choices of both seller and quantity are driven by their valuations, which are private information.

We model seller competition through *competitive search equilibrium* in the spirit of Moen (1997). Specifically, we interpret groups of sellers that post the same price-quantity schedule and feature the same trading probability as "submarkets". In the model, buyers choose which group of sellers to target by choosing the submarket which offers the most attractive combination of price-quantity schedule and trading probability. Seller competition is captured through the fact that sellers must offer contracts that attract buyers to their own submarket in order to increase demand. Within submarkets, all agents commit to trading at the terms posted in that submarket.

Competition is imperfect in our model because search frictions ensure that buyers only meet a finite number of sellers *within* their chosen submarket. To model search frictions within submarkets, we use the general class of search technologies developed in Lester, Visschers, and Wolthoff (2015). In our environment, we have *one-to-many meetings* because buyers can contact either no sellers, one seller, or more than one seller. A "meeting" is simply an opportunity for a buyer to purchase from a seller. We interpret the finite set of sellers a buyer "meets" as the buyer's *choice set*.

After meetings take place, buyers draw a private valuation for each seller in their choice set. We interpret this as buyers learning their preferences *after* observing the goods offered by the sellers they meet, but before purchasing. Buyers then choose to purchase from the seller that maximizes their net utility. Because all sellers offer the same price-quantity schedule within a given submarket, buyers' choice of a specific seller within their choice set is based purely on idiosyncratic preferences. After choosing a seller, buyers determine the quantity of the good to purchase. We focus on incentive-compatible direct revelation mechanisms that induce buyers to reveal their private information to their chosen seller through their choice of quantity.

We establish conditions under which the existence and uniqueness of equilibrium is guaranteed. In equilibrium, there is only one active submarket and all sellers offer the same price-quantity schedule. Importantly, the presence of competitive search ensures that the direct competition between sellers to attract buyers is built into this equilibrium price-quantity schedule. Our general expression for equilibrium quantities can incorporate any degree of competition between monopoly and perfect competition, and it nests both the standard monopoly benchmark and the competitive limit.

Unsurprisingly, the decentralized equilibrium cannot deliver the first-best allocation. As is standard, the equilibrium quantities traded are distorted downwards due to the presence of buyers' private information, while there is no distortion at the top (i.e. at the highest buyer type). Moreover, it is possible that some buyers do not purchase the good, i.e. there may be limited market coverage or participation. We discuss partial coverage in the Appendix and focus our attention on full coverage.

Importantly, the extent of these informational distortions are *endogenous* in our model. In particular, these distortions depend on the seller-buyer ratio, which we interpret as the *degree of competition* because it is the expected number of firms in a buyer's choice set. We prove that the equilibrium price-quantity schedule features quantity discounts and we derive sufficient conditions under which the intensity of nonlinear pricing is *decreasing* in the degree of competition. We find that greater competition leads sellers to reduce the quantity distortions from private information in order to attract more buyers, bringing us closer to the first best. Policy changes that increase competition by lowering the costs of firm entry can thus affect the extent of nonlinear pricing and the quantity distortions due to private information.

The fact that these informational distortions can be influenced by the degree of competition results from a novel feature of our model: the distribution of buyer types is itself *endogenous* and depends on the degree of competition. This arises due to consumer choice via multilateral meetings. With consumer choice, sellers know that any buyer with whom they trade must have chosen their good from that buyer's choice set, so the distribution of "types" is the endogenous distribution of buyers' choices of seller. Greater competition in the sense of more sellers per buyer leads to greater choice for buyers, so the distribution of types for a higher seller-buyer ratio first-order stochastically dominates the distribution for a lower seller-buyer ratio. Importantly, any shift in the distribution of types due to a change in the degree of competition affects the extent of the quantity distortion due to buyers' private information.

In the competitive limit where the seller-buyer ratio becomes large, we find that the quantity distortion is eliminated altogether and we obtain the first best.

2 Related literature

This paper contributes to the large literature on directed and competitive search.[1] In particular, we contribute to the literature on competitive search under private information, including Faig and Jerez (2005), Menzio (2007), Guerrieri (2008), Guerrieri, Shimer, and Wright (2010), Moen and Rosen (2011), Albrecht, Gautier, and Vroman (2014), Davoodalhosseini (2019), Lester, Shourideh, Venkateswaran, and Zetlin-Jones (2019), Roger and Julien (2023), and Auster, Gottardi, and Wolthoff (2025).

Similarly to Auster et al. (2025), we focus on multilateral meetings and how this feature interacts with private information and search frictions. As in Auster et al. (2025), we allow buyers to *simultaneously* contact multiple trading partners. In our environment, however, buyers' private valuations or "types" are realized ex post, i.e. *after* contracts are posted and meetings take place (but before trade occurs). As a result, buyer "types" affect the choices of both seller and quantity within a meeting,

[1]Important earlier contributions include Montgomery (1991), Peters (1991), Shimer (1996), Moen (1997), Peters and Severinov (1997), Acemoglu and Shimer (1999), Julien, Kennes, and King (2000), Burdett, Shi, and Wright (2001), and Shi (2001). See also Rocheteau and Wright (2005), Albrecht, Gautier, and Vroman (2006), Shi (2009), Menzio and Shi (2010), Menzio and Shi (2011), and Galenianos and Kircher (2012). For a survey, see Wright, Kircher, Julien, and Guerrieri (2021). More recent contributions include Rabinovich and Wolthoff (2022), which develops a model of partially directed search based on Lester (2011), and Albrecht, Cai, Gautier, and Vroman (2023a), which considers the role of market makers and the effect of multiple trading partners in competitive search equilibrium.

but do not affect buyers' choice of submarket and there is no market segmentation. Our environment is also somewhat related to Roger and Julien (2023), which develops a competitive search model with competing principals and moral hazard in hidden actions, and Faig and Jerez (2005), which develops a competitive search model with nonlinear pricing and bilateral meetings (i.e. local monopoly) while we introduce buyer choice sets via multilateral or one-to-many meetings (i.e. local oligopoly).[2]

This paper also contributes to the literature on the search-theoretic model of imperfect competition developed in Butters (1977), Varian (1980), and Burdett and Judd (1983). In this class of models, due to search frictions, buyers can only purchase from a finite subset of sellers who the buyer samples or "contacts". This corresponds to the multilateral or one-to-many meetings in our model, which represent buyers' choice sets. In both our environment and these models, the extent of competition for a given buyer is determined by the size of a buyer's choice set.[3] Generally, in these models the distribution of the size of buyers' choice sets is either discrete (one or two buyers) or Poisson. We generalize the assumption of a Poisson distribution by allowing a much wider class of search technologies that nests the Poisson as a special limiting case.

The class of general search technologies we use to model multilateral meetings was developed in Lester et al. (2015), further studied in Cai, Gautier, and Wolthoff (2017), and more recently used in Mangin (2024) and Mangin (2025). Cai, Gautier, and Wolthoff (2025) microfound this class of search technologies and show that they can be used to model heterogeneity across agents with respect to meeting probability. For example, in our environment the search technology could be microfounded in terms of heterogeneity across buyers with respect to search intensity; see Mangin (2025).

This paper is closely related to Lester et al. (2019), which considers the interaction between adverse selection and imperfect competition in a Burdett-Judd style search-theoretic model. The present paper shares with Lester et al. (2019) an interest in the *interaction* between private information and imperfect competition. In the framework of Lester et al. (2019), the *degree of competition* is captured by the probability of

[2]The distribution of the size of buyers' choice sets is related to the distribution of the price count (i.e. the number of quotes a consumer obtains) in Bergemann, Brooks, and Morris (2021), and the distribution of the number of firms in consumers' consideration sets (i.e. the set of firms a consumer considers for their purchase) in Armstrong and Vickers (2022).

[3]Models based on the Burdett and Judd (1983) search-theoretic model of imperfect competition are featured in Kaplan and Menzio (2015), Kaplan and Menzio (2016), Kaplan, Menzio, Rudanko, and Trachter (2019), Burdett and Menzio (2018), Albrecht, Menzio, and Vroman (2023b), Nord (2023), Menzio (2024a), Menzio (2024b), and Mangin and Menzio (2024).

competition (versus monopoly), while in our model it is captured by the seller-buyer ratio or the average size of a buyer's choice set. However, we find that the probability of competition (versus monopoly) also turns out to play a crucial role in our model because this probability directly affects the extent of quantity distortions.

Our model can be interpreted as an imperfectly competitive version of the model of nonlinear monopoly pricing under incomplete information developed in Mussa and Rosen (1978) and Maskin and Riley (1984). There is a small literature on imperfectly competitive nonlinear pricing, which includes Rochet and Stole (1997), Armstrong and Vickers (2001), and Yang and Ye (2008). While different in approach, our model shares with this literature the fact that sellers compete to offer price-quantity schedules that are attractive to buyers, rather than a monopolist determining a price-quantity schedule solely for the purpose of price discrimination without competitors. However, our two-stage search-theoretic approach is novel. We maintain tractability by using competitive search to capture competition among sellers at the first stage when price-quantity schedules are posted, and one-to-many meetings within submarkets to model idiosyncratic buyer choice of seller from a finite choice set in the second stage.[4]

Building on the model developed in this paper, Bajaj and Mangin (2024) develops a search-theoretic model of monetary exchange based on Rocheteau and Wright (2005). The authors allow for the possibility of both multilateral meetings with consumer choice (as in this paper) and bilateral meetings (as in standard monetary search models). The authors examine the effect of greater consumer choice on both market power and the welfare cost of inflation. Surprisingly, they find that a greater degree of consumer choice can have a non-monotonic effect on the market power of firms. When they calibrate the model to U.S. data, they also find that greater consumer choice leads to a quantitatively significant increase in the welfare cost of inflation.

3 Environment

Consider a static environment. There is a continuum of agents divided into two types: *buyers* and *sellers*. Buyers are ex ante identical and sellers are ex ante identical.

[4]The sequential nature of search in our model, in which buyers first choose a submarket using directed or competitive search and then face a "noisy" process of choosing or matching among the random subset of sellers they meet, shares some similarities with the model of sequentially mixed search developed in Shi (2023). However, in our model, buyers' choice of seller within meetings is driven by buyers' private information about their preferences rather than by prices.

The sets of buyers and sellers are denoted B and S respectively. All buyers participate in the market at zero cost, but there is an entry decision by sellers. Only a subset $\bar{S} \subseteq S$ of sellers of measure n enter the market. Sellers may or may not choose to enter at cost $\kappa > 0$ and thus $n \in \mathbb{R}_+$ is endogenous.[5] We normalize $|B| = 1$, so the measure n of sellers who enter is also the seller-buyer ratio.

One-to-many meetings. All sellers meet exactly one buyer, but the number of sellers buyer i meets is a random variable N_i. For any given seller-buyer ratio n, the probability that a buyer meets $j \in \{0, 1, 2, ...\}$ sellers is given by a *search technology* $P_j : \mathbb{N} \to [0, 1]$ where $P_j(n) = \Pr(N_i = j)$ and $\sum_{j=0}^{\infty} jP_j(n) = n$, the expected number of sellers a buyer meets or the average size of a buyer's choice set.

Let $m : \mathbb{R}^+ \to [0, 1]$ be a function that represents the endogenous probability $m(n)$ that a buyer meets at least one seller, i.e. $m(n) \equiv 1 - P_0(n)$. This is the probability that a buyer receives a meeting, so we refer to the function m as the *meeting function.* The probability that a seller receives a meeting is one, but the probability a seller receives a match equals $m(n)/n$, the probability the seller is *chosen* by a buyer.

Buyer's choice of seller. After a meeting with j sellers takes place, the buyer draws a valuation $\theta \in \mathbb{R}_+$ for each of the j sellers they meet. These valuations $(\theta_1, \theta_2, ..., \theta_j)$ are private information for the buyer. We assume that valuations are drawn *after* meetings take place because we interpret this as buyers learning their preferences after observing the goods offered by the sellers they meet (but before purchasing). After learning their valuations, the buyer then chooses *one* seller.[6]

Distribution of valuations. Consumers' private valuations $\theta \in \mathbb{R}_+$ are drawn from a bounded, continuous distribution with cdf G and pdf $g = G'$. The distribution of valuations G is known to all agents. We assume the distribution G is not degenerate and has no mass points, and Assumption 1 is maintained throughout.

Assumption 1. *The distribution of valuations has twice-differentiable cdf G, where $g = G' > 0$, and bounded support $\Theta = [\underline{\theta}, \bar{\theta}] \subseteq \mathbb{R}_+$.*

Buyer and seller utility. Sellers can produce on demand any quantity $q \in \mathbb{R}_+$ of a divisible good at cost $c(q)$, where $c : \mathbb{R}_+ \to \mathbb{R}_+$, and we assume that $c(0) = 0$,

[5] We assume the set S is sufficiently large that $n \leq |S|$ always.

[6] Similarly to discrete choice models, we assume that consumers choose to purchase from a single firm in each meeting. Alternatively, this could be endogenized through specification of preferences.

$c'(q) > 0$, and $c''(q) \geq 0$ for all $q > 0$. A buyer who consumes quantity q of a good with valuation θ receives utility given by a function $\tilde{u} : \mathbb{R}_+^2 \to \mathbb{R}_+$ defined by

$$\tilde{u}(q, \theta) \equiv \theta u(q), \tag{1}$$

where $u : \mathbb{R}_+ \to \mathbb{R}_+$ and we assume that $u(0) = 0$, $u'(0) = \infty$, $u'(q) > 0$, and $u''(q) < 0$ for all $q > 0$. As we discuss later in Section 6.1, our assumption on the form of the utility function $\tilde{u}$ will ensure that the single crossing condition holds.

4 Distribution of buyer types

The distribution of buyers' choices, which we denote by $\tilde{G}$, is the distribution of the valuations of the goods actually *chosen* by buyers. We can interpret a buyer's *type* as their valuation for their chosen good. The distribution of buyers' choices is therefore the distribution of buyer types. Intuitively, it is the outcome of a buyer's choice that represents the private information that is relevant to their chosen seller. From now onwards, we refer to $\tilde{G}$ simply as the *distribution of types.*[7]

The distribution of types $\tilde{G}$ is endogenous. It depends on the equilibrium seller-buyer ratio n and the equilibrium choices made by buyers regarding which seller to purchase from. We will later prove that, in any equilibrium we consider, buyers always choose the highest-valuation seller they meet. Therefore, the distribution of types equals the distribution of the *highest valuation among the sellers a buyer meets*, conditional on meeting a seller. It is this distribution we discuss here.

Throughout the paper, we assume that the search technology P_j is *invariant*, as defined in Lester et al. (2015). The assumption of invariance is useful because the function P_0 captures everything we need to know about the search technology. Examples of invariant meetings technologies include the family of negative binomial distributions, which includes the geometric search technology as a special case and the widely-used Poisson search technology as a limiting case. For a discussion of the intuition behind Assumption 2, see Lester et al. (2015) and Mangin (2024).[8]

[7]Note that the distribution of buyer types $\tilde{G}$ is different than the distribution of purchased goods. In equilibrium with private information, some meetings may not result in trade.

[8]As shown in Cai et al. (2025), any distribution which can be represented as a mixed Poisson distribution satisfies Assumption 2. This is a very wide class of discrete probability distributions.

Assumption 2. *The search technology P_j is* invariant*, i.e. for all $y \in [0,1]$,*

$$\sum_{j=0}^{\infty} P_j(n)y^j = P_0(n(1-y)) \tag{2}$$

where $\mathbb{E}_P(N_i) = n$ and $P_0 : \mathbb{R}^+ \to [0,1]$ is continuous and infinitely differentiable.

Lemma 1 shows that invariant search technologies deliver standard properties for the meeting function m, defined by $m(n) \equiv 1 - P_0(n)$. We refer to $\eta_m(n) \equiv m'(n)n/m(n)$, the elasticity of the meeting function m, as the *meeting elasticity*. The properties highlighted in Lemma 1 will later prove to be useful for some results.

Lemma 1. *If P_j is invariant and $n > 0$, then $m'(n) > 0$, $m''(n) < 0$, $\lim_{n\to 0} m(n) = 0$, $\lim_{n\to 0} m'(n) = 1$, $\lim_{n\to\infty} m(n) = 1$, $\lim_{n\to\infty} m'(n) = 0$, and $\lim_{n\to\infty} m''(n) = 0$. We also have $\eta'_m(n) < 0$, $\lim_{n\to 0} \eta_m(n) = 1$, $\lim_{n\to\infty} \eta_m(n) = 0$, and $\frac{d}{dn}\left(\frac{-m''(n)n}{m'(n)}\right) > 0$.*

Lemma 2 presents an expression for the distribution of types. This expression depends only on the function P_0, which gives the probability a buyer meets no sellers. For example, if the search technology is Poisson then $P_0(x) = e^{-x}$.

Lemma 2. *If P_j is invariant and $n > 0$, the distribution of types has cdf*

$$\tilde{G}(\theta; n) = \frac{P_0(n(1 - G(\theta))) - P_0(n)}{1 - P_0(n)}. \tag{3}$$

1. *In the limit as $n \to 0$, we have $\tilde{G} \to G$ and $\tilde{\theta}(n) \to E_G(\theta)$.*
2. *In the limit as $n \to \infty$, we have $\tilde{G}(\theta; n) \to 0$ for all $\theta \in [\underline{\theta}, \bar{\theta})$, and $\tilde{\theta}(n) \to \bar{\theta}$.*
3. *The distribution of types $\tilde{G}$ first-order stochastically dominates the distribution of valuations G and the average type exceeds the average valuation, $\tilde{\theta}(n) > E_G(\theta)$.*
4. *If $n > n'$, the distribution $\tilde{G}(\theta; n)$ first-order stochastically dominates $\tilde{G}(\theta; n')$.*
5. *For any $f : \Theta \to \mathbb{R}_+$ such that $f' > 0$, $\tilde{f}'(n) > 0$ where $\tilde{f}(n) \equiv \int_{\underline{\theta}}^{\bar{\theta}} f(\theta) d\tilde{G}(\theta; n)$.*

Parts 1 and 2 of Lemma 2 say that, in the limit as $n \to 0$, the distribution of types $\tilde{G}$ converges to the distribution of valuations G, and in the limit as $n \to \infty$ the distribution of types $\tilde{G}$ becomes degenerate at the maximum valuation $\bar{\theta}$.

Part 3 of Lemma 2 states that the distribution of types $\tilde{G}$ first-order stochastically dominates the distribution G, and the average valuation of a *chosen* good $\tilde{\theta}(n) \equiv E_{\tilde{G}}(\theta)$ (i.e. the average buyer *type*) is greater than the average valuation, $E_G(\theta)$.

Part 4 of Lemma 2 says that an increase in the seller-buyer ratio n leads to a first-order stochastic dominance shift in the distribution of types $\tilde{G}(\theta; n)$. Roughly speaking, more seller entry leads to a "better" distribution of buyers' chosen goods.

Part 5 of Lemma 2 implies the average type $\tilde{\theta}(n)$, i.e. the average valuation of a chosen good, is strictly increasing in n, i.e. $\tilde{\theta}'(n) > 0$. Intuitively, $\tilde{\theta}(n)$ is increasing in the seller-buyer ratio because more sellers per buyer means greater choice of seller.

5 First-best allocation

Before we consider competitive search equilibrium, we determine the first-best allocation. To do so, we solve an omniscient planner's problem where the planner has complete information about buyers' valuations and is constrained only by search frictions. We are interested in the first-best allocation as a benchmark only.[9]

The planner knows the search technology P_j and the cost of entry κ, as well as the buyers' valuations for each seller they meet. The planner chooses a seller-buyer ratio n^*, a quantity function $q^* : \Theta \to \mathbb{R}_+$, and a distribution of chosen goods with cdf $\tilde{G} : \Theta \to [0, 1]$, in order to maximize the total surplus created minus the total cost of seller entry, subject to the search frictions. That is, the planner solves:

$$\max_{n \in \mathbb{R}_+, \{q_\theta\}_{\theta \in \Theta}} \left\{ m(n) \int_{\underline{\theta}}^{\bar{\theta}} [\theta u(q_\theta) - c(q_\theta)] \, d\tilde{G}(\theta; n) - n\kappa \right\} \tag{4}$$

where $\tilde{G}$ represents the planner's optimal choice of seller for each buyer.[10]

Define $s_\theta \equiv \theta u(q_\theta) - c(q_\theta)$, the trade surplus (or match surplus) for a good with valuation θ. Let q^*_θ denote the first-best quantity for valuation θ and define $s^*_\theta \equiv \theta u(q^*_\theta) - c(q^*_\theta)$. Assume that $s^*_0 \geq 0$ where $s^*_0 \equiv \underline{\theta} u(q_0) - c(q_0)$ and $q_0 = q(\underline{\theta})$, so there

[9] It is standard in the search literature to refer to this as the *first-best allocation* even though it is constrained by search frictions. For example, see Davoodalhosseini (2019).

[10] The first-best distribution of choices will turn out to be equal to the buyers' distribution of types in equilibrium. Anticipating this, we use the same notation, $\tilde{G}$.

is a (weakly) positive trade surplus for all goods. Define the *expected trade surplus* by

$$\tilde{s}(n;\{q_\theta\}_{\theta\in\Theta}) \equiv \int_{\underline{\theta}}^{\bar{\theta}} [\theta u(q_\theta) - c(q_\theta)] d\tilde{G}(\theta;n). \tag{5}$$

For simplicity of notation, throughout the paper we sometimes suppress the dependence of the expected trade surplus $\tilde{s}(n;\{q_\theta\}_{\theta\in\Theta})$ on the function $q:\Theta\to\mathbb{R}_+$ and let $\tilde{s}(n)$ denote $\tilde{s}(n;\{q_\theta\}_{\theta\in\Theta})$ and $\tilde{s}'(n)$ denote $\partial\tilde{s}(n)/\partial n$.

The following assumption ensures the existence of a social optimum where $n^*>0$. We maintain Assumption 3 throughout the remainder of the paper. Intuitively, this condition says that the expected trade surplus in the limit as $n\to 0$, i.e. $\lim_{n\to 0}\tilde{s}(n)$, must be greater than the entry cost κ.[11] It follows from our earlier assumptions that, for all $\theta\in\Theta$, there exists a unique $q_\theta^*\in\mathbb{R}_+$ such that $\theta u'(q_\theta^*) = c'(q_\theta^*)$.

Assumption 3. *The cost of entry is not too high:* $E_G[\theta u(q_\theta^*) - c(q_\theta^*)] > \kappa$.

Proposition 1 states that there exists a unique first-best allocation $(n^*,\{q_\theta^*\}_{\theta\in\Theta})$ with $n^*>0$ and provides a characterization.

Proposition 1. *There exists a unique first-best allocation and it satisfies:*

1. *For any* $\theta\in\Theta$, *the quantity* $q_\theta^*>0$ *solves*

$$\theta u'(q_\theta^*) = c'(q_\theta^*). \tag{6}$$

2. *The seller-buyer ratio* $n^*>0$ *satisfies*

$$m'(n^*)\tilde{s}(n^*;\{q_\theta^*\}_{\theta\in\Theta}) + m(n^*)\tilde{s}'(n^*;\{q_\theta^*\}_{\theta\in\Theta}) = \kappa. \tag{7}$$

3. *The distribution of types* $\tilde{G}$ *is given by (3).*

Equation (7) is a version of the generalized Hosios condition discussed in Mangin and Julien (2021). Defining the meeting elasticity by $\eta_m(n)\equiv m'(n)n/m(n)$ and the surplus elasticity by $\eta_s(n)\equiv\tilde{s}'(n)n/\tilde{s}(n)$, condition (7) is equivalent to

$$\underbrace{\eta_m(n)}_{\text{meeting elasticity}} + \underbrace{\eta_s(n;\{q_\theta^*\}_{\theta\in\Theta})}_{\text{surplus elasticity}} = \underbrace{\frac{n\kappa}{m(n)\tilde{s}(n;\{q_\theta^*\}_{\theta\in\Theta})}}_{\text{sellers' surplus share}}. \tag{8}$$

[11] Since $\tilde{G}\to G$ as $n\to 0$, as verified in Lemma 2, we have $\lim_{n\to 0}\tilde{s}(n) = E_G[\theta u(q_\theta^*) - c(q_\theta^*)]$

We have not yet discussed equilibrium, but it is useful to refer to the term on the right of (8) as the sellers' surplus share. Given that our equilibrium features free entry of sellers at cost κ, sellers' total expected payoff will be equal to the total cost of seller entry $n\kappa$, and the total surplus created is $m(n)\tilde{s}(n)$. Therefore, the term on the right will be sellers' surplus share in equilibrium. The generalized Hosios condition (8) says that constrained efficiency requires sellers' surplus share to be equal to the meeting elasticity *plus* the surplus elasticity. Since s^*_θ is increasing in θ, Part 5 of Lemma 2 implies that the expected trade surplus $\tilde{s}(n)$ is increasing in the seller-buyer ratio, i.e. $\tilde{s}'(n) > 0$.[12] Therefore, the surplus elasticity $\eta_s(n)$ is positive.

Intuitively, more sellers per buyer means greater choice for buyers, thus increasing the average trade surplus. Equivalently, there is a positive externality arising from the effect of seller entry on the average surplus when there is consumer choice. When the generalized Hosios condition (8) holds, both the standard matching externalities and this novel externality are internalized, delivering constrained efficiency of entry.

6 Competitive search equilibrium

Competitive search is an equilibrium concept developed in Moen (1997) and Shimer (1996). The basic idea is that either buyers or sellers, or sometimes "market makers", post contracts that specify the terms of trade offered. Search is directed in the sense that buyers and sellers choose which *submarket* to enter, where each submarket corresponds to a particular specification of the terms of trade. Commitment is key: buyers and sellers who enter a submarket *commit* to trade at the terms specified within that submarket. Within each submarket, buyers and sellers face search frictions.

We assume that each seller posts a menu of contracts to attract buyers. Sellers take into account the expected relationship between the posted contracts and the seller-buyer ratio n. We restrict sellers to post contracts that are price-quantity schedules of the form (q_θ, t_θ), which specify the quantity of the good q_θ and the payment or transfer t_θ contingent on the buyer's valuation θ for their chosen seller. These contracts (q_θ, t_θ) may depend on n, the expected number of sellers a buyer meets, but not on the actual number of sellers a specific buyer meets (which is not observed by sellers).

Within each submarket, meetings take place, buyers choose sellers, and trade occurs as described in Section 3. In any meeting, the buyer chooses the seller that

[12] It is established in the proof of Proposition 1 that both q^*_θ and s^*_θ are increasing in θ.

maximizes $v_\theta \equiv \theta u(q_\theta) - t_\theta$, the buyer's ex post trade surplus. The equilibrium distribution of types $\tilde{G}$ is determined by buyers' optimal choices of sellers.

Within meetings, buyers' valuations are private information and they cannot be observed directly by any seller (including their chosen seller). However, buyers may choose to reveal their private information to their chosen seller through their choice of contract (q_θ, t_θ) offered by the chosen seller. By the revelation principle, it is without loss of generality to focus on individually rational and incentive-compatible direct mechanisms that induce buyers to truthfully reveal their private information.

Sellers post contracts subject to the following constraints: an individual rationality (IR) constraint and an incentive compatibility (IC) constraint. The IR constraint is

$$\theta u(q_\theta) - t_\theta \geq 0 \tag{9}$$

for all $\theta \in \Theta$. This condition states that buyers must receive a (weakly) positive ex post trade surplus, otherwise they will not trade. The IC constraint is given by

$$\theta u(q_\theta) - t_\theta \geq \theta u(q_{\theta'}) - t_{\theta'} \tag{10}$$

for all $\theta, \theta' \in \Theta$. Intuitively, this condition states that a buyer with valuation θ cannot do better by choosing a contract $(q_{\theta'}, t_{\theta'})$ instead of (q_θ, t_θ).

Restriction of contract space. Our restriction on the contract space warrants a brief discussion. We restrict sellers to post contracts of the form (q_θ, t_θ) because it is more realistic in retail trade for sellers to post this simple form of contract. Given that our model aims to achieve greater realism by introducing consumer choice, it is important to also maintain realism with respect to the form of contracts sellers post. Contracts of the form (q_θ, t_θ) are realistic for retail trade because they are equivalent to (q_θ, p_θ) where p_θ is the unit price. This is equivalent to sellers simply choosing a function $p(q)$ for unit prices based on the quantity. In retail trade, it is common for sellers to set different unit prices depending on the quantity purchased (e.g. through different package sizes and quantity discounts). By focusing on contracts of the form (q_θ, p_θ), or equivalently (q_θ, t_θ), we can also compare our results to the standard theory of nonlinear pricing in Mussa and Rosen (1978) and Maskin and Riley (1984).

At the start of the period, sellers enter and announce the submarkets $\{(q_\theta, t_\theta)\}_{\theta \in \Theta}$, implying an expected seller-buyer ratio n for each submarket. Buyers then choose a submarket in which to trade, in a manner consistent with expectations. Agents trade in their chosen submarket. We let Ω denote the set of open submarkets ω. A submarket ω is characterized by the menu of contracts and implied seller-buyer ratio, $(\{(q_\theta, t_\theta)\}_{\theta\in\Theta}, n)_\omega$. Sellers choose a submarket ω to maximize expected profits:

$$\max_{\omega\in\Omega}\left\{\frac{m(n)}{n}\int_{\underline{\theta}}^{\bar{\theta}}\left[-c(q_\theta)+t_\theta\right]d\tilde{G}(\theta;n)-\kappa\right\} \tag{11}$$

taking into account that buyers' ability to choose a submarket implies that buyers receive the same expected payoff V^b (the "market utility") in all active submarkets:

$$V^b = m(n)\int_{\underline{\theta}}^{\bar{\theta}}\left[\theta u(q_\theta)-t_\theta\right]d\tilde{G}(\theta;n). \tag{12}$$

In addition, the IR and IC constraints given by (9) and (10) must be satisfied.

It is convenient to describe the seller's problem in the following alternative way. Rearranging (11) and (12), the sellers' problem is equivalent to solving the buyer's problem regarding choice of submarket:

$$\max_{\omega\in\Omega}\left\{m(n)\int_{\underline{\theta}}^{\bar{\theta}}\left[\theta u(q_\theta)-t_\theta\right]d\tilde{G}(\theta;n)\right\} \tag{13}$$

subject to the constraint that the measure n of sellers per buyer satisfies the following condition in all submarkets (both active and inactive):

$$\frac{m(n)}{n}\int_{\underline{\theta}}^{\bar{\theta}}\left[-c(q_\theta)+t_\theta\right]d\tilde{G}(\theta;n)\leq\kappa \tag{14}$$

and $n \geq 0$ with complementary slackness. In addition, the IR and IC constraints given by (9) and (10) must be satisfied.[13]

The complementary slackness condition deserves further discussion. It says that $n > 0$ and a submarket is active if and only if (14) holds with equality, i.e. sellers receive zero expected profit after paying the entry cost κ. On the other hand, it says that $n = 0$ and a submarket is inactive if and only if (14) is an inequality, i.e. sellers would receive negative expected profits from entering this submarket even if the prob-

[13]The complementary slackness condition is really a restriction on n as a function of the menu of posted contracts in a submarket, i.e. $n(\{(q_\theta, t_\theta)\}_{\theta\in\Theta})$. However, we omit the dependence on the submarket $\{(q_\theta, t_\theta)\}_{\theta\in\Theta}$ in our notation both for notational simplicity and because we will later prove there is only one active submarket in equilibrium with one implied seller-buyer ratio n.

ability of trade is one. We restrict attention to equilibria in which this complementary slackness condition holds for *all* submarkets, both active and inactive.[14]

6.1 Equilibrium

First, we define competitive search equilibrium. We will later prove that there is a unique solution to this problem and thus all sellers post the same contracts and there is only one active submarket in equilibrium. Anticipating this result, we denote equilibrium by $(\{(q_\theta, t_\theta)\}_{\theta\in\Theta}, n)$. We focus on equilibria with positive entry, $n > 0$.

Definition 1. *A* competitive search equilibrium *is a list* $(\{(q_\theta, t_\theta)\}_{\theta\in\Theta}, n)$ *and a distribution of types* $\{\tilde{G}(\theta; n)\}_{\theta\in\Theta}$ *where* $(q_\theta, t_\theta) \in \mathbb{R}^2_+$ *for all* $\theta \in \Theta$ *and* $n \in \mathbb{R}_+\backslash\{0\}$, *such that* $(\{(q_\theta, t_\theta)\}_{\theta\in\Theta}, n)$ *maximizes (13) subject to (14) plus the IR and IC constraints (9) and (10), and* $\{\tilde{G}(\theta; n)\}_{\theta\in\Theta}$ *represents buyers' optimal choices of sellers.*

In the lead-up to describing equilibrium, we first need to prepare the ground by making some assumptions and providing some definitions. In particular, our proof of the existence and uniqueness of equilibrium under private information requires assumptions regarding the distribution of valuations G and the cost of entry κ.

Single crossing condition. The definition of buyer utility $\tilde{u}(q, \theta) \equiv \theta u(q)$ and the assumption $u'(q) > 0$ together ensure the single crossing property holds because

$$\frac{\partial^2 \tilde{u}(q, \theta)}{\partial q \partial \theta} = u'(q) \geq 0.$$

The fact that the single crossing property is satisfied in our environment is used in Lemma 3, a well-known result that provides sufficient conditions for the IC constraints to hold. Recalling that $v_\theta \equiv \theta u(q_\theta) - t_\theta$, the buyer's ex post trading surplus, this lemma simplifies the IC constraints when there is a continuous distribution of types.

Lemma 3. *If the single crossing property holds, the incentive compatibility (IC) constraint holds for all* $\theta \in \Theta$ *if (i)* q_θ *is non-decreasing in* θ *and (ii)* $v'(\theta) = u(q_\theta)$.

We use Lemma 3 to ensure the IC constraint holds when we solve for the equilibrium as an optimal control problem in the proof found in the Appendix.

[14]This provides an important restriction on beliefs outside of the equilibrium path, as discussed in Shi (2009) and Menzio and Shi (2010).

Virtual valuation function. The virtual valuation function $\psi_G : \Theta \to \mathbb{R}$ is defined by $\psi_G(\theta) \equiv \theta - \frac{1-G(\theta)}{g(\theta)}$. It is common to assume the virtual valuation function is weakly increasing as this is typically required to prove that q_θ is non-decreasing. To ensure this, we assume the distribution of valuations G has an increasing hazard rate, $h_G : \Theta \to \mathbb{R}$ defined by $h_G(\theta) \equiv \frac{g(\theta)}{1-G(\theta)}$. Assumption 4 is maintained throughout.

Assumption 4. *The distribution G has an increasing hazard rate, i.e. $h'_G(\theta) > 0$.*

In our environment, what is important is that the distribution of types $\tilde{G}$ has an increasing hazard rate and an increasing virtual valuation function. This is because sellers know that whenever they are matched with a buyer, this must be because the buyer chose that seller, i.e. they had the highest valuation among the sellers the buyer met. From the seller's perspective, the relevant buyer valuation is effectively a random draw from the distribution of *types*, i.e. the endogenous distribution of valuations for buyers' chosen goods, not the exogenous distribution of valuations G.

The distribution of types $\tilde{G}$ has a hazard rate defined by $h_{\tilde{G}}(\theta; n) \equiv \frac{\tilde{g}(\theta;n)}{1-\tilde{G}(\theta;n)}$. The virtual valuation function of the distribution of types is defined by

$$\psi_{\tilde{G}}(\theta; n) \equiv \theta - \frac{1-\tilde{G}(\theta; n)}{\tilde{g}(\theta; n)}. \tag{15}$$

Given Assumption 4, the following lemma establishes that $\tilde{G}$ has an increasing hazard rate (i.e. increasing in type θ) and an increasing virtual valuation function (i.e. increasing in type θ) for any invariant search technology P_j and any seller-buyer ratio.

Lemma 4. *If P_j is invariant and $n > 0$, the hazard rate $h_{\tilde{G}}(\theta; n)$ and virtual valuation function $\psi_{\tilde{G}}(\theta; n)$ of the distribution of types $\tilde{G}$ have the following properties:*

1. *Both $h_{\tilde{G}}(\theta; n)$ and $\psi_{\tilde{G}}(\theta; n)$ are increasing in buyer type θ, i.e.*

$$\frac{\partial h_{\tilde{G}}(\theta; n)}{\partial \theta} > 0 \text{ and } \frac{\partial \psi_{\tilde{G}}(\theta; n)}{\partial \theta} > 0.$$

2. *Both $h_{\tilde{G}}(\theta; n)$ and $\psi_{\tilde{G}}(\theta; n)$ decreasing in the seller-buyer ratio n, i.e.*

$$\frac{\partial h_{\tilde{G}}(\theta; n)}{\partial n} < 0 \text{ and } \frac{\partial \psi_{\tilde{G}}(\theta; n)}{\partial n} < 0.$$

Lemma 4 also says that the hazard rate and virtual valuation function of the distribution of types $\tilde{G}$ are both decreasing in the seller-buyer ratio n.

Cost of entry. We maintain the following assumption throughout the paper.

Assumption 5. *Let $q_\theta^0 \equiv \lim_{n\to 0} q_\theta(n)$. The entry cost κ is not too high:*

$$E_G[\theta u(q_\theta^0) - c(q_\theta^0)] > \kappa.$$

Assumption 5 is necessary to ensure the existence of equilibrium where $n > 0$. It says the expected trade surplus $\tilde{s}(n)$ must be greater than κ in the limit as $n \to 0$, otherwise no sellers enter. Since we have $\tilde{G} \to G$ in the limit as $n \to 0$ by Lemma 2, we have $\lim_{n\to 0} \tilde{s}(n) = E_G[\theta u(q_\theta^0) - c(q_\theta^0)]$ where $q_\theta^0 \equiv \lim_{n\to 0} q_\theta(n)$.[15]

There are two possible types of equilibria. There may be *full coverage*, i.e. all consumer types choose to purchase.[16] This may occur in either of two cases, depending on the value of the virtual valuation function at $\underline{\theta}$. Alternatively, there may be *partial coverage*, i.e. not all consumer types purchase but only types above a cut-off type.

Lemma 5. *In any equilibrium with seller-buyer ratio $n > 0$, there exists a unique cut-off buyer type $\theta_b(n)$ where $\theta_b(n) \geq \underline{\theta}$ such that*

1. *If $\psi_G(\underline{\theta}) > 0$, then $\theta_b(n) = \underline{\theta}$ and equilibrium is* full coverage *with $q_\theta > 0$ for all $\theta \in \Theta$.*

2. *If $\psi_G(\underline{\theta}) = 0$, then $\theta_b(n) = \underline{\theta}$ and equilibrium is* full coverage *with $q_\theta > 0$ for all $\theta \in (\underline{\theta}, \overline{\theta}]$ and $q_{\underline{\theta}} = 0$.*

3. *If $\psi_G(\underline{\theta}) < 0$, then $\theta_b(n) > \underline{\theta}$ and equilibrium is* partial coverage *with $q_\theta = 0$ for all $\theta \in [\underline{\theta}, \theta_b(n)]$ and $q_\theta > 0$ for all $\theta \in (\theta_b(n), \overline{\theta}]$.*

For simplicity, we normalize $\psi_G(\underline{\theta}) = 0$ and focus on the second case for full coverage. That is, we assume that the virtual valuation function is equal to zero at the minimum type $\underline{\theta}$. For our proofs, however, we consider the general case where we may have $\psi_G(\underline{\theta}) < 0$ and partial coverage (not all consumer types purchase). In the Appendix, we discuss partial coverage equilibria and show how our results extend.

[15]For any $\theta \in \Theta$, the value of q_θ^0 is given by Lemma 10 in the Appendix.

[16]Note that the probability a consumer type is $\underline{\theta}$ is zero, so this does not require $q_{\underline{\theta}} > 0$.

6.2 Full coverage equilibrium

We now present Proposition 2, which establishes the existence and uniqueness of equilibrium with full coverage and provides a characterization.

Proposition 2. *If $\psi_G(\underline{\theta}) = 0$, there exists a unique equilibrium with full coverage.*

1. *We have $q_{\underline{\theta}} = 0$ and $t_{\underline{\theta}} = 0$ and the trading cut-off is $\theta_b(n) = \underline{\theta}$.*
2. *For any $\theta \in (\underline{\theta}, \bar{\theta}]$, the quantity $q_\theta > 0$ solves:*

$$\left(\theta - \eta_m(n)\frac{1 - \tilde{G}(\theta; n)}{\tilde{g}(\theta; n)}\right) u'(q_\theta) = c'(q_\theta) \tag{16}$$

   and the payment $t_\theta > 0$ is given by

$$t_\theta = \theta u(q_\theta) - \int_{\underline{\theta}}^{\theta} u(q_x)dx. \tag{17}$$

3. *The seller-buyer ratio $n > 0$ is strictly decreasing in κ and satisfies*

$$m'(n)\tilde{s}(n; \{q_\theta\}_{\theta\in\Theta}) + m(n)\tilde{s}'(n; \{q_\theta\}_{\theta\in\Theta}) = \kappa. \tag{18}$$

4. *The zero profit condition is satisfied:*

$$\frac{m(n)}{n}\int_{\underline{\theta}}^{\bar{\theta}} \left[-c(q_\theta) + t_\theta\right] d\tilde{G}(\theta; n) = \kappa. \tag{19}$$

5. *The distribution of buyer types $\tilde{G}$ is given by (3).*

The decentralized equilibrium cannot deliver the first-best allocation. This is not surprising due buyers' private information, which distorts the quantities traded. We recover two standard results: there is "no distortion at the top", i.e. $q_\theta = q^*_\theta$ for the highest type $\theta = \bar{\theta}$, but there is downwards distortion below, i.e. $q_\theta < q^*_\theta$ for $\theta < \bar{\theta}$.

Corollary 1. *In any competitive search equilibrium, quantities are distorted downwards relative to the first-best for all $\theta \in (\underline{\theta}, \bar{\theta})$, but there is no distortion at the top, i.e. $q_{\bar{\theta}} = q^*_{\bar{\theta}}$. There may be either under-entry or over-entry of sellers.*

In terms of entry, there may be either under-entry or over-entry relative to the first-best because the distortion in quantities traded due to buyers' private information induces a corresponding distortion in seller entry, which depends on quantities.

6.3 Equilibrium quantities

One way to measure of *market power* is to consider the probability of monopoly. For any given seller-buyer ratio n, the probability $\pi_m(n)$ that a buyer's meeting is a *monopoly* is equal to the probability a buyer meets exactly one seller, which occurs with probability $P_1(n)$, conditional on meeting at least one seller, which occurs with probability $1 - P_0(n)$. The probability $\pi_c(n)$ that a buyer's meeting is *competitive* is equal to the probability a buyer meets two or more sellers, conditional on meeting at least one seller, and therefore we have $\pi_c(n) = 1 - \pi_m(n)$.

For all search technologies that satisfy Assumption 2, we have the following equivalence result, which says the probability of monopoly is equal to the meeting elasticity.[17] All of the properties of the meeting elasticity $\eta_m(n)$ in Lemma 1 thus apply to $\pi_m(n)$.

Lemma 6. *If P_j is invariant and $n > 0$, the probability $\pi_m(n)$ that a buyer's meeting is a monopoly is equal to the meeting elasticity $\eta_m(n)$, i.e. $\pi_m(n) = \eta_m(n)$.*

Therefore, we can interpret expression (16) for equilibrium quantities as:

$$\left(\theta - \underbrace{\pi_m(n)}_{\text{probability of monopoly}} \frac{1 - \tilde{G}(\theta; n)}{\tilde{g}(\theta; n)}\right) u'(q_\theta) = c'(q_\theta). \tag{20}$$

We can now express the equilibrium quantities in the following manner.

Proposition 3. *If $\psi_G(\underline{\theta}) = 0$, the equilibrium quantity for type θ is given by*

$$\left(\underbrace{\pi_c(n)}_{\textit{probability of competition}} \underbrace{\theta}_{\textit{valuation}} + \underbrace{\pi_m(n)}_{\textit{probability of monopoly}} \underbrace{\psi_{\tilde{G}}(\theta; n)}_{\textit{virtual valuation}}\right) u'(q_\theta) = c'(q_\theta)$$

where $\psi_{\tilde{G}}$ is the virtual valuation function of the endogenous distribution of types $\tilde{G}$.

We can interpret the above expression as a kind of weighted average of the competitive limit (which delivers the first-best quantities) and the monopoly benchmark (which is standard except the distribution of types $\tilde{G}$ is endogenous).

As the seller-buyer ratio n varies between zero and infinity, we can nest *any* degree of competition between the two extremes of monopoly ($n \to 0$ and $\pi_m(n) \to 1$) and

[17] Lemma 6 follows from the fact that $-P_0(n)n = P_1(n)$ if Assumption 2 holds. See Condition 2 in Mangin (2024) for a more general condition which is shown to be equivalent to Assumption 2.

perfect competition ($n \to \infty$ and $\pi_c(n) \to 1$). Below, we derive both the competitive limit and the standard monopoly benchmark as limiting cases of Proposition 3.

Competitive limit. In the limit as $n \to \infty$, all meetings are competitive and the probability of monopoly $\pi_m(n)$ converges to zero. We recover the first-best quantities:

$$\theta u'(q_\theta) = c'(q_\theta). \tag{21}$$

We will later discuss the competitive limit in more detail in Section 7.4.

Standard monopoly benchmark. In the limit as $n \to 0$, we have $\tilde{G} \to G$ and therefore $\psi_{\tilde{G}}(\theta; n) \to \psi_G(\theta)$. At the same time, there are no competitive meetings and the probability of monopoly $\pi_m(n)$ converges to one. We thus obtain:

$$\left(\theta - \frac{1 - G(\theta)}{g(\theta)}\right) u'(q_\theta) = c'(q_\theta). \tag{22}$$

We therefore recover, as a limiting case, the standard expression for the equilibrium quantities traded under monopoly pricing with incomplete information.

7 Effects of competition

In this section, we build towards the main result of our paper: in competitive search equilibrium, greater competition through a higher seller-buyer ratio can alleviate the distortions due to private information, bringing us closer to the first best.

While the seller-buyer ratio n is endogenous in our model, in this section we take n as exogenous and interpret an increase in n as an increase in *competition*. This could be endogenized by decreasing the entry cost κ, but we abstract from this here.

For simplicity, we continue to focus our discussion on the full coverage case where we assume $\psi_G(\underline{\theta}) = 0$. We discuss the case of partial coverage in the Appendix, which generalizes the results in this section. The Appendix also includes some additional results regarding the effect of competition on *market coverage*, i.e. the proportion of consumers who successfully purchase (which varies with the degree of competition).

7.1 Informational distortion

First we consider how competition directly affects the informational distortion due to private information. We can define the *informational distortion* by

$$I_{\tilde{G}}(\theta;n) \equiv \frac{1-\tilde{G}(\theta;n)}{\tilde{g}(\theta;n)}. \tag{23}$$

Except for the fact that the distribution of buyer types $\tilde{G}$ is endogenous in our model, this is the "standard" informational distortion under monopoly given by (22). We know the informational distortion $I_{\tilde{G}}(\theta;n)$ is *decreasing* in θ because $\tilde{G}$ has an increasing hazard rate by Lemma 4. Thus, the standard distortion due to private information decreases for higher values of θ, with "no distortion at the top" because $I_{\tilde{G}}(\bar{\theta};n) = 0$.

Now consider the effect of the seller-buyer ratio n on the informational distortion. It follows from Lemma 4 that the informational distortion $I_{\tilde{G}}(\theta;n)$ is *increasing* in the seller-buyer ratio n for any given θ because the hazard rate of $\tilde{G}$ is decreasing in n.

Let $\tilde{I}(n)$ denote the *average informational distortion*, defined by

$$\tilde{I}(n) \equiv \int_{\underline{\theta}}^{\bar{\theta}} I_{\tilde{G}}(\theta;n)d\tilde{G}(\theta;n). \tag{24}$$

The average informational distortion $\tilde{I}(n)$ is also increasing in the seller-buyer ratio.[18] The basic idea is that, as n increases, there are fewer "low types" but these low types are subject to a greater informational distortion. This means that, if we did not consider the effect of seller competition via competitive search, greater competition would increase the average informational distortion. As we will see, however, competitive search has a crucial effect on reducing the extent of the quantity distortion.

7.2 Quantity distortion

To determine the overall effect of greater competition on the quantity distortion, we need to also consider the impact of competitive search, which ensures that sellers compete to attract buyers through their choices of which menus of contracts to post.

[18]Note that substituting $I_{\tilde{G}}(\theta;n)$ from (23) into (24) yields $\tilde{I}(n) \equiv \int_{\underline{\theta}}^{\bar{\theta}}(1-\tilde{G}(\theta;n))d\theta$, which is equal to $\int_{\underline{\theta}}^{\bar{\theta}} \theta\tilde{g}(\theta;n)d\theta - \underline{\theta}$ or $\tilde{\theta}(n) - \underline{\theta}$, which is clearly increasing in n because $\tilde{\theta}'(n) > 0$.

We can define the *quantity distortion* $\delta_{\tilde{G}}(\theta; n)$ by

$$1 - \delta_{\tilde{G}}(\theta; n) \equiv \frac{f(q_\theta)}{f(q_\theta^*)}, \tag{25}$$

where $f(q) \equiv c'(q)/u'(q)$ and $f' > 0$ follows from our assumptions. To justify our measure of the quantity distortion, suppose that $c(q) = cq$ and $u(q) = q^{1-b}/(1-b)$ for $b \in (0, 1)$, which are standard assumptions. Then we have

$$1 - \delta_{\tilde{G}}(\theta; n) = f\left(\frac{q_\theta}{q_\theta^*}\right). \tag{26}$$

Given that f is an increasing function, $1 - \delta_{\tilde{G}}(\theta; n)$ moves in the same direction as q_θ/q_θ^* with respect to n. Thus we can interpret $\delta_{\tilde{G}}(\theta; n)$ as a measure of the quantity distortion. As $q_\theta \to q_\theta^*$, we have $\delta_{\tilde{G}}(\theta; n) \to 0$, and as $q_\theta \to 0$ we have $\delta_{\tilde{G}}(\theta; n) \to 1$.

Using expression (20) for q_θ and expression (21) for q_θ^* plus definition (23) for the informational distortion, we obtain

$$\delta_{\tilde{G}}(\theta; n) = \underbrace{\pi_m(n)}_{\text{competitive effect}} \underbrace{\frac{I_{\tilde{G}}(\theta; n)}{\theta}}_{\text{relative informational distortion}}. \tag{27}$$

The term $\pi_m(n)$ reflects the *competitive effect* on the quantity distortion due to competitive search, while the term $I_{\tilde{G}}(\theta; n)/\theta$ reflects the *relative informational distortion*.

With competitive search, the quantity distortion $\delta_{\tilde{G}}(\theta; n)$ is strictly less than the relative informational distortion because the probability of monopoly is below one, i.e. $\pi_m(n) < 1$ for any $n > 0$. The reason behind the lower quantity distortion under competitive search is the fact that sellers are competing to offer contracts that are attractive to buyers in order to attract more buyers to their own submarket. As a result, sellers offer higher quantities than they otherwise would have in the absence of competitive search. When competition is greater, the probability of monopoly $\pi_m(n)$ is lower, reducing the quantity distortion. This result is clear because we know the probability of monopoly $\pi_m(n)$ is decreasing in n. As the market becomes more competitive and firms compete more intensely to attract buyers to their submarket, the probability $\pi_m(n)$ decreases, thereby reducing the overall quantity distortion.

There are two opposing effects of the seller-buyer ratio n on the quantity distortion. We know the informational distortion $I_{\tilde{G}}(\theta; n)$ is increasing in the seller-buyer ratio, but $\pi_m(n)$ is decreasing in n. It is unclear whether the overall quantity distortion

$\delta_{\tilde{G}}(\theta; n)$ is increasing or decreasing in the degree of competition through n.

In fact, Proposition 4 tells us that the competitive effect – which reduces the quantity distortion by more when n is higher and competition is stronger – always dominates. This competitive effect is sufficiently strong that the overall quantity distortion $\delta_{\tilde{G}}(\theta; n)$ is *decreasing* in the seller-buyer ratio or degree of competition.

Proposition 4. *For any $\theta \in \Theta$, greater competition reduces the quantity distortion $\delta_{\tilde{G}}(\theta; n)$. That is, $\delta_{\tilde{G}}(\theta; n)$ is decreasing in the seller-buyer ratio n.*

The result in Proposition 4 that greater competition reduces the quantity distortion for any given buyer type θ directly implies Corollary 2, which states that greater competition increases the quantity traded $q_\theta(n)$ for any given buyer type.[19]

Corollary 2. *For any $\theta \in \Theta$, greater competition increases the quantity traded $q_\theta(n)$. That is, $q_\theta(n)$ is increasing in the seller-buyer ratio n.*

For any given buyer type θ, the quantity distortion $\delta_{\tilde{G}}(\theta; n)$ is decreasing in the seller-buyer ratio n. However, the distribution of types $\tilde{G}$ is endogenous and is itself changing with n, so we still need to determine whether the *average* quantity distortion is decreasing in n. To do this, we define the *average quantity distortion* by

$$\tilde{\delta}(n) \equiv \int_{\underline{\theta}}^{\bar{\theta}} \delta_{\tilde{G}}(\theta; n) d\tilde{G}(\theta; n).$$

Proposition 5 tells us that the effect of competitive search is sufficiently strong that the overall impact of competition on the average quantity distortion is negative.

Proposition 5. *Greater competition reduces the average quantity distortion $\tilde{\delta}(n)$. That is, $\tilde{\delta}(n)$ is decreasing in the seller-buyer ratio n.*

With competitive search, greater seller entry can therefore reduce the average quantity distortion arising from buyers' private information. This is intuitive because sellers are competing to offer contracts that are attractive to buyers. As competition becomes more and more intense, sellers offer lower and lower quantity distortions.

Corollary 2 implies the *average quantity* increases with greater competition. There are two reasons behind this result. First, we know from Corollary 2 that the quantity traded for any given type θ is increasing in n. Second, greater competition induces a

[19] We write $q_\theta(n)$ instead of q_θ to emphasize the dependence on the seller-buyer ratio n.

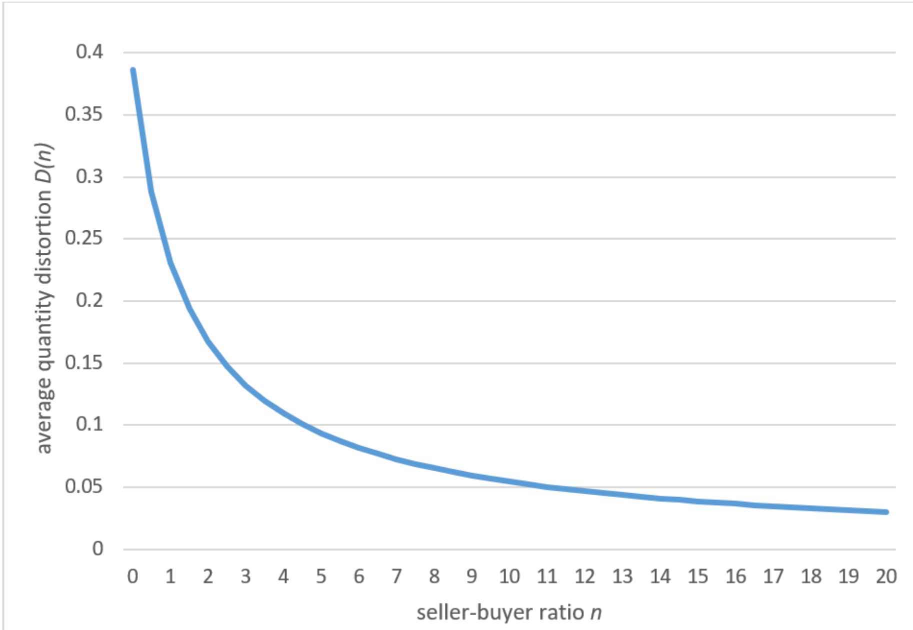

Figure 1: Example of average quantity distortion $\tilde{\delta}(n)$

first-order stochastic dominance shift in the distribution of types by Lemma 2 because buyers have greater choice. Because the equilibrium quantity $q_\theta(n)$ is non-decreasing in θ (as required by Lemma 3), this shift further increases the average quantity.

Corollary 3. *Greater competition increases the average quantity traded $\tilde{q}(n)$. That is, $\tilde{q}(n)$ is increasing in the seller-buyer ratio n.*

Figure 1 illustrates how the average quantity distortion $\tilde{\delta}(n)$ decreases with greater competition. For this example, we assume the search technology P_j is geometric and the distribution G is uniform on $[1, 2]$. For example, if $n = 1$ the quantity distortion is 23.1%, while if $n = 5$ the quantity distortion is 9.3%. As competition intensifies and we reach $n = 20$ sellers per buyer, the quantity distortion is only 3.0%.

The average ratio of equilibrium quantity to the first-best quantity is a monotonic transform of $1 - \tilde{\delta}(n)$ that depends on the parameters of the functions $u(q)$ and $c(q)$. In Figure 2, the orange line depicts the average ratio of the equilibrium quantity to the first-best quantity if $c(q) = cq$ and $u(q) = q^{1-b}/(1-b)$ where $c = 1$ and $b = 1/2$. For comparison, the blue line is $1 - \tilde{\delta}(n)$ where $\tilde{\delta}(n)$ is our measure of the quantity distortion depicted in Figure 1. For example, if $n = 1$ the average ratio of equilibrium quantity to first-best quantity is 65.2%, while if $n = 5$ the average ratio is 85.1%. As

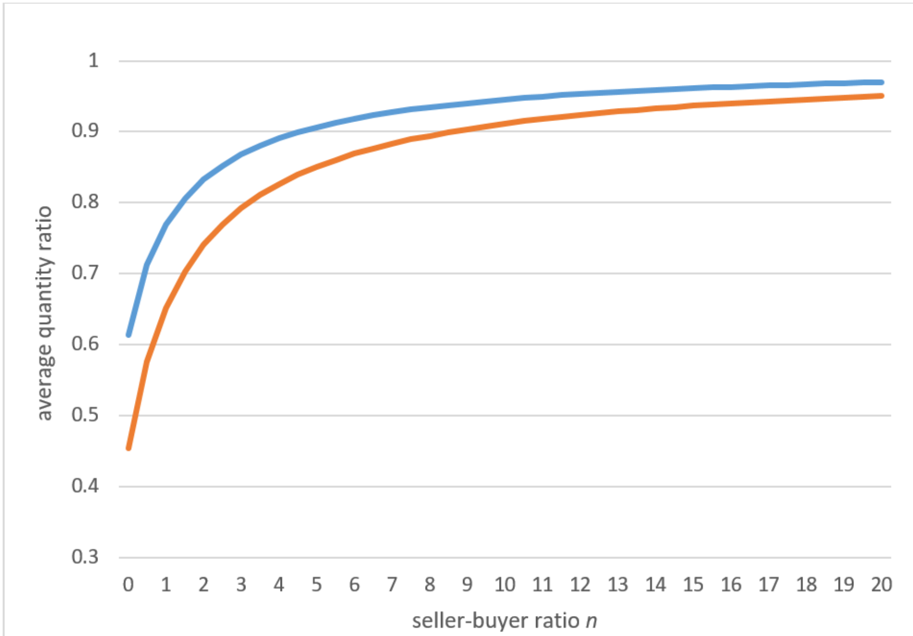

Figure 2: Example of average quantity ratio q_θ/q^*_θ

we reach a highly competitive economy with $n = 20$ sellers per buyer, the average ratio is 95.1% and the quantities traded are becoming very close to the first-best.

7.3 Nonlinear pricing

It is well known that when buyers' valuations are private information, monopolists offer nonlinear price schedules that offer *quantity discounts* in order to induce buyers with higher valuations to purchase greater quantities (Maskin and Riley, 1984). In this section, we verify that sellers offer quantity discounts in our environment where there is competition between sellers and consumer choice. We also consider the effect of competition, i.e. a higher seller-buyer ratio, on the intensity of nonlinear pricing.

Since $q'(\theta) > 0$ for all trading buyer types, we can define a differentiable function $\theta : (0, q_{\bar{\theta}}) \to \mathbb{R}_+$ that gives the valuation θ as a function of quantity purchased. Now let the payment for a given quantity q be defined by $T(q) \equiv t(\theta(q))$ and define the unit price by $p(q) \equiv T'(q)$, the incremental price of the q-th unit of the good.

We say that there are *quantity discounts* whenever $p'(q) < 0$. Proposition 6 verifies that there are indeed quantity discounts, i.e. $p'(q) < 0$ in our environment. This result extends to the partial coverage case where not all consumers choose to purchase.

Proposition 6. *There are quantity discounts in equilibrium, i.e.* $p'(q) < 0$.

Define the *price elasticity* by $\eta_p(q) \equiv \frac{-p'(q)q}{p(q)}$. Proposition 7 provides some sufficient conditions under which the intensity of quantity discounting, as measured by the price elasticity $\eta_p(q)$, is decreasing in the seller-buyer ratio or degree of competition. This result again extends to the partial coverage case described in the Appendix.

Proposition 7. *Suppose that* $c(q) = cq$ *and* $u(q) = q^{1-b}/(1-b)$ *where* $b \in (0,1)$. *Greater competition, i.e. a higher seller-buyer ratio* n, *decreases the intensity of quantity discounts as measured by the price elasticity* $\eta_p(q)$ *if the following holds:*

$$\frac{g'(\theta)\theta}{g(\theta)} \geq -1.$$

The above condition on the density g is sufficient, but not necessary, for the intensity of quantity discounts to be decreasing in the degree of competition. This condition will certainly be true if $G''(\theta) \geq 0$, although this is again not necessary. For example, the sufficient condition $G''(\theta) \geq 0$ holds if G is the uniform distribution.

7.4 Competitive limit

We have seen that greater competition can reduce the distortion in quantities traded. This suggests that if the market is sufficiently competitive, i.e. the seller-buyer ratio is sufficiently high, this distortion may be eliminated altogether.

Consider the limit as the seller-buyer ratio $n \to \infty$. We refer to this as the *competitive limit* (or the "frictionless" limit) because all buyers meet a large number of sellers in each meeting. In the limit as $n \to \infty$, the distribution of types $\tilde{G}$ converges to a degenerate distribution with support $\Theta = \{\bar{\theta}\}$, i.e. all buyers have type $\bar{\theta}$. This is because all buyers are offered a good with the highest valuation in this limiting case.

Proposition 8. *In the competitive limit as* $n \to \infty$,

1. *All consumers have type* $\bar{\theta}$, *i.e. the valuation for their chosen good is* $\bar{\theta}$.

2. *The quantity traded* $q_{\bar{\theta}}$ *satisfies* $\bar{\theta}u'(q_{\bar{\theta}}) = c'(q_{\bar{\theta}})$.

The informational distortion is eliminated and we obtain the first-best allocation in the competitive limit, i.e. $q_{\bar{\theta}} = q^*_{\bar{\theta}}$ for all trades. There are, in fact, two separate reasons for this. First, the probability of monopoly $\pi_m(n)$ goes to zero and thus the quantity distortion in Proposition 3 disappears for *any* given buyer type θ, as discussed in Section 6.3. Second, *all* buyers are of the highest type $\bar{\theta}$ in the competitive limit as $n \to 0$ and there is always "no distortion at the top" for the highest type.

8 Conclusion

This paper introduces consumer choice into a competitive search model of retail trade in which sellers compete to attract buyers by posting nonlinear price schedules. We allow consumers to meet multiple sellers and *choose* a seller with whom to trade. We find that the standard distortion in equilibrium quantities, which arises in monopoly pricing under incomplete information, can be alleviated through greater competition. With greater competition, sellers reduce the quantity distortion in order to attract buyers to their own submarket in competitive search equilibrium. In the competitive limit where the seller-buyer ratio becomes large, the effects of private information are eliminated altogether and we obtain the first-best allocation.

In future research, it would be interesting to apply this framework to the question of how nonlinear pricing is affected by the nature of the search technology, i.e. the distribution of the size of buyer's choice sets. For example, does greater *dispersion* in the number of sellers each buyer meets have a positive or a negative effect on consumers? Can reducing dispersion bring us closer to achieving the first-best allocation, even in the presence of private information? We leave these questions for future research.

9 Appendix A: Partial coverage

In this Appendix, we generalize our results by considering partial coverage equilibria, i.e. equilibria in which not all consumer types choose to purchase.

9.1 Partial coverage equilibrium

Proposition 9 establishes the existence and uniqueness of equilibrium with partial coverage and provides a characterization.

Before presenting Proposition 9, it will be useful to define $\rho(\theta;n) \equiv 1 - \tilde{G}(\theta;n)$, the probability that a buyer's type is greater than θ. We also define $\varepsilon_\rho(\theta;n) \equiv -\theta\rho'(\theta;n)/\rho(\theta;n)$, the elasticity of $\rho(\theta;n)$ with respect to θ, where $\rho'(\theta;n) \equiv \frac{\partial\rho(\theta;n)}{\partial\theta}$. When we discuss market coverage below, we will see that $\varepsilon_\rho(\theta;n)$ can be interpreted as the *elasticity of market coverage* with respect to the trading cut-off type θ.

Proposition 9. *If $\psi_G(\underline{\theta}) < 0$, there exists a unique equilibrium with partial coverage.*

1. *We have $q_\theta = 0$ and $t_\theta = 0$ for all $\theta \in [\underline{\theta}, \theta_b(n)]$ where $\theta_b(n) > \underline{\theta}$.*

2. *For any $\theta \in (\theta_b(n), \bar{\theta}]$, the quantity $q_\theta > 0$ solves:*

$$\left(\theta - \varepsilon_\rho(\theta_b(n);n)\frac{1-\tilde{G}(\theta;n)}{\tilde{g}(\theta;n)}\right)u'(q_\theta) = c'(q_\theta) \quad (28)$$

   and the payment $t_\theta > 0$ is given by

$$t_\theta = \theta u(q_\theta) - \int_{\underline{\theta}}^{\theta} u(q_x)dx. \quad (29)$$

3. *The seller-buyer ratio $n > 0$ is strictly decreasing in κ and satisfies*

$$m'(n)\tilde{s}(n;\{q_\theta\}_{\theta\in\Theta}) + m(n)\tilde{s}'(n;\{q_\theta\}_{\theta\in\Theta}) = \kappa. \quad (30)$$

4. *The zero profit condition is satisfied:*

$$\frac{m(n)}{n}\int_{\underline{\theta}}^{\bar{\theta}}\left[-c(q_\theta) + t_\theta\right]d\tilde{G}(\theta;n) = \kappa. \quad (31)$$

5. *The distribution of buyer types $\tilde{G}$ is given by (3).*

Again, the decentralized equilibrium cannot deliver the first-best allocation. Buyers' private information distorts the quantities traded, with may be either positive but less than optimal, or zero. We again recover two standard results: there is no distortion at the top (i.e. for the highest type) but downwards distortion below.

9.2 Market coverage

We can define *market coverage* $\mu(\theta_b(n); n)$ as the proportion of consumers that successfully purchase. Market coverage is endogenous and depends on the seller-buyer ratio or degree of competition. Specifically, market coverage is equal to the meeting probability for buyers $m(n)$, multiplied by the probability that a buyer's type (i.e. the buyer's valuation for their chosen seller) is above the trading cut-off $\theta_b(n)$. That is,

$$\mu(\theta_b(n); n) = m(n)(1 - \tilde{G}(\theta_b(n); n)). \tag{32}$$

In the case of full coverage, we have $\theta_b(n) = \underline{\theta}$ and therefore $\mu(\theta_b(n); n) = m(n)$. While market coverage is not equal to one in this case due to search frictions, this is the highest coverage possible for a given search technology and seller-buyer ratio.

Our first result is that greater competition increases the level of market coverage. Before presenting this result, we provide a useful lemma which says that the trading cut-off $\theta_b(n)$ is decreasing in the seller-buyer ratio n.

Lemma 7. *The trading cut-off $\theta_b(n)$ is decreasing in n, i.e. $\theta_b'(n) < 0$.*

This result tells us that an increase in competition through a higher seller-buyer ratio n decreases the trading cutoff, which means that goods for which consumers have *lower* valuations can trade in equilibrium when there are more competing firms.

The next result tells us that greater competition increases market coverage.

Proposition 10. *Greater competition increases market coverage $\mu(\theta_b(n); n)$. That is, the proportion of buyers who purchase is increasing in the seller-buyer ratio n.*

There are two different reasons behind this result. First, a higher seller-buyer ratio increases the meeting probability for buyers by reducing the search frictions buyers face. Second, a higher seller-buyer ratio increases the trading probability for a buyer conditional on a meeting occurring. This is because more sellers means a higher expected valuation for the buyer's choice, which is more likely to be greater than a given trading cut-off $\theta_b(n)$, while at the same time greater competition means that the trading cut-off is itself lower. Overall, this result tells us that a wider range of consumers (i.e. with lower valuations) purchase when competition is greater.

9.3 Quantity distortion

Using expression (28) for q_θ plus definition (23) for the informational distortion,

$$\delta_{\tilde{G}}(\theta; n) = \underbrace{\varepsilon_\rho(\theta_b(n); n)}_{\text{competitive effect}} \underbrace{\frac{I_{\tilde{G}}(\theta; n)}{\theta}}_{\text{relative informational distortion}}$$

for all $\theta > \theta_b(n)$, and $\delta_{\tilde{G}}(\theta; n) = 1$ for $\theta \leq \theta_b(n)$. The term $\varepsilon_\rho(\theta_b(n); n)$ reflects the *competitive effect* on the quantity distortion due to competitive search, while the term $I_{\tilde{G}}(\theta; n)/\theta$ reflects the *relative informational distortion.*

With competitive search, the quantity distortion $\delta_{\tilde{G}}(\theta; n)$ is strictly less than the relative informational distortion because the competitive effect $\varepsilon_\rho(\theta_b(n); n) < 1$ for any $n > 0$.[20] As we saw for the full coverage case, the reason behind the lower quantity distortion under competitive search is the fact that sellers are competing to attract more buyers to their own submarket by offering lower quantity distortions.

Lemma 8 says that when the seller-buyer ratio n is higher and competition is greater, the competitive effect $\varepsilon_\rho(\theta_b(n); n)$ is lower, reducing the quantity distortion.

Lemma 8. *Greater competition decreases the competitive effect $\varepsilon_\rho(\theta_b(n); n)$. That is, the term $\varepsilon_\rho(\theta_b(n); n)$ is decreasing in the seller buyer-ratio n.*

As in the full coverage case, there are two opposing effects of the seller-buyer ratio n on the quantity distortion. We know the informational distortion $I_{\tilde{G}}(\theta; n)$ is increasing in the seller-buyer ratio n. At the same time, Lemma 8 says that the competitive effect $\varepsilon_\rho(\theta_b(n); n)$ is decreasing in n. It is again unclear whether the overall quantity distortion $\delta_{\tilde{G}}(\theta; n)$ is increasing or decreasing in n.

Proposition 11 tells us that, for any trading buyer type $\theta > \theta_b(n)$, the competitive effect always dominates. This competitive effect is sufficiently strong that the quantity distortion $\delta_{\tilde{G}}(\theta; n)$ is decreasing in the seller-buyer ratio n for all traded goods, which generalizes the result we saw for the full coverage case to partial coverage equilibrium.

[20] To see this, in the limit as $n \to 0$ we have $\varepsilon_\rho(\theta_b(n); n) = 1$ because θ_b^0 solves $\psi_G(\theta_b) = 0$ by Lemma 10 whenever $\psi_G(\underline{\theta}) \leq 0$, which we have assumed. Also, we have $\varepsilon_\rho(\theta_b(n); n)$ decreasing in n by Lemma 8 so $\varepsilon_\rho(\theta_b(n); n) < 1$ for any $n > 0$.

Proposition 11. *For any* $\theta \in (\theta_b(n), \bar{\theta}]$*, greater competition reduces the quantity distortion* $\delta_{\tilde{G}}(\theta; n)$*. That is,* $\delta_{\tilde{G}}(\theta; n)$ *is decreasing in the seller-buyer ratio* n*.*

The result in Proposition 11 that greater competition reduces the quantity distortion for any trading buyer type directly implies Corollary 4, which states that greater competition increases the quantity traded $q_\theta(n)$ for any trading buyer type θ.

Corollary 4. *For any* $\theta \in (\theta_b(n), \bar{\theta}]$*, greater competition increases the quantity traded* $q_\theta(n)$*. That is,* $q_\theta(n)$ *is increasing in the seller-buyer ratio* n*.*

For any trading type θ, the quantity distortion $\delta_{\tilde{G}}(\theta; n)$ is decreasing in the seller-buyer ratio n. At the same time, the distribution of types $\tilde{G}$ is changing with n, so it remains to determine whether the *average* quantity distortion is decreasing in n.

Proposition 12 tells us that the effect of competitive search is sufficiently strong that the overall impact of competition on the average quantity distortion is negative.

Proposition 12. *Greater competition reduces the average quantity distortion* $\tilde{\delta}(n)$*. That is,* $\tilde{\delta}(n)$ *is decreasing in the seller-buyer ratio* n*.*

Corollary 4 implies that greater competition increases the average quantity traded. As discussed for the full coverage case, this is driven by two separate reasons: the increase in the quantity traded for any given type θ and the first-order stochastic dominance shift in the distribution of types that is induced by greater choice. In partial coverage equilibrium, there is also an additional reason why the average quantity traded increases: more consumer types purchase with greater competition.

Corollary 5. *Greater competition increases the average quantity traded* $\tilde{q}(n)$*. That is,* $\tilde{q}(n)$ *is increasing in the seller-buyer ratio* n*.*

As we saw for the full coverage case, greater competition reduces the average quantity distortion due to buyers' private information. This is intuitive because sellers are competing to offer contracts that are attractive to buyers. As competition becomes more and more intense, sellers offer lower and lower quantity distortions.

Propositions 6 and 7 also apply to the partial coverage case. That is, we have quantity discounts in equilibrium and the intensity of nonlinear pricing is decreasing in the degree of competition if the sufficient conditions in Proposition 7 hold.

In the competitive limit, we obtain Proposition 8, which says that all buyers have the highest type $\bar{\theta}$ and therefore equilibrium is always full coverage, and the equilibrium quantities traded are the first-best quantities (i.e. "no distortion at the top").

10 Appendix B: Proofs

First, Lemma 9 summarizes some properties of invariant search technologies. Given that P_j is invariant, we know from Cai et al. (2025) that P_0 is a completely monotone function and can thus be represented as a Laplace transform. Lemma 9 follows.[21]

Lemma 9. *If P_j is an invariant meeting technology, then*

1. *We have $P_0'(x) < 0$ and $P_0''(x) > 0$ for all $x \in \mathbb{R}^+ \backslash \{0\}$.*
2. *We have $\lim_{x\to 0} P_0(x) = 1$ and $\lim_{x\to 0} P_0'(x) = -1$.*
3. *We have $\lim_{x\to\infty} P_0(x) = 0$, $\lim_{x\to\infty} P_0'(x) = 0$, and $\lim_{x\to\infty} P_0''(x) = 0$.*

Proof of Lemma 1

The properties of m follow immediately from $m(n) = 1 - P_0(n)$ and Lemma 9. Given that P_j is invariant and P_0 is a completely monotone function, we can apply Lemma 8 from Campbell, Ushchev, and Zenou (2024) to obtain $\eta_m'(n) < 0$ and $\frac{d}{dx}\left(\frac{-m''(x)x}{m'(x)}\right) > 0$. The fact that $\lim_{n\to 0} \eta_m(n) = 1$ follows from L'Hopital's rule. We also have $\lim_{n\to\infty} \eta_m(n) = \lim_{n\to\infty} \frac{P_1(n)}{1-P_0(n)} = 0$ since $P_1(n) = -nP_0(n)$. Finally, the fact that $\eta_m'(n) < 0$ and $\lim_{n\to 0} \eta_m(n) = 1$ implies that $\eta_m(n) < 1$. ■

Proof of Lemma 2

The distribution of the maximum of $j \geq 1$ draws is $(G(\theta))^j$, and weighting by the probability $P_j(n)$ that exactly j sellers meet a buyer, conditional on $j \geq 1$, yields

$$\tilde{G}(\theta; n) = \frac{\sum_{j=1}^{\infty} P_j(n)(G(\theta))^j}{m(n)}. \tag{33}$$

Given that we assume the search technology P_j is invariant, we have $\sum_{j=0}^{\infty} P_j(n)y^j = P_0(n(1-y))$ and substituting into the above yields (3).

Part 1. Taking the limit as $n \to 0$, we have

$$\lim_{n\to 0} \tilde{G}(\theta; n) = \lim_{n\to 0}\left(\frac{P_0(n(1-G(\theta))) - P_0(n)}{m(n)}\right) = G(\theta)$$

[21] See, for example, Cai et al. (2025) and Mangin (2025) for further details.

using L'Hopital's rule and the fact that $\lim_{z\to 0} P_0(z) = 1$ and $\lim_{z\to 0} P_0'(z) = -1$ by Lemma 9. Therefore, $\tilde{\theta}(n) \to E_G(\theta)$.

Part 2. Taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \tilde{G}(\theta; n) = \lim_{n\to\infty} \left(\frac{P_0(n(1 - G(\theta))) - P_0(n)}{m(n)} \right) = 0$$

for any $\theta \in [\underline{\theta}, \bar{\theta})$ and $\lim_{n\to\infty} \tilde{G}(\bar{\theta}; n) = 1$ because $\lim_{z\to\infty} P_0(z) = 0$ by Lemma 9.

Part 3. For $n > 0$, we have $\tilde{G}(\theta; n) < G(\theta)$ for all $\theta \in \Theta$. To see this, let $w_j(n) = P_j(n)/m(n)$. Using (33), $\tilde{G}(\theta; n) = \sum_{j=1}^{\infty} w_j(n)(G(\theta))^j$. Since $\tilde{G}(\theta; n)$ is a weighted average of the term $(G(\theta))^j$ for all $j \geq 1$, and $(G(\theta))^j < G(\theta)$ for all $j > 1$ and $\theta \in (\underline{\theta}, \bar{\theta})$, and $G(\theta)^j = G(\theta)$ for $j = 1$ and $\theta = \underline{\theta}$ or $\theta = \bar{\theta}$, we have $\tilde{G}(\theta; n) < G(\theta)$. So $\tilde{G}(\theta; n)$ first order stochastically dominates $G(\theta)$ and $\tilde{\theta}(n) > E_G(\theta)$.

Part 4. Consider any $f : \Theta \to \mathbb{R}_+$ such that $f' > 0$. Re-stating Lemma 1 in terms of $P_0(x)$ gives $\frac{d}{dx}\left(\frac{-P_0''(x)x}{P_0'(x)}\right) > 0$. For any n_1 and n_2 such that $n_1 > n_2$, Part 5 implies that $\tilde{f}(n_1) > \tilde{f}(n_2)$, i.e. $\int_{\underline{\theta}}^{\bar{\theta}} f(\theta) d\tilde{G}(\theta; n_1) > \int_{\underline{\theta}}^{\bar{\theta}} f(\theta) d\tilde{G}(\theta; n_2)$. Thus $\tilde{G}(\theta; n_1) \leq \tilde{G}(\theta; n_2)$ and $\tilde{G}(\theta; n_1)$ first order stochastically dominates $\tilde{G}(\theta; n_2)$.

Part 5. Applying Leibniz' integral rule gives us

$$\tilde{f}'(n) = \int_{\underline{\theta}}^{\bar{\theta}} f(\theta) \frac{\partial \tilde{g}(\theta; n)}{\partial n} d\theta.$$

First, we show that there exists a unique cutoff $\hat{\theta} \in \Theta$ such that $\frac{\partial \tilde{g}(\theta;n)}{\partial n} = 0$, and we have $\frac{\partial \tilde{g}(\theta;n)}{\partial n} > 0$ for $\theta > \hat{\theta}$ and $\frac{\partial \tilde{g}(\theta;n)}{\partial n} < 0$ for $\theta < \hat{\theta}$. To start with, we have

$$\tilde{g}(\theta; n) = \frac{-ng(\theta)P_0'(n(1 - G(\theta)))}{m(n)}. \tag{34}$$

Differentiating (34) with respect to n, we obtain

$$\frac{\partial \tilde{g}(\theta; n)}{\partial n} = \frac{g(\theta)P_0'(x)}{m(n)} \left(\frac{-xP_0''(x)}{\mathbb{P}_0'(x)} - (1 - \eta_m(n)) \right)$$

where $x = n(1 - G(\theta))$. Since $P_0'(x) < 0$ by Lemma 9, $\frac{\partial \tilde{g}(\theta;n)}{\partial n} > 0$ if and only if

$$\frac{-P_0''(x)x}{P_0'(x)} < 1 - \eta_m(n).$$

Re-stating Lemma 1 in terms of $P_0(x)$ gives $\frac{d}{dx}\left(\frac{-P_0''(x)x}{P_0'(x)}\right) > 0$. So there exists a unique solution x, and therefore a unique solution θ, such that the above holds with equality. Defining $\hat{\theta}$ as the solution to this equality, we have $\frac{\partial \tilde{g}(\theta;n)}{\partial n} > 0$ if and only if $\theta > \hat{\theta}$, so

$$\tilde{f}'(n) \equiv \int_{\underline{\theta}}^{\hat{\theta}} f(\theta) \frac{\partial \tilde{g}(\theta;n)}{\partial n} d\theta + \int_{\hat{\theta}}^{\bar{\theta}} f(\theta) \frac{\partial \tilde{g}(\theta;n)}{\partial n} d\theta.$$

We therefore have $\tilde{f}'(n) > 0$ if and only if

$$\int_{\hat{\theta}}^{\bar{\theta}} f(\theta) \frac{\partial \tilde{g}(\theta;n)}{\partial n} d\theta > -\int_{\underline{\theta}}^{\hat{\theta}} f(\theta) \frac{\partial \tilde{g}(\theta;n)}{\partial n} d\theta > 0. \tag{35}$$

Given that $f' > 0$, and both sides of (35) are positive, a sufficient condition is

$$\int_{\hat{\theta}}^{\bar{\theta}} f(\hat{\theta}) \frac{\partial \tilde{g}(\theta;n)}{\partial n} d\theta \geq -\int_{\underline{\theta}}^{\hat{\theta}} f(\hat{\theta}) \frac{\partial \tilde{g}(\theta;n)}{\partial n} d\theta,$$

which holds if and only if $\int_{\hat{\theta}}^{\bar{\theta}} \frac{\partial \tilde{g}(\theta;n)}{\partial n} d\theta \geq -\int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial \tilde{g}(\theta;n)}{\partial n} d\theta$ or $\int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial \tilde{g}(\theta;n)}{\partial n} d\theta \geq 0$. By Leibniz' rule, $\int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial \tilde{g}(\theta;n)}{\partial n} d\theta = \frac{\partial}{\partial n} \int_{\underline{\theta}}^{\bar{\theta}} \tilde{g}(\theta;n) d\theta = 0$, since $\int_{\underline{\theta}}^{\bar{\theta}} \tilde{g}(\theta;n) d\theta = 1$, so $\tilde{f}'(n) > 0$. ■

Proof of Proposition 1

The first-order condition with respect to q_θ is

$$m(n)[\theta u'(q_\theta) - c'(q_\theta)]\tilde{g}(\theta;n) = 0 \tag{36}$$

and the first order-condition with respect to n is

$$m'(n)\tilde{s}(n;\{q_\theta\}_{\theta\in\Theta}) + m(n)\tilde{s}'(n;\{q_\theta\}_{\theta\in\Theta}) = \kappa. \tag{37}$$

We can verify that $s_\theta^* = \theta u(q_\theta^*) - c(q_\theta^*)$ is strictly increasing in θ. Differentiating s_θ^*,

$$\frac{ds_\theta^*}{d\theta} = u(q_\theta^*) + [\theta u'(q_\theta^*) - c'(q_\theta^*)] \frac{dq^*}{d\theta}.$$

Since $\theta u'(q_\theta^*) - c'(q_\theta^*) = 0$ by (36) if $n^* > 0$, we have $\frac{ds_\theta^*}{d\theta} = u(q_\theta^*) > 0$ for all $\theta \in (\underline{\theta}, \bar{\theta}]$. Given that s_θ^* is strictly increasing in θ and $s_0^* \geq 0$ where $s_0^* \equiv \underline{\theta} u(q_0) - c(q_0)$, we have $s_\theta^* \geq 0$ for all $\theta \in \Theta$. Therefore, all chosen goods $\theta \in \Theta$ are traded (except possibly $\underline{\theta}$). For all $\theta \in \Theta$, the quantity q_θ satisfies $\theta u'(q_\theta) = c'(q_\theta)$.

Existence and uniqueness of the first-best allocation with $n^* > 0$ follows from the existence and uniqueness proofs for Proposition 2 provided Assumption 5 holds, except $q_\theta^0 = q_\theta^*$ since q_θ^* does not depend directly on n, so Assumption 3 suffices.

Since s_θ^* is strictly increasing in θ, at the first-best the seller with the highest valuation θ is chosen. The distribution of types $\tilde{G}$ is thus equal to (3). ■

Proof of Lemma 4

First, we show that the hazard rate $h_{\tilde{G}}(\theta; n)$ is increasing in the buyer type θ, i.e. $\frac{\partial h_{\tilde{G}}(\theta;n)}{\partial \theta} > 0$, and therefore the virtual valuation function is also increasing in θ, i.e. we have $\frac{\partial \psi_{\tilde{G}}(\theta;n)}{\partial \theta} < 0$. Starting with (3) and letting $x = n(1 - G(\theta))$, we have

$$\tilde{G}(\theta; n) = \frac{P_0(x) - P_0(n)}{m(n)}$$

and therefore, using $m(n) = 1 - P_0(n)$, we obtain

$$\tilde{G}(\theta; n) = \frac{m(n) - m(x)}{m(n)}$$

Next, differentiating with respect to θ yields

$$\tilde{g}(\theta; n) = \frac{ng(\theta)m'(x)}{m(n)}. \tag{38}$$

Therefore, we obtain

$$\frac{1 - \tilde{G}(\theta; n)}{\tilde{g}(\theta; n)} = \frac{m(x)}{ng(\theta)m'(x)},$$

which, using the fact that $x = n(1 - G(\theta))$, is equivalent to

$$\frac{1 - \tilde{G}(\theta; n)}{\tilde{g}(\theta; n)} = \frac{m(x)}{m'(x)x} \left(\frac{1 - G(\theta)}{g(\theta)} \right). \tag{39}$$

The hazard rate of the distribution $\tilde{G}(\theta; n)$ is thus given by

$$h_{\tilde{G}}(\theta; n) = \eta_m(n(1 - G(\theta)))h_G(\theta) \tag{40}$$

where $\eta_m(x) = \frac{m'(x)x}{m(x)}$ and $h_G(\theta) = \frac{1-G(\theta)}{g(\theta)}$. The result follows immediately from (40) plus Lemma 1, which implies that $\eta_m'(x) < 0$ and therefore $\eta_m(x)$ is increasing in θ, plus our assumption that G has an increasing hazard rate, i.e. $h_G'(\theta) > 0$.

Next, we verify the hazard rate $h_{\tilde{G}}(\theta; n)$ is decreasing in the seller-buyer ratio n, i.e. $\frac{\partial h_{\tilde{G}}(\theta;n)}{\partial n} < 0$, and therefore the virtual valuation function is also decreasing in n, i.e. $\frac{\partial \psi_{\tilde{G}}(\theta;n)}{\partial n} < 0$. From (40) above, we can write $h_{\tilde{G}}(\theta; n) = \eta_m(n(1 - G(\theta)))h_G(\theta)$.

Given that $\eta_m'(x) < 0$, it is clear that the hazard rate is decreasing in n. ■

Proof of Lemma 5

To prove this result, we use Lemma 10 and the fact that Lemma 7 in Appendix A tells us that $\theta_b'(n) < 0$ provided that $\theta_b(n) > \underline{\theta}$, which is a lower bound for $\theta_b(n)$.

Lemma 10. *For all $\theta \in (\theta_b^0, \bar{\theta}]$, the quantity $q_\theta^0 \equiv \lim_{n\to 0} q_\theta(n)$ solves*

$$\left(\theta - \frac{1 - G(\theta)}{g(\theta)}\right) u'(q_\theta) = c'(q_\theta).$$

1. *If $\psi_G(\underline{\theta}) > 0$, then $\theta_b^0 = \underline{\theta}$ and $q_\theta^0 > 0$ for all $\theta \in \Theta$.*
2. *If $\psi_G(\underline{\theta}) = 0$, then $\theta_b^0 = \underline{\theta}$ and $q_\theta^0 = 0$ for $\theta = \underline{\theta}$ and $q_\theta^0 > 0$ for all $\theta \in (\underline{\theta}, \bar{\theta}]$.*
3. *If $\psi_G(\underline{\theta}) < 0$, then $\theta_b^0 > 0$ and $q_\theta^0 = 0$ for all $\theta \in [\underline{\theta}, \theta_b^0]$ and $q_\theta^0 > 0$ for all $\theta \in (\theta_b^0, \bar{\theta}]$ where θ_b^0 is the unique solution to $\psi_G(\theta) = 0$.*

Proof. In the limit as $n \to 0$, we have $\tilde{G}(\theta; n) \to G(\theta)$ by Lemma 2. Also, as $n \to 0$, we have $m(n) \to 0$ and $\eta_m(n) \to 1$ by Lemma 1. It is clear that $\delta(n) \to \infty$ as $n \to 0$. Lemma 10 follows from this fact plus expressions (60) and (61). ■

Part 1. If $\psi_G(\underline{\theta}) > 0$, then $\theta_b^0 = \underline{\theta}$ by Lemma 10. Therefore, Lemma 7 implies that $\theta_b(n) = \underline{\theta}$ for any $n > 0$, and $q_\theta > 0$ for all $\theta \in \Theta$. This follows from the fact that q_θ is strictly increasing in θ for $\theta > \theta_b(n)$ by Lemma 14.

Part 2. If $\psi_G(\underline{\theta}) = 0$, then $\theta_b^0 = \underline{\theta}$ by Lemma 10. Therefore, Lemma 7 implies that $\theta_b(n) = \underline{\theta}$ for any $n > 0$, and $q_\theta > 0$ for all $\theta \in (\underline{\theta}, \bar{\theta}]$ and $q_{\underline{\theta}} = 0$. This uses the fact that q_θ is strictly increasing in θ for $\theta > \theta_b(n)$ by Lemma 14.

Part 3. If $\psi_G(\underline{\theta}) < 0$, then by Lemma 10 we have $\theta_b^0 > \underline{\theta}$ and $q_\theta^0 = 0$ for all $\theta \in [\underline{\theta}, \theta_b^0]$ and $q_\theta^0 > 0$ for all $\theta \in (\theta_b^0, \bar{\theta}]$ where θ_b^0 is the unique solution to $\psi_G(\theta) = 0$. For any $n > 0$, the fact that $\theta_b'(n) < 0$ implies that $\theta_b(n) < \theta_b^0$ where $\psi_G(\theta_b^0) = 0$. ■

Proof of Propositions 2 and 9

We solve for the equilibrium in two stages. First, we take n as given and solve for sellers' posted contracts $\{(q_\theta, t_\theta)\}_{\theta\in\Theta}$ (inner maximization problem). Second, we solve for n (outer maximization problem) given $\{(q_\theta, t_\theta)\}_{\theta\in\Theta}$.

We first solve the inner and outer maximization problems. Next, we prove Parts 1 to 8. Finally, we prove existence and uniqueness of equilibrium.

Stage 1. Inner maximization problem

Taking $n > 0$ as given (we later prove this), the problem is to maximize (13) subject to (14) at equality, plus the IC constraint (10) and the IR constraint (9). Ignoring constants, the inner maximization problem is:

$$\max_{\{(q_\theta, t_\theta)\}_{\theta \in \Theta}} \left\{ m(n) \int_{\underline{\theta}}^{\bar{\theta}} \left[\theta u(q_\theta) - t_\theta\right] d\tilde{G}(\theta; n) \right\}, \tag{41}$$

subject to

$$\frac{m(n)}{n} \int_{\underline{\theta}}^{\bar{\theta}} \left[-c(q_\theta) + t_\theta\right] d\tilde{G}(\theta; n) = \kappa, \tag{42}$$

and, for all $\theta, \theta' \in \Theta$,

$$\theta u(q_\theta) - t_\theta \geq \theta u(q_{\theta'}) - t_\theta, \tag{43}$$

$$\theta u(q_\theta) - t_\theta \geq 0, \tag{44}$$

$$t_\theta, q_\theta \geq 0. \tag{45}$$

To solve problem (41), we transform this problem using Lemma 3. Recalling that $v_\theta \equiv \theta u(q_\theta) - t_\theta$, the buyer's ex post trading surplus, Lemma 3 simplifies the IC constraints because the single crossing property (6.1) holds. In particular, Lemma 3 says the IC constraints hold if q_θ is non-decreasing and $v'(\theta) = u(q_\theta)$.

We can use $v_\theta \equiv \theta u(q_\theta) - t_\theta$ and Lemma 3 to re-write the problem as an optimal control problem where q_θ is the control variable, v_θ is the state variable, and δ is the Lagrange multiplier associated with the seller entry constraint (42).

We take n and δ as given and later solve for these. Using $v_\theta \equiv \theta u(q_\theta) - t_\theta$ to eliminate t_θ in the above, and substituting in the constraint (42), the problem becomes

$$\max_{\{(q_\theta, v_\theta)\}_{\theta \in \Theta}} \left\{ m(n) \int_{\underline{\theta}}^{\bar{\theta}} \left\{(1-\delta) v_\theta + \delta \left[\theta u(q_\theta) - c(q_\theta)\right]\right\} \tilde{g}(\theta; n) d\theta - \delta n k \right\}, \tag{46}$$

subject to $v_0 = 0$ and q_θ is non-decreasing, and for all $\theta \in \Theta$,

$$\dot{v}_\theta = u(q_\theta), \tag{47}$$

$$q_\theta, v_\theta \geq 0. \tag{48}$$

The inner maximization problem is a standard optimal control problem with q_θ as the control variable and v_θ as the state variable. We can therefore apply the Maximum Principle to find the necessary conditions for the optimal path of the control and state variables. To solve the inner maximization problem, we ignore the condition that q_θ is non-decreasing and later verify that it holds in Lemma 14.

Ignoring constants, the current value Hamiltonian for this problem is:

$$H = m(n)\{(1-\delta)v_\theta + \delta\left[\theta u(q_\theta) - c(q_\theta)\right]\}\tilde{g}(\theta;n) + \lambda_\theta u(q_\theta) \tag{49}$$

where λ_θ is the costate variable, and the Lagrangian is:

$$L = m(n)\{(1-\delta)v_\theta + \delta\left[\theta u(q_\theta) - c(q_\theta)\right]\}\tilde{g}(\theta;n) + \lambda_\theta u(q_\theta) + \hat{\theta}_\theta q_\theta + \eta_\theta v_\theta$$

where $\hat{\theta}_\theta$ and η_θ are the Lagrangian multipliers associated with the non-negativity constraint and IR constraint respectively.

The first-order conditions and the transversality condition are as follows:

$$\frac{\partial L}{\partial q_\theta} = m(n)\delta\left[\theta u'(q_\theta) - c'(q_\theta)\right]\tilde{g}(\theta;n) + \lambda_\theta u'(q_\theta) + \hat{\theta}_\theta = 0, \tag{50}$$

$$\frac{\partial L}{\partial v_\theta} = (1-\delta)m(n)\tilde{g}(\theta;n) + \eta_\theta = -\dot{\lambda}_\theta, \tag{51}$$

$$\frac{\partial L}{\partial \lambda_\theta} = \dot{v}_\theta = u(q_\theta), \tag{52}$$

$$\lambda_{\bar{\theta}} v_{\bar{\theta}} = 0. \tag{53}$$

For the inequality constraints, the conditions are:

$$\hat{\theta}_\theta \geq 0, \; \hat{\theta}_\theta q_\theta = 0, \tag{54}$$

$$\eta_\theta \geq 0, \; \eta_\theta v_\theta = 0. \tag{55}$$

The following lemma provides an expressions for λ_θ.

Lemma 11. *For all $\theta \in [\underline{\theta}, \bar{\theta}]$, we have the following:*

$$\lambda_\theta = m(n)(1-\delta)[1 - \tilde{G}(\theta;n)] + \int_\theta^{\bar{\theta}} \eta_x dx. \tag{56}$$

Proof. Start with the fact that

$$(1-\delta)m(n)\tilde{g}(\theta;n) + \eta_\theta = -\dot{\lambda}_\theta$$

from the first-order condition (51) above. Integrating both sides over $[\theta, \bar{\theta}]$, we obtain

$$-\int_\theta^{\bar{\theta}} \dot{\lambda}_x dx = \int_\theta^{\bar{\theta}} (1-\delta)m(n)\tilde{g}(x;n)dx + \int_\theta^{\bar{\theta}} \eta_x dx$$

and therefore

$$-(\lambda_{\bar{\theta}} - \lambda_\theta) = m(n)(1-\delta)\int_\theta^{\bar{\theta}} \tilde{g}(x;n)dx + \int_\theta^{\bar{\theta}} \eta_x dx.$$

The transversality condition $\lambda_{\bar{\theta}} v_{\bar{\theta}} = 0$ implies $\lambda_{\bar{\theta}} = 0$ since $v_{\bar{\theta}} > 0$. Setting $\lambda_{\bar{\theta}} = 0$ in the above yields

$$\lambda_\theta = m(n)(1-\delta)\int_\theta^{\bar{\theta}} \tilde{g}(x;n)dx + \int_\theta^{\bar{\theta}} \eta_x dx, \tag{57}$$

and using $\int_\theta^{\bar{\theta}} \tilde{g}(x;n)dx = [\tilde{G}(x;n)]_\theta^{\bar{\theta}} = 1 - \tilde{G}(\theta;n)$ yields (56). ■

Lemma 12. *If $\psi_G(\underline{\theta}) = 0$, we have the following:*

$$\delta = \frac{1}{1-\underline{\theta}\tilde{g}(\underline{\theta};n)}\left(1 + \frac{\int_{\underline{\theta}}^{\bar{\theta}} \eta_x dx}{m(n)}\right). \tag{58}$$

Proof. To start with, we have

$$m(n)\delta\left[\theta u'(q_\theta) - c'(q_\theta)\right]\tilde{g}(\theta;n) + \lambda_\theta u'(q_\theta) + \hat{\theta}_\theta = 0$$

from the first-order condition (50) for q_θ. Dividing both sides by $u'(q_\theta)$, we obtain

$$m(n)\delta\left[\theta - \frac{c'(q_\theta)}{u'(q_\theta)}\right]\tilde{g}(\theta;n) + \lambda_\theta = \frac{-\hat{\theta}_\theta}{u'(q_\theta)}.$$

Taking the limit as $q_\theta \to 0$, and using $\lim_{q\to 0} u'(q) = +\infty$ and $\lim_{q\to 0}\frac{c'(q)}{u'(q)} = 0$ yields

$$\lim_{q\to 0}\left[m(n)\delta\left[\theta - \frac{c'(q)}{u'(q)}\right]\tilde{g}(\theta;n) + \lambda_\theta + \frac{\hat{\theta}_\theta}{u'(q)}\right] = m(n)\delta\theta\tilde{g}(\theta;n) + \lambda_\theta = 0$$

for any $\theta \leq \theta_b(n)$ and therefore $\lambda_\theta = -m(n)\delta\theta\tilde{g}(\theta; n)$ for any $\theta \leq \theta_b(n)$. In particular,

$$\lambda_{\underline{\theta}} = -m(n)\delta\underline{\theta}\tilde{g}(\underline{\theta}; n).$$

Next, applying Lemma 11 to the special case $\theta = \underline{\theta}$, we have

$$\lambda_{\underline{\theta}} = m(n)(1-\delta) + \int_{\underline{\theta}}^{\bar{\theta}} \eta_x dx.$$

Equating these two expressions, we obtain (58). ■

To determine q_θ for all $\theta \in \Theta$, we need to determine δ. By Lemma 5, there are three cases to consider. For all three cases, for any $\theta \in (\theta_b(n), \bar{\theta}]$, we have $q_\theta > 0$ and therefore $\hat{\theta}_\theta = 0$, so q_θ solves

$$m(n)\delta\left[\theta u'(q_\theta) - c'(q_\theta)\right]\tilde{g}(\theta; n) \;= -\lambda_\theta u'(q_\theta). \tag{59}$$

Using Lemma 11, plus $\int_\theta^{\bar{\theta}} \eta_x dx = 0$ for all $\theta \geq \theta_b(n)$ since $\eta_\theta = 0$ for $\theta > \theta_b(n)$,

$$\lambda_\theta = m(n)(1-\delta)[1 - \tilde{G}(\theta; n)]$$

for any $\theta \in \Theta$. Substituting into (59), we obtain

$$(\theta - \phi(\theta; n))u'(q_\theta) = c'(q_\theta) \tag{60}$$

where

$$\phi(\theta; n) = \left(\frac{\delta - 1}{\delta}\right)\left(\frac{1 - \tilde{G}(\theta; n)}{\tilde{g}(\theta; n)}\right). \tag{61}$$

If $\psi_G(\underline{\theta}) = 0$, then $\theta_b(n) = \underline{\theta}$ and Lemma 12 implies that the value of δ is

$$\delta = \frac{1}{1 - \underline{\theta}\tilde{g}(\underline{\theta}; n)}.$$

Using the decomposition (39), this simplifies to

$$\delta = \frac{1}{1 - \underline{\theta}g(\underline{\theta})\eta_m(n)}. \tag{62}$$

If $\psi_G(\underline{\theta}) = 0$, then $\underline{\theta}g(\underline{\theta}) = 1$ and the above expression for δ simplifies.

If $\psi_G(\underline{\theta}) < 0$, we have $\theta_b(n) - \phi(\theta_b(n); n) = 0$ from Lemma 13 below. Using this fact, plus expression (61), the value of δ is given by the following expression:

$$\delta = \frac{1}{1 - \frac{\theta_b(n)\tilde{g}(\theta_b(n);n)}{1-\tilde{G}(\theta_b(n);n)}}. \tag{63}$$

Lemma 13. *If $\psi_G(\underline{\theta}) < 0$, we have $\theta = \phi(\theta; n)$ for all $\theta \leq \theta_b(n)$.*

Proof. Suppose that $\psi_G(\underline{\theta}) < 0$. For $\theta = \theta_b(n)$, we have $q_a = 0$. Using (60),

$$\lim_{\theta \to \theta_b(n)} (\theta - \phi(\theta; n)) = \lim_{\theta \to \theta_b(n)} \left[1 - \frac{\phi(\theta; n)}{\theta}\right] \theta = \lim_{q \to 0} \frac{c'(q)}{u'(q)} = 0$$

since $\lim_{q \to 0} \frac{c'(q)}{u'(q)} = 0$ by assumption. Therefore, by continuity of the function q_θ, we have either $\frac{\phi(\theta_b(n);n)}{\theta_b(n)} = 1$, or equivalently $\theta = \phi(\theta_b(n); n)$, or $\theta_b(n) = 0$. We know that $\theta_b(n) > \underline{\theta}$ if $\psi_G(\underline{\theta}) < 0$ and $\underline{\theta} \geq 0$, so we have $\theta = \phi(\theta; n)$ for all $\theta \leq \theta_b(n)$. ■

Stage 2. Outer maximization problem

The outer maximization problem we solve next is

$$\max_{n,\delta} \left\{ J(n, \delta) - \delta n k \right\}, \tag{64}$$

where we define

$$J(n, \delta) \equiv \max_{\{(q_\theta, v_\theta)\}_{\theta \in \Theta}} \left\{ m(n) \int_{\underline{\theta}}^{\bar{\theta}} \left\{ (1-\delta) v_\theta + \delta \left[\theta u(q_\theta) - c(q_\theta) \right] \right\} \tilde{g}(\theta; n) d\theta \right\}, \tag{65}$$

subject to $v_0 = 0$ and, for all $\theta \in \Theta$, constraints (47) and (48).

To solve the outer maximization problem, the function $J(n, \delta)$ is equivalent to

$$J(n, \delta) = \max_{\{(q_\theta, v_\theta)\}_{\theta \in \Theta}} \left\{ \begin{array}{c} \int_{\underline{\theta}}^{\bar{\theta}} m(n) \{ (1-\delta) v_\theta + \delta \left[\theta u(q_\theta) - c(q_\theta) \right] \} \tilde{g}(\theta; n) d\theta \\ + \int_{\underline{\theta}}^{\bar{\theta}} \left[\eta_\theta v_\theta + \lambda_\theta u(q_\theta) + \theta_\theta q_\theta \right] d\theta \end{array} \right\}. \tag{66}$$

Define $\tilde{s}(n) \equiv \int_{\underline{\theta}}^{\bar{\theta}} s_\theta d\tilde{G}(\theta; n)$ and $\tilde{v}(n) \equiv \int_{\underline{\theta}}^{\bar{\theta}} v_\theta d\tilde{G}(\theta; n)$. Returning to our original formulation to eliminate δ, the above problem is equivalent to $\max_n \hat{J}(n)$ where

$$\hat{J}(n) = \max_{\{(q_\theta, v_\theta)\}_{\theta \in \Theta}} \left\{ m(n) \tilde{v}(n) + \int_{\underline{\theta}}^{\bar{\theta}} \left[\eta_\theta v_\theta + \lambda_\theta u(q_\theta) + \theta_\theta q_\theta \right] d\theta \right\}$$

subject to the constraint

$$\frac{m(n)}{n}[\tilde{s}(n) - \tilde{v}(n)] \leq \kappa \tag{67}$$

and $n \geq 0$ with complementary slackness. Using the envelope theorem, the first-order condition for n is

$$m'(n)\tilde{v}(n) + m(n)\tilde{v}'(n) = 0. \tag{68}$$

Proof of Parts 1 to 5 for Propositions 2 and 9

Part 1. Follows from Lemma 5.

Part 2. From above, for any $\theta \in [\theta_b(n), \bar{\theta}]$, we have

$$(\theta - \phi(\theta; n))u'(q_\theta) = c'(q_\theta)$$

where, using expression (61) for $\phi(\theta; n)$, we have

$$\phi(\theta; n) = \left(\frac{\delta - 1}{\delta}\right)\left(\frac{1 - \tilde{G}(\theta; n)}{\tilde{g}(\theta; n)}\right) \tag{69}$$

where δ is given by (62) for Proposition 2 and (63) for Proposition 9.

Also, $\dot{v}_\theta = u(q_\theta)$ implies $v_\theta - v_0 = \int_{\underline{\theta}}^{\theta} u(q_x)dx$, so $v_\theta = \int_{\underline{\theta}}^{\theta} u(q_x)dx$ since $v_0 = 0$. We can derive t_θ from v_θ using the fact that $v_\theta \equiv \theta u(q_\theta) - t_\theta$.

Part 3. The first-order condition for $n > 0$ given by (68) can be written as

$$m'(n)\tilde{s}(n) + m(n)\tilde{s}'(n) = \kappa, \tag{70}$$

using the constraint (67) at equality. More precisely, this is equivalent to

$$m'(n)\tilde{s}(n; \{q_\theta\}_{\theta\in\Theta}) + m(n)\tilde{s}'(n; \{q_\theta\}_{\theta\in\Theta}) = \kappa.$$

The fact that n is strictly decreasing in κ is proven in Lemma 17 below.

Part 4. The zero profit condition is given by (67), using the definition of v_θ.

Part 5. Since v_θ is increasing in θ, the highest valuation is always chosen by buyers and this is their "type". Therefore the cdf of buyer types is given by (3). ■

Proof that q_θ is non-decreasing for Propositions 2 and 9

Finally, we verify that q_θ is non-decreasing, as required for Lemma 3.

Lemma 14. *The function $q(.)$ is non-decreasing for all $\theta \in \Theta$. In addition, $q'(\theta) > 0$ for all $\theta \in (\theta_b(n), \bar{\theta}]$.*

Proof. For all $\theta < \theta_b(n)$, we have $q_\theta = 0$ and $q'(\theta) = 0$. For all $\theta \in [\theta_b(n), \bar{\theta}]$, implicit differentiation of

$$(\theta - \phi(\theta; n))u'(q_\theta) = c'(q_\theta)$$

yields

$$q'(\theta) = \frac{-[1 - \phi'(\theta)]u'(q_\theta)}{[\theta - \phi(\theta; n)]u''(q_\theta) - c''(q_\theta)} \tag{71}$$

where $\phi(\theta; n)$ is given by (69). Since $u'(q_\theta) > 0$ and $u''(q_\theta) < 0$ and $c''(q_\theta) > 0$ and $\theta - \phi(\theta; n) > 0$, to establish $q'(\theta) > 0$ it suffices to show that $\phi'(\theta) < 1$. We know from Lemma 4 that $\tilde{G}$ has an increasing hazard rate, so the right term in (69) is decreasing in θ. Therefore, we have $\phi'(\theta) < 0$ and $q'(\theta) > 0$ follows from (71). ■

Proof of existence and uniqueness for Propositions 2 and 9

We first prove existence and uniqueness of the solution to the inner maximization problem and then prove the same for the outer maximization problem.

Inner maximization. We prove that, given n from the outer maximization problem, the solution to the inner maximization problem exists and is unique.

Existence. A solution to the problem exists because the set of admissible paths is non-empty and compact, and there exists an admissible path for which the objective is finite. For example, the path $q_\theta = 0$ and $v_\theta = (\theta - 1)u(q_\theta)$ for all $\theta \in \Theta$ is admissible (since $v_0 = 0$, $q_\theta \geq 0$, $v_\theta \geq 0$, and $\dot{v}_\theta = u(q_\theta) + (\theta - 1)u'(q_\theta)q'(\theta) = u(q_\theta)$, and $q'(\theta) \geq 0$). Also, the objective is finite under this path. Finally, the set of feasible paths is compact since $q_\theta \in [0, q^*_{\bar{\theta}}]$ where $q^*_{\bar{\theta}}$ solves $\bar{\theta}u'(q_{\bar{\theta}}) = c'(q_{\bar{\theta}})$ and $v_\theta \in [0, v_{\bar{\theta}}]$ where $v_{\bar{\theta}} = u(q^*_{\bar{\theta}})[\bar{\theta} - \underline{\theta}]$.

Uniqueness. The Hamiltonian $H(q_\theta, v_\theta, \lambda_\theta)$ given by (49), where λ_θ is the co-state variable given by the Maximum Principle, is strictly concave in the control and state variables (q_θ, v_θ) for all θ. So, the solution is an optimum that solves the inner maximization problem and is unique. To establish strict concavity, differentiating $H(q_\theta, v_\theta, \lambda_\theta)$ with respect to q_θ yields

$$\begin{aligned}\frac{\partial H}{\partial q_\theta} &= m(n)\delta[u'(q_\theta) - c'(q_\theta)]\tilde{g}(\theta; n) + \lambda_\theta u'(q_\theta), \\ \frac{\partial^2 H}{\partial q_\theta^2} &= m(n)\delta[u''(q_\theta) - c''(q_\theta)]\tilde{g}(\theta; n) + \lambda_\theta u''(q_\theta) \equiv -X,\end{aligned}$$

where $X > 0$, since $u''(q_\theta) < 0$ and $c''(q_\theta) > 0$. Differentiating $H(q_\theta, v_\theta, \lambda_\theta)$ with respect to v_θ, we obtain $\frac{\partial H}{\partial v_\theta} = m(n)(1-\delta)\tilde{g}(\theta; n)$ and $\frac{\partial^2 H}{\partial v_\theta^2} = 0$. Finally, $\frac{\partial^2 H}{\partial v_\theta \partial q_\theta} = 0$, so we get the Hessian matrix, $\mathbb{H} = \begin{bmatrix} -X & 0 \\ 0 & 0 \end{bmatrix}$. Since $\mathbf{x}^T \mathbb{H} \mathbf{x} < 0$ for all $\mathbf{x} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$, the Hessian $\mathbb{H}$ is negative definite and the Hamiltonian is strictly concave in (q_θ, v_θ).

Outer maximization. We prove that, given $\{(q_\theta, v_\theta)\}_{\theta \in \Theta}$ from the inner maximization problem, the solution n to the outer maximization problem exists and is unique, and n is an interior solution with $n > 0$ if Assumption 5 holds. To establish this result, we first prove that there exists a non-empty set of solutions n, denoted by $N(\kappa)$, that solves the problem. We then show that equilibrium is unique if $n > 0$ for all $n \in N(\kappa)$, and finally we prove that $n > 0$ for any $n \in N(\kappa)$.

Taking $\{(q_\theta, v_\theta)\}_{\theta \in \Theta}$ as given by the inner maximization problem, and ignoring constants, the outer maximization problem is equivalent to

$$\max_n \left\{ m(n) \int_{\underline{\theta}}^{\bar{\theta}} [\theta u(q_\theta) - t_\theta] \, d\tilde{G}(\theta; n) \right\}, \tag{72}$$

subject to

$$\frac{m(n)}{n} \int_{\underline{\theta}}^{\bar{\theta}} [-c(q_\theta) + t_\theta] \, d\tilde{G}(\theta; n) \leq \kappa \tag{73}$$

and $n \geq 0$ with complementary slackness, where $\{(q_\theta, v_\theta)\}_{\theta \in \Theta}$ solves the inner maximization.

Lemma 15. *The set of solutions $N(\kappa)$ is nonempty and upper hemicontinuous.*

Proof. Since $m(n)$ is a bijection, we can rewrite (72) in terms of m as follows:

$$\max_m \left\{ m \int_{\underline{\theta}}^{\bar{\theta}} [\theta u(q_\theta) - t_\theta] \, d\tilde{G}(\theta; m) \right\}.$$

The objective function is continuous and, without loss of generality, we can restrict m to the compact set $m \in [0, 1]$. The constraint (73) is therefore $m \in \Gamma(\kappa)$ for all $\kappa \geq 0$, where $\Gamma(\kappa)$ is a continuous and compact-valued correspondence. Applying the Theorem of the Maximum (Theorem 3.6 in Stokey, Lucas, and Prescott, 1989), the correspondence that gives the set of solutions for m is nonempty and upper hemicontinuous, so $N(\kappa)$ is nonempty and upper hemicontinuous. ■

Lemma 16 establishes that any strictly positive solution $n \in N(\kappa)$ must be unique.

Lemma 16. *If* $N^+ \subseteq N(\kappa)$ *and* $N^+ \subseteq \mathbb{R}_+ \backslash \{0\}$, *then* $N^+ = \{n\}$.

Proof. Consider any solution $n \in N(\kappa)$ such that $n > 0$. Defining $\Phi(n) \equiv m(n)\tilde{v}(n)$, the solutions n satisfy the first-order condition (68), which says $\Phi'(n) = 0$. We show that $\Phi''(n) < 0$ and thus any solution is unique. Using (34), we have

$$\Phi(n) = -\int_{\underline{\theta}}^{\bar{\theta}} P_0'(n(1 - G(\theta)))v_\theta g(\theta) d\theta.$$

Using Leibniz's integral rule, plus the envelope theorem,

$$\Phi'(n) = \int_{\underline{\theta}}^{\bar{\theta}} -P_0'(n(1 - G(\theta)))v_\theta g(\theta) d\theta - \int_{\underline{\theta}}^{\bar{\theta}} n(1 - G(\theta))P_0''(n(1 - G(\theta)))v_\theta g(\theta) d\theta.$$

By integration by parts on the second integral in $\Phi'(n)$ above, we obtain

$$\Phi'(n) = \int_{\underline{\theta}}^{\bar{\theta}} -P_0'(n(1 - G(\theta)))(1 - G(\theta))v'(\theta) d\theta - P_0'(n)v(\underline{\theta}) > 0. \tag{74}$$

Differentiating (74), we find that

$$\Phi''(n) = -\left(\int_{\underline{\theta}}^{\bar{\theta}} P_0''(n(1 - G(\theta))(1 - G(\theta))^2 v'(\theta) d\theta + P_0''(n)v(\underline{\theta}) \right) < 0. \tag{75}$$

The fact that $\Phi''(n) < 0$ follows from the fact that $P_0''(x) > 0$ by Lemma 9, plus the fact that $v'(\theta) = u(q_\theta) \geq 0$ for all θ and $v'(\theta) > 0$ for some θ and also $v(\underline{\theta}) = 0$. ■

From Lemma 15, for any given $\kappa \geq 0$, there exists a non-empty set of solutions $N(\kappa)$ that solves problem (72). We now prove that, for any $n \in N(\kappa)$, we have $n \in \mathbb{R}_+ \backslash \{0\}$ if Assumption 5 holds. Also, the function $n(\kappa)$ is strictly decreasing in κ.

Lemma 17. *Any solution $n \in N(\kappa)$ is interior, i.e. $n \in \mathbb{R}_+ \backslash \{0\}$. The function $n(\kappa)$ is strictly decreasing in κ.*

Proof. First, we show there exists an interior solution $n > 0$. Define $\Lambda(n) \equiv m(n)\tilde{s}(n)$. The first-order condition (70) says $\Lambda'(n) = \kappa$. We prove there exists $n > 0$ such that $\Lambda'(n) = \kappa$ if Assumption 5 holds. We have $\lim_{n\to\infty} \Lambda'(n) = 0$, and

$$\lim_{n\to 0} \Lambda'(n) = \int_{\underline{\theta}}^{\bar{\theta}} \lim_{n\to 0} s(\theta; q_\theta(n)) dG(\theta)$$

where $\lim_{n\to 0} s(\theta; q_\theta(n)) = s(\theta; \lim_{n\to 0} q_\theta(n))$. If the following condition holds:

$$E_G[\theta u(q_\theta^0) - c(q_\theta^0)] > \kappa \tag{76}$$

where $q_\theta^0 \equiv \lim_{n\to 0} q_\theta(n)$, there exists $n > 0$ such that $\Lambda'(n) = \kappa$ provided that $\Lambda''(n) < 0$ (proven below). Lemma 10 describes how to calculate $q_\theta^0 \equiv \lim_{n\to 0} q_\theta(n)$.

Next, any interior solution $n > 0$ is better than $n = 0$. Define the value function:

$$V(\kappa) \equiv \max_n \left\{ m(n) \int_{\underline{\theta}}^{\bar{\theta}} [\theta u(q_\theta) - t_\theta]\, d\tilde{G}(\theta; n) \right\}.$$

We have $V(\kappa) \equiv \max_n \{m(n)\tilde{v}(n)\}$. If $n = 0$ then $V(\kappa) = 0$. If $n > 0$, $V(\kappa) \equiv \max_n \{m(n)\tilde{s}(n) - n\kappa\}$ using constraint (73) with equality. Letting $\Lambda(n) = m(n)\tilde{s}(n)$, we have $V(\kappa) > 0$ if $\Lambda(n) - n\kappa > 0$. Thus the candidate solution $n > 0$ is better than $n = 0$ if $\Lambda(n) > n\kappa$ for $n > 0$. Using the fact that $\Lambda'(n) = \kappa$, it suffices to show that $\Lambda''(n) < 0$ and $\frac{\Lambda'(n)n}{\Lambda(n)} < 1$ for $n > 0$. Similarly to Lemma 16, using (34) yields

$$\Lambda(n) = -\int_{\underline{\theta}}^{\bar{\theta}} nP_0'(n(1 - G(\theta)))s_\theta g(\theta) d\theta$$

and, using Leibniz's integral rule, plus the envelope theorem, yields

$$\Lambda'(n) = \int_{\underline{\theta}}^{\bar{\theta}} -P_0'(n(1 - G(\theta)))s_\theta g(\theta)d\theta - \int_{\underline{\theta}}^{\bar{\theta}} n(1 - G(\theta))P_0''(n(1 - G(\theta)))s_\theta g(\theta)d\theta.$$

Therefore, letting $x = n(1 - G(\theta))$, we have

$$\frac{\Lambda'(n)n}{\Lambda(n)} = 1 + \frac{\int_{\underline{\theta}}^{\bar{\theta}} xP_0''(x)s_\theta g(\theta)d\theta}{\int_{\underline{\theta}}^{\bar{\theta}} P_0'(x)s_\theta g(\theta)d\theta}.$$

Because $P_0''(x) > 0$ and $P_0'(x) < 0$ by Lemma 9, we have $\frac{\Lambda'(n)n}{\Lambda(n)} < 1$ for $n > 0$.

Finally, $\Phi(n) = \Lambda(n) - n\kappa$ for $n > 0$, so $\Phi'(n) = \Lambda'(n) - \kappa$ and $\Phi''(n) = \Lambda''(n)$. Since $\Phi''(n) < 0$ from the proof of Lemma 16, we have $\Lambda''(n) < 0$. It follows that, for any $n \in N(\kappa)$, we have $n > 0$. Since we assume $\kappa > 0$, this implies $n \in \mathbb{R}_+ \backslash \{0\}$. Since n is unique by Lemma 16, there is a function $n : \mathbb{R}_+ \backslash \{0\} \to \mathbb{R}_+ \backslash \{0\}$ such that $n(\kappa)$ solves $\Lambda'(n) = \kappa$. Clearly, n is strictly decreasing in κ since $\Lambda''(n) < 0$. ■

Proof of Corollary 1

In general, the equilibrium quantities traded are below the first-best for all $\theta \in [\underline{\theta}, \bar{\theta})$. If $\theta = \bar{\theta}$, then $q_\theta = q_\theta^*$ because $1 - \tilde{G}(\theta; n) = 0$. The equilibrium n satisfies

$$m'(n)\tilde{s}(n; \{q_\theta\}_{\theta \in \Theta}) + m(n)\tilde{s}'(n; \{q_\theta\}_{\theta \in \Theta}) = \kappa$$

and the first-best n^* satisfies

$$m'(n^*)\tilde{s}(n^*; \{q_\theta^*\}_{\theta \in \Theta}) + m(n^*)\tilde{s}'(n^*; \{q_\theta^*\}_{\theta \in \Theta}) = \kappa.$$

We know from above that $q_\theta^* > q_\theta$ for any $\theta \in [\underline{\theta}, \bar{\theta})$, but we cannot infer anything about whether there is under-entry ($n < n^*$), over-entry ($n > n^*$), or first-best entry ($n = n^*$). We can find examples of equilibria for each of these three possibilities. ■

Proof of Propositions 4 and 11

We must show that $\delta_{\tilde{G}}(\theta; n)$ is decreasing in n where

$$\delta_{\tilde{G}}(\theta; n) = \varepsilon_\rho(\theta_b(n); n)\frac{I_{\tilde{G}}(\theta; n)}{\theta}.$$

Using expression (80) gives us $\frac{I_{\tilde{G}}(\theta;n)}{\theta} = \frac{1}{\varepsilon_\rho(\theta;n)}$, so we obtain

$$\delta_{\tilde{G}}(\theta; n) = \frac{\varepsilon_\rho(\theta_b(n); n)}{\varepsilon_\rho(\theta; n)}.$$

Next, expression (81) delivers

$$\delta_{\tilde{G}}(\theta; n) = \frac{h_G(\theta_b(n))}{h_G(\theta)} \frac{\eta_m(x(\theta_b(n); n))}{\eta_m(x(\theta; n))}$$

where $x(\theta; n) = n(1 - G(\theta))$. Differentiating with respect to n, we have

$$\frac{\partial}{\partial n}\delta_{\tilde{G}}(\theta; n) = \frac{d}{dn}\left(\frac{h_G(\theta_b(n))}{h_G(\theta)}\right) \frac{\eta_m(x(\theta_b(n); n))}{\eta_m(x(\theta; n))} + \frac{h_G(\theta_b(n))}{h_G(\theta)} \frac{d}{dn}\left(\frac{\eta_m(x(\theta_b(n); n))}{\eta_m(x(\theta; n))}\right).$$

Since $h'_G(\theta) > 0$ by Assumption 4 and $\theta'_b(n) < 0$ by Lemma 7, the first term in above derivative is negative. The second term has the same sign as

$$\frac{d}{dn}\frac{\eta_m(x(\theta_b(n); n))}{\eta_m(x(\theta; n))} = \frac{\eta'_m(x_b)\frac{d}{dn}x(\theta_b(n); n)}{\eta_m(x)} - \frac{\eta_m(x_b)\eta'_m(x)\frac{d}{dn}x(\theta; n)}{\eta_m(x)^2}$$

where x denotes $x(\theta; n)$ and x_b denotes $x(\theta_b(n); n)$. So $\frac{\partial}{\partial n}\delta_{\tilde{G}}(\theta; n) < 0$ provided that

$$\eta'_m(x_b)\eta_m(x)\frac{d}{dn}x(\theta_b(n); n) - \eta_m(x_b)\eta'_m(x)\frac{d}{dn}x(\theta; n) < 0.$$

Now, we have $\frac{d}{dn}x(\theta; n) = -ng(\theta)$, so we require the following:

$$\eta'_m(x_b)\eta_m(x)\frac{d}{dn}x(\theta_b(n); n) + \eta_m(x_b)\eta'_m(x)ng(\theta) < 0.$$

Given that $\eta'_m < 0$ by Lemma 1, it suffices to show that $\frac{d}{dn}x(\theta_b(n); n) \geq 0$. Given that $x(\theta_b(n); n) = n(1 - G(\theta_b(n)))$, differentiating yields

$$\frac{d}{dn}x(\theta_b(n); n) = -ng(\theta_b(n))\theta'_b(n) + 1 - G(\theta_b(n)).$$

We have $\theta'_b(n) < 0$ by Lemma 7 and thus $\frac{d}{dn}x(\theta_b(n); n) > 0$, so $\frac{\partial}{\partial n}\delta_{\tilde{G}}(\theta; n) < 0$. ■

Proof of Corollaries 2 and 4

By definition of the quantity distortion, we have

$$\theta[1 - \delta_{\tilde{G}}(\theta; n)] \equiv \frac{c'(q_\theta)}{u'(q_\theta)}. \tag{77}$$

By assumption, $u''(q) < 0$ and $c''(q) \geq 0$, so the right-hand side is increasing in q_θ. The left-hand side is increasing in n because $\frac{\partial}{\partial n}\delta_{\tilde{G}}(\theta; n) < 0$, so the right-hand side must also be increasing in n. Therefore, we must have $q_\theta(n)$ increasing in n. ■

Proof of Propositions 5 and 12

Differentiating $\tilde{\delta}(n)$ using Leibniz' integral formula yields

$$\tilde{\delta}'(n) = \int_{\underline{\theta}}^{\bar{\theta}} \tilde{g}(\theta; n) \frac{\partial}{\partial n} \delta_{\tilde{G}}(\theta; n) d\theta + \int_{\underline{\theta}}^{\bar{\theta}} \delta_{\tilde{G}}(\theta; n) \frac{\partial}{\partial n} \tilde{g}(\theta; n) d\theta.$$

Given that $\frac{\partial}{\partial n} \delta_{\tilde{G}}(\theta; n) < 0$ by Propositions 4 and 11, it suffices to show that

$$\int_{\underline{\theta}}^{\bar{\theta}} \delta_{\tilde{G}}(\theta; n) \frac{\partial}{\partial n} \tilde{g}(\theta; n) d\theta < 0. \tag{78}$$

Holding n fixed, define $\delta_n(\theta) \equiv \delta_{\tilde{G}}(\theta; n)$ where $\delta_n'(\theta) < 0$ because $\frac{\partial}{\partial \theta} \delta_{\tilde{G}}(\theta; n) < 0$. Rearranging and applying Leibniz' integral rule, inequality (78) is equivalent to

$$\frac{\partial}{\partial n} \int_{\underline{\theta}}^{\bar{\theta}} \left(-\delta_n(\theta)\right) d\tilde{G}(\theta; n) > 0. \tag{79}$$

By Part 5 of Lemma 2, we know that for any $f : \Theta \to \mathbb{R}_+$ such that $f' > 0$, we have $\tilde{f}'(n) > 0$ where $\tilde{f}(n) \equiv \int_{\underline{\theta}}^{\bar{\theta}} f(\theta) d\tilde{G}(\theta; n)$. Letting $f(\theta) = -\delta_n(\theta)$, it is clear that $f' > 0$ because $\delta_n'(\theta) < 0$. Thus inequality (79) follows and we have $\tilde{\delta}'(n) < 0$. ■

Proof of Corollaries 3 and 5

Corollaries 3 and 5 follow from Corollaries 2 and 4 by applying Leibniz' integral rule and then using Part 5 of Lemma 2, using reasoning directly analogous to that used in the above proof of Propositions 5 and 12. ■

Proof of Proposition 6

By definition, $T(q) = t(\theta(q))$, so we have $T'(q) = t'(\theta)\theta'(q)$, or $t'(\theta) = T'(q)q'(\theta)$. At the same time, the payment $t_\theta = \theta u(q_\theta) - v_\theta$, so $t'(\theta) = u(q_\theta) + \theta u'(q_\theta)q'(\theta) - v'(\theta)$. We know that $v'(\theta) = u(q_\theta)$, so we have $t'(\theta) = \theta u'(q_\theta)q'(\theta)$. Therefore, $T'(q) = \theta(q)u'(q)$ and $p(q) = \theta(q)u'(q)$. In terms of the quantity distortion, (77) implies

$$\theta u'(q_\theta) = \frac{c'(q_\theta)}{1 - \delta_{\tilde{G}}(\theta; n)}.$$

So, we conclude that

$$T'(q) = \frac{c'(q)}{1 - \delta_{\tilde{G}}(\theta(q); n)}.$$

Differentiating $T'(q)$ with respect to q, we obtain

$$T''(q) = \frac{c''(q)}{1-\delta_{\tilde{G}}(\theta(q);n)} + \frac{c'(q)\frac{\partial}{\partial\theta}\delta_{\tilde{G}}(\theta(q);n)}{[1-\delta_{\tilde{G}}(\theta(q);n)]^2 q'(\theta)}.$$

The first term in $T''(q)$ is weakly negative because we assume $c''(q) \leq 0$. For the second term, we know that $q'(\theta) > 0$, so the denominator is positive, plus the numerator is negative because $\frac{\partial}{\partial\theta}\delta_{\tilde{G}}(\theta;n) < 0$. Therefore, $T''(q) < 0$ or equivalently $p'(q) < 0$. ■

Proof of Proposition 7

By assumption, we have $c(q) = cq$, and therefore

$$T''(q) = \frac{c\frac{\partial}{\partial\theta}\delta_{\tilde{G}}(\theta(q);n)}{[1-\delta_{\tilde{G}}(\theta(q);n)]^2 q'(\theta)}$$

and we have

$$T'(q) = \frac{c}{1-\delta_{\tilde{G}}(\theta(q);n)}.$$

Therefore, we obtain

$$\frac{T''(q)q}{T'(q)} = \frac{\frac{\partial}{\partial\theta}\delta_{\tilde{G}}(\theta(q);n)q}{[1-\delta_{\tilde{G}}(\theta(q);n)]q'(\theta)}.$$

Also, if $c(q) = cq$ we have

$$q'(\theta) = \frac{-[1-\phi'(\theta)]u'(q_\theta)}{[\theta-\phi(\theta;n)]u''(q_\theta)}$$

where $\theta\delta_{\tilde{G}}(\theta;n) = \phi(\theta;n)$, so we have

$$q'(\theta) = \frac{-[1+J(n;\theta)]u'(q_\theta)}{\theta u''(q_\theta)}$$

where we define

$$J(n;\theta) \equiv \frac{-\theta\frac{\partial}{\partial\theta}\delta_{\tilde{G}}(\theta;n)}{1-\delta_{\tilde{G}}(\theta;n)}.$$

By assumption, $u(q) = q^{1-b}/(1-b)$, which implies $-u''(q)q/u'(q) = b$ and

$$q'(\theta) = \frac{q_a(1+J(n;\theta))}{ab}$$

and

$$\frac{-T''(q)q}{T'(q)} = \frac{b}{1/J(n;\theta)+1}.$$

Therefore, $\frac{-T''(q)q}{T'(q)}$ is decreasing in n if and only if $\frac{\partial J(n;\theta)}{\partial n} < 0$.

Differentiating, we have

$$\begin{aligned}
\frac{\partial \delta_{\tilde{G}}(\theta;n)}{\partial \theta} &= \varepsilon_\rho(\theta_b(n);n)\frac{d}{d\theta}\left(\frac{1-\tilde{G}(\theta;n)}{\theta\tilde{g}(\theta;n)}\right) \\
&= \frac{-\varepsilon_\rho(\theta_b(n);n)}{\theta}\left[1+\left(1+\frac{\tilde{g}'(\theta;n)\theta}{\tilde{g}(\theta;n)}\right)\left(\frac{1-\tilde{G}(\theta;n)}{\theta\tilde{g}(\theta;n)}\right)\right] \\
&= -\frac{1}{\theta}\left[\varepsilon_\rho(\theta_b(n);n)+\left(1+\frac{\tilde{g}'(\theta;n)\theta}{\tilde{g}(\theta;n)}\right)\delta_{\tilde{G}}(\theta;n)\right].
\end{aligned}$$

Therefore, we have

$$J(n;\theta) = \frac{-\theta\frac{\partial}{\partial\theta}\delta_{\tilde{G}}(\theta;n)}{1-\delta_{\tilde{G}}(\theta;n)} = \frac{\varepsilon_\rho(\theta_b(n);n)+\left(1+\frac{\tilde{g}'(\theta;n)\theta}{\tilde{g}(\theta;n)}\right)\delta_{\tilde{G}}(\theta;n)}{1-\delta_{\tilde{G}}(\theta;n)}.$$

By Propositions 4 and 11, we know that $\delta_{\tilde{G}}(\theta;n)$ is decreasing in n, and $\varepsilon_\rho(\theta_b(n);n)$ is decreasing in n by Lemma 8. To ensure that we have $\frac{\partial J(n;\theta)}{\partial n} < 0$, it therefore suffices to have $\frac{\tilde{g}'(\theta;n)\theta}{\tilde{g}(\theta;n)} \geq -1$. Starting with expression (38), we have

$$\tilde{g}(\theta;n) = \frac{n}{m(n)}g(\theta)m'(x)$$

where $x = n(1-G(\theta))$, and therefore

$$\tilde{g}'(\theta;n) = \frac{n}{m(n)}[g'(\theta)m'(x) - ng(\theta)^2m''(x)].$$

So we have

$$\begin{aligned}
\frac{\tilde{g}'(\theta;n)\theta}{\tilde{g}(\theta;n)} &= \frac{\left[g'(\theta)m'(x)-ng(\theta)^2m''(x)\right]\theta}{g(\theta)m'(x)} \\
&= \frac{g'(\theta)\theta}{g(\theta)} - \frac{ng(\theta)m''(x)\theta}{m'(x)} \\
&= \frac{g'(\theta)\theta}{g(\theta)} + \left(\frac{g(\theta)\theta}{1-G(\theta)}\right)\left(\frac{-m''(x)x}{m'(x)}\right) \\
&> \frac{g'(\theta)\theta}{g(\theta)}
\end{aligned}$$

where the last step follows from the fact that $\frac{-m''(x)x}{m'(x)} > 0$. Therefore, to ensure $\frac{\partial J(n;\theta)}{\partial n} < 0$, it suffices for the density g to satisfy the condition $\frac{g'(\theta)\theta}{g(\theta)} \geq -1$. ■

Proof of Lemma 7

Consider the zero-profit condition (19), which can be written as

$$\int_{\underline{\theta}}^{\bar{\theta}} (-c(q_\theta) + t_\theta) \frac{m(n)\tilde{g}(\theta; n)}{n} d\theta = \kappa.$$

Using the fact that $q_\theta = t_\theta = 0$ for all $\theta \leq \theta_b(n)$, we can write

$$\int_{\theta_b(n)}^{\bar{\theta}} \pi_\theta \frac{m(n)\tilde{g}(\theta; n)}{n} d\theta = \kappa$$

where $\pi_\theta \equiv -c(q_\theta) + t_\theta$. Using Leibniz' integral rule to differentiate with regard to n,

$$\int_{\theta_b(n)}^{\bar{\theta}} \frac{d}{dn} \left(\pi_\theta \frac{m(n)\tilde{g}(\theta; n)}{n} \right) d\theta - \lim_{\theta \to \theta_b} \pi_\theta \frac{m(n)\tilde{g}(\theta(n); n)}{n} \theta_b'(n) = 0.$$

Rearranging, and using the envelope theorem, $\theta_b'(n)$ has the same sign as $\frac{d}{dn}\left(\frac{m(n)\tilde{g}(\theta;n)}{n}\right)$. Using expression (38) for $\tilde{g}(\theta; n)$, we have

$$\frac{d}{dn} \left(\frac{m(n)\tilde{g}(\theta; n)}{n} \right) = \frac{d}{dn} \left[g(\theta) m'(x(\theta; n)) \right]$$

where $x(\theta; n) = n(1 - G(\theta))$ and

$$\frac{d}{dn} \left[g(\theta) m'(x(\theta; n)) \right] = g(\theta) m''(x(\theta; n))(1 - G(\theta)).$$

Therefore, since $m'' < 0$ by Lemma 1, $\frac{d}{dn}\left(\frac{m(n)\tilde{g}(\theta;n)}{n}\right) < 0$ for $\theta < \bar{\theta}$ and $\theta_b'(n) < 0$ provided that we have $\theta_b(n) > \underline{\theta}$, which is a lower bound for the value of $\theta_b(n)$. ■

Proof of Proposition 10

To prove that market coverage $\mu(\theta_b(n); n)$ is increasing in n, recall that

$$\mu(\theta_b(n); n) = m(n)(1 - \tilde{G}(\theta_b(n); n)).$$

Starting with expression (3) for $\tilde{G}(\theta; n)$, we have

$$1 - \tilde{G}(\theta; n) = \frac{1 - P_0(n(1 - G(\theta)))}{m(n)}.$$

We can therefore rewrite $\mu(\theta_b(n); n)$ as follows:

$$\mu(\theta_b(n); n) = 1 - P_0(n(1 - G(\theta_b(n)))).$$

It follows from Lemma 7 that $\theta_b'(n) < 0$. Therefore, $G(\theta_b(n))$ is decreasing in n and $n(1 - G(\theta_b(n)))$ is increasing in n. We know that $P_0'(x) < 0$ from Lemma 9, so $P_0(n(1 - G(\theta_b(n)))$ is decreasing in n. Thus $\mu(\theta_b(n); n)$ is increasing in n. ■

Proof of Lemma 8

Differentiating $\varepsilon_\rho(\theta_b(n); n)$ with respect to n,

$$\frac{d}{dn}\varepsilon_\rho(\theta_b(n); n) = \theta_b'(n)\frac{\partial}{\partial\theta}\varepsilon_\rho(\theta_b(n); n) + \frac{\partial}{\partial n}\varepsilon_\rho(\theta_b(n); n).$$

By definition of $\varepsilon_\rho(\theta; n)$, we have

$$\varepsilon_\rho(\theta; n) = \frac{\theta\tilde{g}(\theta; n)}{1 - \tilde{G}(\theta; n)} = \frac{\theta}{I_{\tilde{G}}(\theta; n)}. \tag{80}$$

Using expression (40) gives us

$$\varepsilon_\rho(\theta; n) = \theta\eta_m(x(\theta; n))h_G(\theta) \tag{81}$$

where $x(\theta; n) = n(1 - G(\theta))$. Thus, we have $\frac{\partial}{\partial n}\varepsilon_\rho(\theta; n) < 0$ because $\eta_m' < 0$ by Lemma 1 and $\frac{\partial}{\partial n}x(\theta; n) > 0$. We also know that $\theta_b'(n) < 0$ from Lemma 7, so it suffices to show that $\frac{\partial}{\partial\theta}\varepsilon_\rho(\theta; n) > 0$. Differentiating with respect to θ, we have

$$\frac{\partial}{\partial\theta}\eta_m(x(\theta; n))h_G(\theta) = -\eta_m'(x(\theta; n))ng(\theta)h_G(\theta) + \eta_m(x)h_G'(\theta) > 0$$

because $\eta_m' < 0$ by Lemma 1 and $h_G'(\theta) > 0$ by Assumption 4. Therefore, we have $\frac{\partial}{\partial\theta}\varepsilon_\rho(\theta; n) > 0$ and thus we have proven that $\frac{d}{dn}\varepsilon_\rho(\theta_b(n); n) < 0$, as required. ■

References

D. Acemoglu and R. Shimer. Holdups and Efficiency with Search Frictions. *International Economic Review*, 40(4):827–849, 1999.

J. Albrecht, P. Gautier, and S. Vroman. Equilibrium Directed Search with Multiple Applications. *Review of Economic Studies*, 73(4):869–891, 2006.

J. Albrecht, P. Gautier, and S. Vroman. Efficient Entry in Competing Auctions. *American Economic Review*, 104(10):3288–96, 2014.

J. Albrecht, X. Cai, P. Gautier, and S. Vroman. On the foundations of competitive search equilibrium with and without market makers. *Journal of Economic Theory*, 208:105605, 2023a.

J. Albrecht, G. Menzio, and S. Vroman. Vertical differentiation in frictional product markets. *Journal of Political Economy Macroeconomics*, 1(3):586–632, 2023b.

M. Armstrong and J. Vickers. Competitive price discrimination. *RAND Journal of economics*, pages 579–605, 2001.

M. Armstrong and J. Vickers. Patterns of competitive interaction. *Econometrica*, 90 (1):153–191, 2022.

S. Auster, P. Gottardi, and R. Wolthoff. Simultaneous search and adverse selection. *Review of Economic Studies*, 2025.

A. Bajaj and S. Mangin. Consumer choice, market power, and inflation. *Working Paper*, 2024.

D. Bergemann, B. Brooks, and S. Morris. Search, information, and prices. *Journal of Political Economy*, 129(8):2275–2319, 2021.

G. Bornstein and A. Peter. Nonlinear pricing and misallocation. *American Economic Review (forthcoming)*, 2025.

K. Burdett and K. L. Judd. Equilibrium price dispersion. *Econometrica*, pages 955–969, 1983.

K. Burdett and G. Menzio. The (q, s, s) pricing rule. *The Review of Economic Studies*, 85(2):892–928, 2018.

K. Burdett, S. Shi, and R. Wright. Pricing and Matching with Frictions. *Journal of Political Economy*, 109(5):1060–1085, 2001.

G. Butters. Equilibrium Distributions of Sales and Advertising Prices. *Review of Economic Studies*, 44:465–491, 1977.

X. Cai, P. A. Gautier, and R. P. Wolthoff. Search frictions, competing mechanisms and optimal market segmentation. *Journal of Economic Theory*, 169:453–473, 2017.

X. Cai, P. Gautier, and R. Wolthoff. Spatial search. *Journal of Economic Theory*, page 105976, 2025.

A. Campbell, P. Ushchev, and Y. Zenou. The network origins of entry. *Journal of Political Economy*, 132(11):3867–3916, 2024.

S. M. Davoodalhosseini. Constrained efficiency with adverse selection and directed search. *Journal of Economic Theory*, 183:568–593, 2019.

M. Faig and B. Jerez. A theory of commerce. *Journal of Economic Theory*, 122(1): 60–99, 2005.

M. Galenianos and P. Kircher. On the Game-Theoretic Foundations of Competitive Search Equilibrium. *International Economic Review*, 53(1):1–21, 2012.

V. Guerrieri. Inefficient unemployment dynamics under asymmetric information. *Journal of Political Economy*, 116(4):667–708, 2008.

V. Guerrieri, R. Shimer, and R. Wright. Adverse selection in competitive search equilibrium. *Econometrica*, 78(6):1823–1862, 2010.

B. Julien, J. Kennes, and I. King. Bidding for Labor. *Review of Economic Dynamics*, 3(4):619–649, 2000.

G. Kaplan and G. Menzio. The morphology of price dispersion. *International Economic Review*, 56(4):1165–1206, 2015.

G. Kaplan and G. Menzio. Shopping externalities and self-fulfilling unemployment fluctuations. *Journal of Political Economy*, 124(3):771–825, 2016.

G. Kaplan, G. Menzio, L. Rudanko, and N. Trachter. Relative price dispersion: Evidence and theory. *American Economic Journal: Microeconomics*, 11(3):68–124, August 2019.

B. Lester. Information and prices with capacity constraints. *American Economic Review*, 101(4):1591–1600, 2011.

B. Lester, L. Visschers, and R. Wolthoff. Meeting technologies and optimal trading mechanisms in competitive search markets. *Journal of Economic Theory*, 155:1–15, 2015.

B. Lester, A. Shourideh, V. Venkateswaran, and A. Zetlin-Jones. Screening and adverse selection in frictional markets. *Journal of Political Economy*, 127(1):338–377, 2019.

S. Mangin. When is competition price-increasing? The impact of expected competition on prices. *The RAND Journal of Economics*, 55(4):627–657, 2024.

S. Mangin. Extreme value theory with heterogeneous agents. *Econometrica (forthcoming)*, 2025.

S. Mangin and B. Julien. Efficiency in search and matching models: A generalized hosios condition. *Journal of Economic Theory*, 193:105208, 2021.

S. Mangin and G. Menzio. Private and social learning in frictional product markets. *Working paper*, 2024.

E. Maskin and J. Riley. Monopoly with incomplete information. *The RAND Journal of Economics*, 15(2):171–196, 1984.

G. Menzio. A theory of partially directed search. *Journal of Political Economy*, 115 (5):748–769, 2007.

G. Menzio. Search theory of imperfect competition with decreasing returns to scale. *Journal of Economic Theory*, 218:105827, 2024a.

G. Menzio. Markups: A search-theoretic perspective. Technical report, National Bureau of Economic Research, 2024b.

G. Menzio and S. Shi. Block recursive equilibria for stochastic models of search on the job. *Journal of Economic Theory*, 145(4):1453–1494, 2010.

G. Menzio and S. Shi. Efficient search on the job and the business cycle. *Journal of Political Economy*, 119(3):468–510, 2011.

E. Moen. Competitive Search Equilibrium. *Journal of Political Economy*, 105(2): 385–411, 1997.

E. R. Moen and A. Rosen. Incentives in competitive search equilibrium. *The Review of Economic Studies*, 78(2):733–761, 2011.

J. Montgomery. Equilibrium Wage Dispersion and Interindustry Wage Differentials. *Quarterly Journal of Economics*, 106(1):163–179, 1991.

M. Mussa and S. Rosen. Monopoly and product quality. *Journal of Economic Theory*, 18(2):301–317, 1978.

L. Nord. Shopping, demand composition, and equilibrium prices. *Working paper*, 2023.

M. Peters. Ex Ante Price Offers in Matching Games: Non-Steady States. *Econometrica*, 59(5):1425–1454, 1991.

M. Peters and S. Severinov. Competition among Sellers Who Offer Auctions Instead of Prices. *Journal of Economic Theory*, 75(1):141–179, 1997.

S. Rabinovich and R. Wolthoff. Misallocation inefficiency in partially directed search. *Journal of Economic Theory*, 206:105559, 2022.

J.-C. Rochet and L. Stole. Competitive nonlinear pricing. *SSRN 2396105*, 1997.

G. Rocheteau and R. Wright. Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium. *Econometrica*, 73(1):175–202, 2005.

G. Roger and B. Julien. Moral hazard and efficiency in a frictional market. *American Economic Journal: Microeconomics*, 15(1):693–730, February 2023.

S. Shi. Frictional Assignment I: Efficiency. *Journal of Economic Theory*, 98(2):232–260, 2001.

S. Shi. Directed Search for Equilibrium Wage-Tenure Contracts. *Econometrica*, 77 (2):561–584, 2009.

S. Shi. Sequentially mixed search and equilibrium price dispersion. *Journal of Economic Theory*, 213:105735, 2023.

R. Shimer. PhD Thesis. *M.I.T.*, 1996.

N. Stokey, R. Lucas, and E. C. Prescott. *Recursive Methods in Economic Dynamics*. Harvard University Press, 1989.

H. R. Varian. A model of sales. *American Economic Review*, 70(4):651–659, 1980.

R. Wright, P. Kircher, B. Julien, and V. Guerrieri. Directed search and competitive search equilibrium: A guided tour. *Journal of Economic Literature*, 59(1):90–148, 2021.

H. Yang and L. Ye. Nonlinear pricing, market coverage, and competition. *Theoretical Economics*, 3(1):123–153, 2008.