Efficiency in Search and Matching Models: A Generalized Hosios Condition*

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Abstract

When is the level of entry of buyers or sellers *efficient* in markets with search and matching frictions? This paper generalizes the well-known Hosios condition for constrained efficiency to dynamic search and matching environments where the expected match output depends on the market tightness. We provide a number of examples of such environments. The generalized Hosios condition is simple and intuitive: entry is constrained efficient when buyers' surplus share equals the matching elasticity plus the *surplus elasticity* (i.e. the elasticity of the expected joint match surplus with respect to buyers). In search models of the labor market, for example, this implies that vacancy creation and equilibrium unemployment are not constrained efficient unless firms are compensated for the effect of firm entry on both employment and average labor productivity.

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1 Introduction

The well-known Hosios rule specifies a precise condition under which entry is *constrained efficient* in markets featuring search and matching frictions. Buyer (or seller) entry is "constrained efficient" when the equilibrium level of entry is the same as that which would be chosen by a social planner who is limited by the same frictions as those found in the decentralized market. The original version of the rule introduced in Hosios (1990) states that buyer entry is constrained efficient only when buyers' share of the total joint surplus equals the elasticity of the *matching* function with respect to buyers. This condition has proven to be widely applicable across a broad range of search and matching models. For example, in search-theoretic models of the labor market the Hosios condition tells us when the equilibrium level of vacancy creation – and therefore the unemployment rate – is constrained efficient.[1]

While the Hosios condition applies to a wide range of search models, it does not apply in settings where the expected match output is *endogenous* in the sense that it depends on the market tightness.[2] In such environments, the standard Hosios condition can imply either inefficiently high or inefficiently low levels of entry. This is particularly important in search-and-matching models featuring Nash bargaining because such markets are generically inefficient. The Hosios condition does not hold endogenously, but is instead often *imposed* as a way of determining the bargaining parameter: firms' bargaining power is simply set equal to the "efficient" level given by the matching elasticity.

This paper makes two contributions. First, we generalize the Hosios rule to environments where the expected match output depends on the market tightness. Second, we provide a number of different examples of such environments to demonstrate the wide applicability of this general condition.

Dependence of the expected match output on the market tightness can arise naturally in markets where either buyers or sellers are *heterogeneous* prior to

[1]There is a vast literature on search-theoretic models of the labor market. The classic survey is Rogerson, Shimer, and Wright (2005). See also the recent survey on directed and competitive search by Wright, Kircher, Julien, and Guerrieri (2017).

[2]We use the term "match output" because our examples focus on labor markets, but the term *output* can be interpreted more broadly to cover any trade or productive activity.

matching.[3] We identify two distinct channels. With one-on-one or bilateral meetings, the expected match output may depend on market tightness when there is a participation decision by heterogeneous buyers (or sellers) and the market composition is endogenous. We call this the *composition channel*. With many-on-one or multilateral meetings, there is an additional channel: the expected match output may depend on market tightness because agents face a choice regarding potential trading partners.[4] We call this the *selection channel*.

The Hosios condition ensures that decentralized markets internalize the search externalities that arise through the frictional matching process. However, when the expected match output depends on the market tightness, a novel externality arises. We call this the *output externality* and it can arise through either the composition or the selection channel. Depending on the specific environment, the expected match output may be either increasing or decreasing in the market tightness and therefore the externality may be either positive or negative. The Hosios condition does not incorporate this new externality and it may therefore result in either over-entry or under-entry relative to the social optimum. For example, in labor markets featuring Nash bargaining, imposing the Hosios condition may entail setting workers' bargaining parameter too high, leading to inefficiently high unemployment because firms are not compensated for the effect of vacancy creation on average labor productivity.

Consider an environment with buyer entry. An equilibrium allocation is constrained efficient when buyers are paid their marginal contribution to the social surplus. If the expected output per match is exogenous, buyers need only be paid for their effect on the total *number* of matches and the standard Hosios condition applies: entry is constrained efficient only when buyers' surplus share equals the matching elasticity. If the expected match output is endogenous, however, buyers must be compensated for their effect on both the total number of matches and the expected *value* of the joint surplus created by each match.

[3] We focus on one-sided heterogeneity and do not consider search and matching environments with two-sided heterogeneity and assortative matching such as Shimer and Smith (2000, 2001), Shi (2001), Shimer (2005), and Eeckhout and Kircher (2010a).

[4] With bilateral meetings, the heterogeneity must be *ex ante*. However, with multilateral meetings, buyers and sellers need not be ex ante heterogeneous: they can be identical prior to *meetings* provided there is some heterogeneity prior to *matching*.

We show that entry is constrained efficient only when buyers' surplus share equals the matching elasticity plus the *surplus elasticity* (i.e. the elasticity of the expected match surplus with respect to buyers).

We call this simple condition the *generalized Hosios condition.* To the best of our knowledge, this condition is new to the literature. When the generalized Hosios condition holds, both the standard search externalities and the output externality are fully internalized by a decentralized market. Like the original version, the generalized Hosios condition is highly intuitive. Moreover, we show that the simple rule that arises in a static environment carries over directly to dynamic settings with enduring matches (as found in the labor market).

As a guiding principle, Hosios (1990) suggests that when we want to determine the efficiency properties of a particular model, the question we need to ask is "whether the unattached agents who participate in the corresponding matching process receive more or less than their social marginal product" (p. 296). This guiding principle remains true. However, Hosios states that *all* we need to do to answer this question is determine the equilibrium surplus sharing rule and the matching technology, and then apply what is now known as the "Hosios rule". In fact, when the expected match output is endogenous, this rule must be generalized. In addition to considering the surplus shares and the matching technology, we must also consider the *output technology* which determines how changes in the market tightness affect the expected match output. That is, we need the generalized Hosios condition.

Outline. This paper proceeds as follows. In Section 2, we present our key result: the generalized Hosios condition. We first discuss a static economy and then derive the main result for a dynamic economy with enduring matches. Section 3 provides a number of different examples of search and matching models to which the generalized Hosios condition applies.

In Section 3, we first consider some examples in which prices are determined by Nash bargaining. In these examples, the generalized Hosios condition applies but it holds only in a knife-edge special case and we do not generally have constrained efficiency. Next, we consider some examples in which prices are determined through directed or competitive search. In these examples, the generalized Hosios condition holds endogenously and we have constrained efficiency. Finally, we provide an example in which the generalized Hosios condition applies but it does not hold and therefore the economy is *not* constrained efficient – even though it features competitive search. We discuss the relevant literature throughout the paper. All proofs are found in the Appendix.

2 Generalized Hosios Condition

To build intuition, we first consider a static environment and then consider a dynamic economy with enduring matches.

2.1 Static economy

There is a fixed measure N_S of risk-neutral sellers and a large number of risk-neutral potential buyers. If N_B is the measure of risk-neutral buyers who enter, the market tightness, or buyer/seller ratio, is denoted by $\theta \equiv N_B/N_S$. Buyers and sellers are matched according to a constant-returns-to-scale matching function. The matching probabilities for sellers and buyers are denoted respectively by $m(\theta)$ and $m(\theta)/\theta$. We call the function $m(.)$ the *matching technology* and assume that it satisfies the following standard properties.

Assumption 1. *The function $m(.)$ has the following properties: (i) $m'(\theta) > 0$ and $m''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$, (ii) $\lim_{\theta\to 0} m(\theta) = 0$, (iii) $\lim_{\theta\to 0} m'(\theta) = 1$, (iv) $\lim_{\theta\to\infty} m(\theta) = 1$, (v) $\lim_{\theta\to\infty} m'(\theta) = 0$, and (vi) $m(\theta)/\theta$ is strictly decreasing in θ for all $\theta \in \mathbb{R}_+$.*

Let $y(\theta)$ denote the *expected match output.* When the expected match output is exogenous, we have $y(\theta) = \bar{y}$ for all $\theta \in \mathbb{R}_+$. In general, we allow the expected match output $y(\theta)$ to be *endogenous* in the sense that it depends directly on the market tightness. We call the function $y(.)$ the *output technology.*

There is free entry of buyers, each paying a cost $c > 0$ to enter, and $b \geq 0$ is the outside option of sellers. We assume that $y(\theta) > b$ for all $\theta \in \mathbb{R}_+$.

Suppose the social planner is constrained by both the matching technology $m(.)$ *and* the output technology $y(.)$. Taking both functions $m(.)$ and $y(.)$ as given, the social planner chooses a level of buyer entry N_B, or equivalently a market tightness θ, that maximizes the total social surplus. If there exists a unique solution, we denote it by θ^P and call it the social optimum. We say that a decentralized equilibrium allocation is *constrained efficient* if and only if $\theta^* = \theta^P$ where θ^* is the equilibrium market tightness and θ^P is the social optimum.

The social surplus per seller $\Omega(\theta)$ is

$$\Omega(\theta) = m(\theta)y(\theta) + (1 - m(\theta)b - c\theta. \tag{1}$$

The first-order condition for the social planner's problem is

$$\Omega'(\theta) = m'(\theta)y(\theta) + m(\theta)y'(\theta) - m'(\theta)b - c = 0. \tag{2}$$

We can define $f(\theta) \equiv m(\theta)y(\theta)$, the expected market output of a seller. This function is a natural extension of the matching function to environments with endogenous match output.[5] In terms of $f(\theta)$, the first-order condition is

$$\Omega'(\theta) = f'(\theta) - m'(\theta)b - c = 0. \tag{3}$$

To ensure the existence of a unique solution to the social planner's problem, we make the following assumption.

Assumption 2. *The function $f(.)$ has the following properties: (i) $\lim_{\theta\to 0} f(\theta) = 0$; (ii) $\lim_{\theta\to 0} f'(\theta) - b > c$; (iii) $\lim_{\theta\to\infty} f'(\theta) \leq 0$; and (iv) $f''(\theta) - m''(\theta)b < 0$ for all $\theta \in \mathbb{R}_+$.*

Assumptions 1 and 2 imply that $\lim_{\theta\to 0} \Omega'(\theta) > 0$, $\lim_{\theta\to\infty} \Omega'(\theta) < 0$, and $\Omega''(\theta) < 0$. Applying the intermediate value theorem, there exists a unique social optimum $\theta^P > 0$ where $\Omega'(\theta^P) = 0$. Note that if all of the conditions except (iv) are satisfied, we have existence but not uniqueness.

[5] In many of the examples we present, $f(.)$ is actually the primitive function and we define $y(\theta) \equiv f(\theta)/m(\theta)$. For example, see Sections 3.2, 3.5, 3.6, and 3.8.

The expected joint match surplus created by each match is $s(\theta) \equiv y(\theta) - b$.[6] In terms of $s(\theta)$, the first-order condition for the social planner's problem is

$$\Omega'(\theta) = m'(\theta)s(\theta) + m(\theta)s'(\theta) - c = 0. \tag{4}$$

Rearranging (4), the social planner's solution θ^P satisfies the following:

$$\frac{m'(\theta)\theta}{m(\theta)} + \frac{s'(\theta)\theta}{s(\theta)} = \frac{c\theta}{m(\theta)s(\theta)}. \tag{5}$$

Now let $\eta_m(\theta) \equiv m'(\theta)\theta/m(\theta)$, the elasticity of the matching probability $m(\theta)$ with respect to θ.[7] We call this the *matching elasticity*. Let $\eta_s(\theta) \equiv s'(\theta)\theta/s(\theta)$, the elasticity of the expected joint match surplus, $s(\theta)$. We call this the *surplus elasticity*. Substituting into (5), the social optimum θ^P solves

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{buyers' surplus share}}. \tag{6}$$

Since there is free entry of buyers, the expected payoff per buyer equals the cost of entry c and the term on the right of (6) equals buyers' total *surplus share*. Condition (6) says that the social planner chooses the market tightness θ^P that sets the buyers' surplus share equal to the matching elasticity *plus* the surplus elasticity. Since θ^P is unique, we have constrained efficiency if and only if θ^* also satisfies condition (6).

We call this the *generalized Hosios condition* because it generalizes the standard Hosios condition to static environments with both matching frictions and an expected match output that depends directly on the market tightness. When the expected match output is *exogenous*, $\eta_s(\theta^*) = 0$ and we recover the standard Hosios condition: the matching elasticity with respect to buyers must equal their surplus share. In general, if the expected match surplus depends on the market tightness, buyers' surplus share must equal the matching elasticity *plus* the surplus elasticity.

[6] Ljungqvist and Sargent (2017) call this the *fundamental surplus*.

[7] Note that $\eta_m(\theta) < 1$ follows from our assumption that $m(\theta)/\theta$ is strictly decreasing.

Since the buyers' surplus share and the sellers' surplus share add to one, the social optimum θ^P must also satisfy

$$1 - \underbrace{\eta_m(\theta)}_{\text{matching elasticity}} - \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{\pi(\theta)}{m(\theta)s(\theta)}}_{\text{sellers' surplus share}} \tag{7}$$

where $\pi(\theta)$ is the expected payoff for sellers. If there is free entry of *sellers* at cost κ instead of free entry of buyers, then substituting $\pi(\theta) = \kappa$ into the above equation delivers the generalized Hosios condition for seller entry.

Discussion. In search and matching models with free entry of buyers, there are two standard externalities related to the frictional matching process: the congestion and thick market externalities. The former is a negative externality that arises because a higher buyer/seller ratio reduces the matching probability of each buyer. The latter is a positive externality that arises because a higher buyer/seller ratio increases the matching probability of each seller. In general, these search externalities are fully captured by the standard Hosios condition through the matching elasticity term.

In environments where the expected match output depends on market tightness, a novel externality arises. Depending on the specific environment, a higher buyer/seller ratio may either increase or decrease the expected match output. We call this the *output externality* and it may be either positive or negative. Under the standard Hosios condition, buyers' entry decisions fail to internalize the output externality and entry is not constrained efficient. To ensure that entry is efficient, we need the generalized Hosios condition. When this condition is satisfied, buyers' entry decisions internalize both the search externalities *and* the output externality. The standard externalities are captured by the matching elasticity, while the output externality is reflected in the surplus elasticity.

2.2 Dynamic economy

Consider a continuous-time dynamic environment. In period t, there is measure one of risk-neutral sellers and a large number of risk-neutral potential buyers. The measure of risk-neutral buyers who enter is denoted by v_t. There is a measure u_t of unmatched sellers in period t and the market tightness is defined by $\theta_t \equiv v_t/u_t$.

There is free entry of buyers who pay a cost $c > 0$ each period. At the start of each period, buyer-seller matches are destroyed at an exogenous rate $\delta \in (0, 1]$.[8] Future payoffs are discounted at a rate $r > 0$.

In continuous time, $m(\theta_t)$ and $m(\theta_t)/\theta_t$ are now arrival rates rather than matching probabilities for buyers and sellers respectively, and thus Assumption 1 needs to be amended.

Assumption 1a. *The function* $m(.)$ *has the following properties: (i)* $m'(\theta) > 0$ *and* $m''(\theta) < 0$ *for all* $\theta \in \mathbb{R}_+$*, (ii)* $\lim_{\theta\to 0} m(\theta) = 0$*, (iii)* $\lim_{\theta\to 0} m'(\theta) = +\infty$*, (iv)* $\lim_{\theta\to +\infty} m(\theta) = +\infty$*, (v)* $\lim_{\theta\to\infty} m'(\theta) = 0$*, and (vi)* $m(\theta)/\theta$ *is strictly decreasing in* θ *for all* $\theta \in \mathbb{R}_+$*.*

The expected output for a new match *created* in period t is $y(\theta_t)$. The flow payoff for unmatched sellers is $b \geq 0$ for all t where $b < y(\theta_t)$ for all $\theta_t \in \mathbb{R}_+$. Let y_t denote the average match output across all matches during period t. Note that y_t is not equal to $y(\theta_t)$, since y_t is a weighted average across *all* active matches, i.e. both newly formed matches and existing matches that have survived from previous periods. In the Appendix, we derive the following continuous time law of motion for y_t.

$$\dot{y}_t = \underbrace{(y(\theta_t) - y_t)}_{\text{difference in expected output}} \underbrace{\frac{m(\theta_t)u_t}{1 - u_t}}_{\text{share of } new \text{ matches}} \tag{8}$$

Intuitively, $\dot{y}_t$ is equal to the difference in expected output between new and existing matches, weighted by the share of total matches that are new.

Now let Ω denote the social surplus, given by

$$\Omega = \int_0^\infty e^{-rt}((1 - u_t)y_t + bu_t - c\theta_t u_t)dt. \tag{9}$$

[8] In the special case $\delta = 1$, the economy features non-enduring matches, i.e. all matches are destroyed at the end of each period. All of the following results therefore apply to economies with non-enduring matches.

Given initial conditions u_0 and y_0, the social planner chooses θ_t for all $t \in \mathbb{R}_+$ to maximize (9) subject to the following constraints:

$$\dot{u}_t = \delta(1 - u_t) - m(\theta_t)u_t \tag{10}$$

and

$$\dot{y}_t = (y(\theta_t) - y_t)\frac{m(\theta_t)u_t}{1 - u_t}. \tag{11}$$

In the proof of Proposition 1, we solve the current value Hamiltonian for this problem. We focus on steady state solutions where $\dot{u}_t = \dot{y}_t = 0$ and $\dot{\theta}_t = 0$. Before presenting Proposition 1, we first determine the steady state expected joint match surplus $s(\theta)$ since the solution will be stated in terms of this.

Let V_S and V_B denote the steady state asset values for matched sellers and buyers respectively, and let U_S and U_B denote the steady state asset values for unmatched sellers and buyers respectively. In steady state, the expected joint match surplus is $s(\theta) \equiv V_B + V_S - U_B - U_S$. Using the Bellman equations, and the fact that $U_B = 0$ with free entry of buyers, Lemma 1 provides a useful expression for the expected match surplus $s(\theta)$ in the dynamic economy.[9]

Lemma 1. *In steady state, the expected joint match surplus $s(\theta)$ is*

$$s(\theta) = \frac{y(\theta) - b + c\theta}{r + \delta + m(\theta)}. \tag{12}$$

We are now in a position to present a necessary condition for efficiency.

Proposition 1. *Any steady state social optimum θ^P must satisfy*

$$\eta_m(\theta) + \frac{y'(\theta)\theta}{(r + \delta)s(\theta)} = \frac{c\theta}{m(\theta)s(\theta)}. \tag{13}$$

In its current form, it is unclear how to reconcile condition (13) with the intuitive condition (6) that we found in the static economy. In fact, condition (13) turns out to be *equivalent* to the generalized Hosios condition (6). Using

[9]Note that in the static economy, $V_B + V_S = y(\theta)$ and $U_S = b$, so we have $s(\theta) = y(\theta) - b$.

expression (12) for $s(\theta)$, we can write $\eta_s(\theta) \equiv s'(\theta)\theta/s(\theta)$ as the elasticity of the numerator minus the elasticity of the denominator:

$$\eta_s(\theta) = \frac{(y'(\theta)+c)\theta}{y(\theta)-b+c\theta} - \frac{m'(\theta)\theta}{r+\delta+m(\theta)}. \tag{14}$$

Using (14) and (12), it can be shown that condition (13) is equivalent to

$$\eta_m(\theta) + \eta_s(\theta) = \frac{c\theta}{m(\theta)s(\theta)} \tag{15}$$

and condition (15) is equivalent to

$$m'(\theta)s(\theta) + m(\theta)s'(\theta) = c. \tag{16}$$

We define the function $\Lambda(.)$ by $\Lambda(\theta) \equiv m(\theta)s(\theta)$. To ensure existence and uniqueness of a social optimum, we make the following assumption. Note that if all of the conditions except (iv) are satisfied, we have existence but not uniqueness.

Assumption 2a. *The function $\Lambda(.)$ has the following properties: (i) $\lim_{\theta\to 0} \Lambda(\theta) = 0$; (ii) $\lim_{\theta\to 0} \Lambda'(\theta) > c$; (iii) $\lim_{\theta\to\infty} \Lambda'(\theta) \leq 0$; and (iv) $\Lambda''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$.*

If Assumption 2a holds, it follows from the intermediate value theorem that there exists a unique θ^P that satisfies $\Lambda'(\theta) = c$, i.e. that satisfies condition (16) and therefore also the necessary condition (13). Lemma 2 follows by using Arrow's sufficiency theorem to prove that θ^P is indeed a global maximum for Ω. See the Appendix for details.

Lemma 2. *There exists a unique social optimum $\theta^P > 0$.*

Proposition 2 generalizes the standard Hosios condition to dynamic environments with both matching frictions and an endogenous match output that depends on the market tightness. To achieve constrained efficiency, buyers' surplus share must equal the *matching elasticity* plus the *surplus elasticity.* When this condition holds, buyers' entry decisions fully internalize both the standard externalities due to the matching process and the output externality. The matching elasticity captures the standard matching externalities, while the output externality is reflected in the surplus elasticity term.

Proposition 2 (**Generalized Hosios Condition**). *A steady state equilibrium allocation is constrained efficient if and only if*

$$\underbrace{\eta_m(\theta^*)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta^*)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta^*}{m(\theta^*)s(\theta^*)}}_{\text{buyers' surplus share}}. \tag{17}$$

Depending on the specific environment, the surplus elasticity may be either positive or negative. This means that simply applying the standard Hosios condition may result in either over-entry or under-entry of buyers relative to the social optimum. Corollary 1 tells us that the direction of the inefficiency depends *only* on the output technology $y(\theta)$. In particular, the direction of the inefficiency depends on whether the expected match output $y(\theta)$ is increasing or decreasing in the buyer/seller ratio at the equilibrium θ^*.

Corollary 1. *There is under-entry (over-entry) of buyers under the standard Hosios condition if and only if* $y'(\theta^*) > (<)\ 0$.

When $y'(\theta^*) > 0$, the output externality arising from buyer entry is *positive* and the standard Hosios condition results in under-entry. Alternatively, if $y'(\theta^*) < 0$, the output externality from buyer entry is *negative* and the standard Hosios condition results in over-entry. If $y'(\theta^*) = 0$, there is no output externality and entry is constrained efficient under the standard Hosios condition.

When there is seller entry instead of buyer entry, the direction of the effect of entry is reversed. If there is free entry of *sellers* at cost κ, the analogue of condition (17) is

$$1 - \underbrace{\eta_m(\theta^*)}_{\text{matching elasticity}} - \underbrace{\eta_s(\theta^*)}_{\text{surplus elasticity}} = \underbrace{\frac{\kappa}{m(\theta^*)s(\theta^*)}}_{\text{sellers' surplus share}}. \tag{18}$$

Corollary 2. *There is over-entry (under-entry) of sellers under the standard Hosios condition if and only if* $y'(\theta^*) > (<)\ 0$.

When $y'(\theta^*) > 0$, the output externality arising from seller entry is *negative* since $\theta = N_B/N_S$ and therefore $y(.)$ is decreasing in the measure of sellers N_S. In this case, the standard Hosios condition results in over-entry. If $y'(\theta^*) < 0$, the output externality from seller entry is *positive* and the standard Hosios condition results in under-entry. If $y'(\theta^*) = 0$, there is no output externality and entry is constrained efficient under the standard Hosios condition.

3 Examples

In this section, we discuss a number of examples of different search and matching environments to illustrate the wide applicability of the generalized Hosios condition. We focus mainly on labor market environments in which sellers and buyers are unemployed workers and firms (or vacancies), but the ideas apply more generally to any kinds of buyers and sellers. For simplicity, we focus mainly on static environments for these examples but the efficiency results extend to dynamic environments as shown in Section 2.

First, we discuss examples where prices are determined by Nash bargaining, and then later we consider directed and competitive search. With Nash bargaining, the generalized Hosios condition typically holds only in a knife-edge special case. With directed or competitive search, the generalized Hosios condition typically holds *endogenously*, i.e. entry is always constrained efficient. We do provide one example, however, where the generalized Hosios condition applies but it does not hold endogenously and hence the economy is not constrained efficient, even though prices are determined by competitive search.

3.1 Nash bargaining with endogenous match output

For now, we simply *assume* that the expected match output is a function of market tightness. In the following sections, we will see how dependence of the expected match output on market tightness can arise naturally through either the *composition* channel or the *selection* channel.

Consider a static Diamond-Mortensen-Pissarides (DMP) style environment where meetings are bilateral and wages are determined by Nash bargaining.[10] The measure of vacancies or firms is V, the measure of unemployed workers is U, and the labor market tightness is $\theta \equiv V/U$. While the environment is otherwise standard, we assume that the expected output per match depends directly on the market tightness θ.

There is free entry of firms (or vacancies) at a cost $c > 0$. The matching probabilities for workers and firms are $m(\theta)$ and $m(\theta)/\theta$ respectively where $m(.)$ satisfies Assumption 1. Workers' bargaining parameter is β and the value of non-market activity is b where $y(\theta) > b$ for all $\theta \in \mathbb{R}_+$.

The expected match surplus is $s(\theta) = y(\theta) - b$. Under certain conditions,[11] there exists a unique equilibrium $\theta^* > 0$ that satisfies

$$\frac{m(\theta)}{\theta}(1-\beta)(y(\theta)-b) = c \tag{19}$$

or equivalently, the equilibrium $\theta^* > 0$ satisfies

$$1-\beta = \frac{c\theta}{m(\theta)s(\theta)}. \tag{20}$$

If Assumption 2 is satisfied, there exists a unique social optimum $\theta^P > 0$. Applying the generalized Hosios condition in Proposition 2, and using (20), the economy is constrained efficient if and only if θ^* satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{1-\beta}_{\text{firms' bargaining power}}. \tag{21}$$

Intuitively, the economy is efficient only when firms are paid for their contribution to both the *number* of matches and the *value* of the expected match surplus. Corollary 1 says that the standard Hosios condition may result in either *under-entry* or *over-entry* of firms, depending on whether the expected match output is increasing or decreasing in the market tightness, i.e. whether

[10] The classic references are Mortensen and Pissarides (1994) and Pissarides (2000).

[11] Specifically, let $\Lambda(\theta) = m(\theta)s(\theta)$. If $\eta_\Lambda(\theta) < 1$ for all $\theta \in \mathbb{R}_+$, $\lim_{\theta\to\infty} \Lambda(\theta)/\theta = 0$, and $c < (1-\beta)\lim_{\theta\to 0} \Lambda(\theta)/\theta$, then there exists a unique equilibrium $\theta^* > 0$. Note that $\eta_\Lambda(\theta) < 1$ if and only if $\Lambda(\theta)/\theta$ is strictly decreasing.

the output externality is positive or negative, i.e. $y'(\theta^*) > 0$ or $y'(\theta^*) < 0$.

Example 3.1.1

Consider the special case where the expected match output is *exogenous*, i.e. $y(\theta) = \bar{y}$ for all $\theta \in \mathbb{R}_+$. Output per match may be either constant or stochastic provided that the *expected* match output does not depend on the market tightness θ. For example, we could have one-on-one or bilateral meetings and match-specific productivities y drawn from an exogenous distribution F with $E_F(y) = \bar{y}$. According to (21), we have constrained efficiency only when the equilibrium θ^* satisfies the following well-known condition:

$$\eta_m(\theta) = 1 - \beta. \tag{22}$$

Clearly, the standard Hosios condition is a special case of (21). When the expected match output is exogenous, we have constrained efficiency only when the matching elasticity $\eta_m(\theta)$ equals firms' bargaining power at the equilibrium θ^*. As is well-known, constrained efficiency obtains only in a knife-edge case.

Example 3.1.2

Suppose that match output is given by $y(\theta) = A\theta^\gamma$ and $\gamma \in [0, 1)$, $b = 0$, and $\eta_m(\theta) = \eta$ where $\eta + \gamma < 1$. Clearly, the expected match output $y(\theta)$ is increasing in the market tightness and $\eta_s(\theta) = \gamma$. In this case, the generalized Hosios condition (21) is particularly simple and we have constrained efficiency if and only if

$$\eta + \gamma = 1 - \beta. \tag{23}$$

In general, there is no reason why this condition would hold since both the matching elasticity η *and* the surplus elasticity γ are independent of both the Nash bargaining parameter β and each other. To ensure constrained efficiency, we must impose the generalized Hosios condition by setting firms' bargaining power $1 - \beta$ equal to $\eta + \gamma$. Under the standard Hosios condition, firm entry will be inefficiently low because it sets the bargaining power β of workers too high. Since $y'(\theta^*) > 0$, this is consistent with Corollary 1.

3.2 Nash bargaining with ex post firm heterogeneity

In this example, unlike the previous examples, both the fact that the expected match output $y(\theta)$ depends directly on the market tightness θ, and the specific properties of the function $y(.)$, are not assumptions: the function $y(.)$ and its properties arise *endogenously*.

When there are multilateral or many-on-one meetings (e.g. many buyers meet one seller), dependence of the expected match output on market tightness can arise when there is *ex post* heterogeneity (i.e. after meetings but before trade) on the "many" side of the market.[12] For example, when a seller can choose one buyer among many heterogeneous buyers in a meeting, a greater number of buyers per seller means that sellers can be more *selective*, thereby increasing the expected match output. We call this the *selection channel*.

Consider a simple model with identical workers and ex ante identical firms. There is free entry of firms at a cost $c > 0$. The measure of vacancies or firms is V, the measure of unemployed workers is U, and the labor market tightness is $\theta \equiv V/U$. After entering, firms draw a productivity. The probability that a firm is low productivity is α and the probability that a firm is high productivity is $1 - \alpha$. Low productivity firms produce output x_L and high productivity firms produce output x_H.

Unemployed workers and firms are matched according to a Poisson meeting technology where $P_n(\theta) = \frac{\theta^n e^{-\theta}}{n!}$ is the probability that n firms approach a given worker. The matching probabilities for workers and firms are $m(\theta) = 1-e^{-\theta}$ and $m(\theta)/\theta$ respectively, where $m(.)$ satisfies Assumption 1. After firms approach, each worker observes the productivity of each firm that approaches and chooses to work for one of them. For simplicity, workers' value of non-market activity is $b = 0$.

If wages are determined by Nash bargaining with workers' bargaining power β, then $w_H = \beta x_H$ for high productivity firms and $w_L = \beta x_L$ for low produc-

[12] The heterogeneity among buyers must be of a kind that affects the match "output".

tivity firms. Workers always choose to work for the highest productivity firm that approaches. Since workers are only hired by low productivity firms if *all* of the n firms approaching are low productivity, the probability of a worker being hired by a low productivity firm, conditional on being hired, is

$$\Pr(x = x_L) = \frac{\sum_{n=1}^{\infty} P_n(\theta)\alpha^n}{1 - P_0(\theta)} = \frac{e^{-\theta(1-\alpha)} - e^{-\theta}}{1 - e^{-\theta}} \tag{24}$$

and the probability of being hired by a high productivity firm is

$$\Pr(x = x_H) = \frac{1 - e^{-\theta(1-\alpha)}}{1 - e^{-\theta}}. \tag{25}$$

The expected market output per worker $f(\theta) = m(\theta)y(\theta)$ is

$$f(\theta) = (1 - e^{-\theta(1-\alpha)})x_H + (e^{-\theta(1-\alpha)} - e^{-\theta})x_L \tag{26}$$

and the expected match output, or average labor productivity, is

$$y(\theta) = \frac{(1 - e^{-\theta(1-\alpha)})x_H + (e^{-\theta(1-\alpha)} - e^{-\theta})x_L}{1 - e^{-\theta}}. \tag{27}$$

The expected match surplus is $s(\theta) = y(\theta)$ since $b = 0$. It is straightforward to show that the expected match output is increasing in the market tightness, i.e. $y'(\theta) > 0$. Intuitively, this is because a greater number of firms per worker allows workers to be more *selective* in multilateral meetings.

The equilibrium market tightness θ^* is determined by the zero profit condition (19) where $b = 0$ and $y(\theta)$ is given by (27). There exists a unique social optimum $\theta^P > 0$ if Assumption 2 is satisfied. Applying the generalized Hosios condition in Proposition 2, the economy is constrained efficient if and only if θ^* satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{1 - \beta}_{\text{firms' bargaining power}}. \tag{28}$$

Since $y'(\theta^*) > 0$ in this environment, Corollary 1 tells us that applying the standard Hosios rule would result in *under-entry* of firms or, equivalently, inefficiently high unemployment. Vacancy creation is lower than socially optimal under the standard Hosios condition because workers' bargaining power β is too high: it does not incorporate the fact that greater entry of firms leads not only to lower unemployment for workers, but also a higher average labor productivity. Under the generalized Hosios condition, however, firms' entry decisions internalize both the effect on unemployment and the positive output externality that arises here. Since wages are determined by Nash bargaining, this condition holds only in a knife-edge special case.

3.3 Nash bargaining with ex ante firm heterogeneity

When there is *ex ante* heterogeneity among buyers or sellers, dependence of the expected match output on market tightness can arise naturally through market composition. If the market tightness influences the individual entry decisions of buyers or sellers that are ex ante heterogeneous with respect to characteristics that affect match output, then average output per match will depend on market tightness. We call this the *composition channel*.

Albrecht, Navarro, and Vroman (2010) consider an environment where workers are ex ante heterogeneous with respect to their market productivity and there is both firm entry and a labor force participation decision.[13] The authors show that such an environment can violate the standard Hosios rule: when workers' bargaining parameter satisfies the standard Hosios condition, there is *over-entry* of firms relative to the social optimum. To illustrate the use of the generalized Hosios condition, we consider a related but simpler environment that features ex ante *firm* heterogeneity instead of worker heterogeneity.[14]

Suppose there is a measure U of unemployed workers and a measure M of firms that may choose to search. Firms' productivities y are distributed according to a twice differentiable distribution with cdf G and density g where

[13] Related literature following Albrecht et al. (2010) includes Gavrel (2011), Charlot, Malherbet, and Ulus (2013), and Masters (2015). See also Albrecht, Navarro, and Vroman (2009).

[14] Julien and Mangin (2017) show that the environment in Albrecht et al. (2010) with labor force participation features both an output externality and a *participation externality*, which makes it more complicated than this example with firm entry.

$G(0) = 0$ and $g(y) > 0$ for all $y \in [0, 1]$. Firms learn their own productivity before deciding whether to pay the entry cost $c > 0$ and search. Wages are determined by generalized Nash bargaining where workers' bargaining parameter is β and the value of non-market activity is zero. We assume that $c < 1 - \beta$.

Let V be the measure of *searching* firms and define $\theta = V/U$. Meetings are bilateral and the probabilities of matching for workers and firms are $m(\theta)$ and $m(\theta)/\theta$ respectively. A firm with productivity y chooses to pay the cost c to search for a worker if and only if

$$\frac{m(\theta)}{\theta}(1-\beta)y > c \tag{29}$$

and therefore the cut-off productivity for firm entry is

$$y^* = \frac{c\theta}{(1-\beta)m(\theta)} \tag{30}$$

and average labor productivity is $y(\theta) = E(y|y \geq y^*)$, which is given by

$$y(\theta) = \int_{y^*}^{1} \frac{yg(y)}{1 - G(y^*)} dy. \tag{31}$$

The cut-off productivity y^* is increasing in θ since $m(\theta)/\theta$ is decreasing. This is intuitive: as the market tightness increases, the probability of finding a worker is lower so only high productivity firms choose to pay the cost c and search. At the same time, the average match output $y(\theta)$ is increasing in the cut-off productivity y^* and therefore $y'(\theta) > 0$ for all $\theta \in \mathbb{R}_+$.

The equilibrium θ^* satisfies

$$\theta = (1 - G(y^*))\frac{M}{U} \tag{32}$$

where y^* is given by (30). Defining $R(\theta) \equiv 1 - G(y^*)$, the proportion of firms that choose to search, the equilibrium condition (32) is equivalent to

$$\frac{R(\theta)}{\theta} = \frac{U}{M}. \tag{33}$$

Using (30) and Assumption 1, we have $\lim_{\theta \to 0} R(\theta)/\theta = \infty$ and $\lim_{\theta \to \infty} R(\theta)/\theta = 0$. Also, $R'(\theta) < 0$ and therefore there exists a unique equilibrium $\theta^* > 0$.

The expected match surplus is $s(\theta) = y(\theta)$ since $b = 0$. If Assumption 2 is satisfied, there exists a unique social optimum θ^P and we can apply the generalized Hosios condition in Proposition 2. We have constrained efficiency if and only if θ^* satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{firms' surplus share}}. \tag{34}$$

Unlike the previous example, firms' surplus share $c\theta/m(\theta)s(\theta)$ does not equal $1-\beta$ here. Instead, using (30), the right-hand side of (34) equals $(1-\beta)y^*/y(\theta)$.

In this environment, the *composition channel* gives rise to endogenous match output that depends on the market tightness. The threshold y^* is increasing in θ, leading to a positive output externality from firm entry, i.e. $y'(\theta) > 0$. Since $y'(\theta^*) > 0$, Corollary 1 implies that there is under-entry of firms under the standard Hosios condition. While it has been known at least since Shimer and Smith (2001) that the standard Hosios condition does not apply in environments with *ex ante* heterogeneity, Proposition 2 provides us with a generalized version of the Hosios condition that does apply here.

3.4 Competitive search with endogenous match output

Unlike DMP style models with generalized Nash bargaining, models with directed or competitive search are typically constrained efficient (Shimer (1996); Moen (1997)). In such models, firms internalize the search externalities arising from the matching process and the standard Hosios condition typically holds *endogenously*.[15] Early papers on directed or competitive search include Montgomery (1991), Peters (1991), Acemoglu and Shimer (1999b,a), Julien, Kennes,

[15] Guerrieri (2008) develops a dynamic competitive search model with informational asymmetries and identifies a new externality that means the decentralized equilibrium is not always constrained efficient. Guerrieri, Shimer, and Wright (2010), Moen and Rosen (2011), and Julien and Roger (2015) also consider competitive search with informational frictions.

and King (2000), Burdett, Shi, and Wright (2001), Shi (2001, 2002).[16] For a detailed survey, see Wright et al. (2017).

Consider a simple competitive search model in the spirit of Moen (1997). There is a continuum of submarkets indexed by $i \in [0,1]$ and free entry of firms at a cost $c > 0$. Workers in submarket i post the same wage w_i and face the same market tightness θ_i, the ratio of firms to workers in that submarket. Firms' search is *directed* by observing the posted wages and deciding which submarkets to enter. Within each submarket i, the matching probabilities for workers and firms are $m(\theta_i)$ and $m(\theta_i)/\theta_i$ respectively, where $m(.)$ satisfies Assumption 1.

Suppose that the expected match output $y(\theta_i)$ in submarket i depends on the market tightness θ_i in that submarket. The value of non-market activity is b where $y(\theta_i) > b$ for all $\theta_i \in \mathbb{R}_+$. The expected match surplus in submarket i is $s(\theta_i) = y(\theta_i) - b$ and $f(\theta_i) = m(\theta_i)y(\theta_i)$ where $f(.)$ satisfies Assumption 2.

In competitive search models where the match surplus is exogenous, agents simply trade off prices against the probability of matching. Here, agents trade off prices against both the probability of matching *and* the expected match surplus. The fact that agents can do so is what delivers constrained efficiency.

The expected payoff for firms operating in submarket i with wage w_i and market tightness θ_i is given by

$$\Pi(\theta_i, w_i) = \frac{m(\theta_i)}{\theta_i}(y(\theta_i) - w_i), \tag{35}$$

and the expected payoff for workers in submarket i with market tightness θ_i is

$$V(\theta_i, w_i) = m(\theta_i)w_i + (1 - m(\theta_i))b. \tag{36}$$

Workers in submarket i choose a wage w_i^* and market tightness θ_i^* that solve

$$\max_{w_i, \theta_i \in \mathbb{R}_+} (m(\theta_i)w_i + (1 - m(\theta_i))b) \tag{37}$$

subject to $\Pi(\theta_i, w_i) \leq c$ and $\theta_i \geq 0$ with complementary slackness. To induce

[16] Hosios (1990) also considers an example based on Peters (1984) that is similar to directed search and is constrained efficient.

participation by firms in submarket i, i.e. $\theta_i > 0$, the constraint $\Pi(\theta_i, w_i) \leq c$ is binding and we have

$$\frac{m(\theta_i)}{\theta_i}(y(\theta_i) - w_i) = c. \tag{38}$$

Solving for w_i as a function of θ_i using (38), we obtain

$$w(\theta_i) = y(\theta_i) - \frac{c\theta_i}{m(\theta_i)}. \tag{39}$$

Choosing a wage w_i^* is thus equivalent to choosing a market tightness θ_i^* where

$$\theta_i^* = \arg\max_{\theta_i \in \mathbb{R}_+} \left(m(\theta_i)w(\theta_i) + (1 - m(\theta_i))b\right) \tag{40}$$

and using (39), this is equivalent to

$$\theta_i^* = \arg\max_{\theta_i \in \mathbb{R}_+} \left(m(\theta_i)y(\theta_i) + (1 - m(\theta_i))b - c\theta_i\right) \tag{41}$$

and θ_i^* is the unique solution to the first-order condition

$$m'(\theta_i)s(\theta_i) + m(\theta_i)s'(\theta_i) = c, \tag{42}$$

or equivalently, θ_i^* solves

$$\underbrace{\eta_m(\theta_i)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta_i)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta_i}{m(\theta_i)s(\theta_i)}}_{\text{firms' surplus share}}. \tag{43}$$

That is, the generalized Hosios condition holds *within each active submarket i*.

If we consider a symmetric equilibrium in which firms are indifferent across submarkets and all workers post the same wage, then $\theta_i^* = \theta^*$ for all submarkets i and Proposition 2 tells us that firm entry is constrained efficient.[17]

In this example, as in Section 3.1, we have simply assumed an arbitrary

[17] While we consider only a static model in this example, the generalized Hosios condition also holds endogenously if we consider a dynamic competitive search model with endogenous match output. Details available on request.

output technology $y(.)$. The output externality from buyer entry may be either positive or negative and the standard Hosios condition may result in either under-entry or over-entry, depending on the specific environment. In the following section, we present a competitive search model in which the output technology $y(.)$ arises through the selection channel.

3.5 Competing auctions with ex post heterogeneity

In a competing auctions environment, a large number of sellers compete to attract buyers by posting auctions. Following the seminal work of Peters and Severinov (1997), recent papers that use competing auctions include Albrecht, Gautier, and Vroman (2012, 2014, 2016); Kim and Kircher (2015); Lester, Visschers, and Wolthoff (2015); and Mangin (2017). Competing auctions models with buyer heterogeneity are essentially competitive search models that feature both private information and endogenous expected match output.

Unlike the previous example, both the fact that the expected match output $y(\theta)$ depends directly on the market tightness θ, and the specific properties of the function $y(.)$, are not assumptions but instead arise *endogenously.* As in Section 3.2, the fact that meetings are many-on-one or multilateral is crucial: such meetings give rise to the possibility of choice among potential trading partners. Through the auction mechanism, sellers "select" the buyer with the highest valuation and thus the *selection channel* endogenizes the expected match output $y(\theta)$. As a result, we cannot simply apply the Hosios rule in its traditional form. Instead, the constrained efficiency of buyer entry in competing auctions environments is a direct application of the generalized Hosios condition.

Consider the labor market environment in Mangin (2017). Workers are identical sellers who auction their labor using second-price auctions and post reservation wages to attract firms. Firms are ex ante identical buyers who pay a cost $c > 0$ to enter and search for workers. The labor market tightness is $\theta \equiv V/U$, the ratio of vacancies or firms to unemployed workers. The meeting technology is Poisson and $P_n(\theta) = \frac{\theta^n e^{-\theta}}{n!}$ is the probability that n firms approach a given worker. The matching probability for workers is $m(\theta) = 1 - e^{-\theta}$, which satisfies Assumption 1.

Firms' valuations y of workers' labor are match-specific productivity draws that are private information. Valuations are drawn *ex post* (i.e. after meetings) independently from a distribution with cdf G that is twice differentiable with density $g = G' > 0$, a finite mean, and support $[y_0, \infty)$ where $y_0 \geq 0$.

Let $w(\theta; r)$ be the expected wage when a worker posts reservation wage r. For any given reservation wage $r \in \mathbb{R}^+$, the market tightness $\theta^*(r)$ must satisfy

$$\frac{m(\theta)}{\theta}(y(\theta) - w(\theta; r)) \leq c \tag{44}$$

and $\theta^*(r) \geq 0$, with complementary slackness. Workers' reservation wage r^* maximizes their expected payoff, anticipating the effect on firm entry:

$$r^* = \arg \max_{r \in [0,\infty)} \ (m(\theta^*(r))w(\theta^*(r); r) + (1 - m(\theta^*(r)))b). \tag{45}$$

We consider symmetric equilibria where workers post the same reservation wage. If $c < E_G(y) - b$, there exists a unique equilibrium function $\theta^*(.)$ where $\theta^* \equiv \theta^*(r^*) > 0$ and workers' reservation wage r^* equals the value of non-market activity, $b \in [0, y_0]$.[18] The equilibrium $\theta^* \in \mathbb{R}_+$ satisfies

$$\int_{y_0}^{\infty} e^{-\theta(1-G(y))}(1 - G(y))dy + e^{-\theta}(y_0 - b) = c \tag{46}$$

and the expected *output per worker* $f(\theta)$ is given by

$$f(\theta) = \int_{y_0}^{\infty} \theta e^{-\theta(1-G(y))} y dG(y). \tag{47}$$

The properties of $f(.)$ are summarized in Proposition 1 of Mangin (2017) and the expected output per match $y(\theta)$ is defined as $y(\theta) \equiv f(\theta)/m(\theta)$.

To establish constrained efficiency, it is easier to work directly with $f(\theta)$. Let $\eta_f(\theta) \equiv f'(\theta)\theta/f(\theta)$, the elasticity of $f(\theta)$ with respect to θ. It is straightforward

[18] In a labor market setting, it is important that we relax the standard "no gap" assumption found in Peters and Severinov (1997) and Albrecht et al. (2014) by allowing $b < y_0$ (i.e. sellers' valuation can be strictly less than the minimum buyers' valuation). This enables us to nest the directed search model in Example 3.5.1 as a special case.

to show that the generalized Hosios condition is equivalent to[19]

$$\eta_f(\theta) = \frac{c\theta}{f(\theta)} + \frac{\eta_m(\theta)b}{y(\theta)}. \tag{48}$$

Differentiating (47) and simplifying yields

$$f'(\theta) = \int_{y_0}^{\infty} e^{-\theta(1-G(y))}(1-G(y))dy + y_0 e^{-\theta}. \tag{49}$$

Since $f''(\theta) - bm''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$, if $c < E_G(y) - b$ then Assumption 2 is satisfied and there exists a unique social optimum $\theta^P > 0$. Using (49) and (46), condition (48) clearly holds at θ^* and we therefore have constrained efficiency.

Mangin (2017) proves that $y'(\theta) > 0$ for all $\theta \in \mathbb{R}_+$ if G is *well-behaved,* i.e. if it satisfies a mild regularity condition that is satisfied by almost all standard distributions. Corollary 1 therefore implies that the standard Hosios condition would result in *under-entry* of firms, since $y'(\theta^*) > 0$ and the output externality is positive. Instead, the generalized Hosios condition is required in order for constrained efficiency to hold. The auction mechanism ensures this condition holds endogenously since firms are compensated for the effect of firm entry on both employment *and* average labor productivity $y(\theta)$.

In this environment, the *selection channel* gives rise to a positive output externality from firm entry because the auction mechanism selects the most productive firm at each meeting. If firms were simply chosen at random, the selection channel would be shut down and the expected match output would not depend on the market tightness. In this way, the nature of the output technology $y(.)$, which transforms the market tightness into expected match output, depends on features of the decentralized market which the social planner takes as given when determining the efficient level of firm entry.

[19] Using the fact that $s(\theta) = y(\theta) - b$, we can express the generalized Hosios condition in terms of $\eta_y(\theta) \equiv y'(\theta)\theta/y(\theta)$, and since $y(\theta) = f(\theta)/m(\theta)$, we have $\eta_y(\theta) = \eta_f(\theta) - \eta_m(\theta)$. Substituting into (17) and simplifying yields (48).

Example 3.5.1

If G is degenerate, we recover the large economy version of the directed search model found in Julien et al. (2000) where workers post second-price auctions. All firms have the same productivity $y_0 = \bar{y}$ and pay a cost $c > 0$ to search. The matching probability for workers is $m(\theta) = 1 - e^{-\theta}$. In equilibrium, workers set reserve prices equal to their outside option b and there exists a unique equilibrium market tightness θ^*. If $c < \bar{y} - b$, then $\theta^* > 0$ satisfies

$$e^{-\theta}(\bar{y} - b) = c. \tag{50}$$

We can easily recover the constrained efficiency of directed search models such as Julien et al. (2000) by applying condition (17). In this case, it is just the standard Hosios condition: entry is efficient if and only if θ^* satisfies

$$\frac{\theta e^{-\theta}}{1 - e^{-\theta}} = \frac{c\theta}{m(\theta)s(\theta)} \tag{51}$$

since $\eta_m(\theta) = \theta e^{-\theta}/(1 - e^{-\theta})$. Substituting $s(\theta) = \bar{y} - b$ into (51) and rearranging, we have constrained efficiency since θ^* satisfies (50).

Example 3.5.2

Suppose that G is a Pareto distribution, $G(y) = 1 - y^{-1/\lambda}$ for $y \in [1, \infty)$ and $\lambda \in (0, 1)$. To enter and search for a worker, firms must hire one unit of capital at cost $c > 0$. For simplicity, let $b = 0$. The parameter λ is a measure of the fatness of the tails of the distribution G. A higher λ implies fatter tails.

Mangin (2017) shows that a "frictionless" limit of this economy delivers a familiar benchmark: a Cobb-Douglas aggregate production function with constant factor shares. In general, we obtain an aggregate production function that directly incorporates matching frictions. Letting $f(\theta)$ be output per capita,

$$f(\theta) = \theta^{\lambda}\gamma(1 - \lambda, \theta) \tag{52}$$

where $\gamma(1-\lambda,\theta) \equiv \int_0^\theta t^{-\lambda}e^{-t}dt$. Expected match output is $y(\theta) \equiv f(\theta)/m(\theta)$. Observe that $y(\theta) \sim A\theta^\lambda$ where $A = \Gamma(1-\lambda)$ in the limit as $\theta \to \infty$.

Substituting into (46), the equilibrium $\theta^* > 0$ satisfies

$$\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta} = c \tag{53}$$

if $c < 1/(1-\lambda)$. Since G is well-behaved, $y'(\theta) > 0$ for all $\theta \in \mathbb{R}_+$. Using the fact that $b = 0$ and $f'(\theta) = \lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}$, we have $f'(\theta^*) = c$ and it is easy to see that (48) holds and we therefore have constrained efficiency.

3.6 Competing auctions with seller entry

Albrecht et al. (2014) examines the efficiency of *seller* entry in a competing auctions environment. The authors consider both ex ante and ex post buyer heterogeneity, as well as seller heterogeneity, and they prove that seller entry is always constrained efficient. Although they do not explicitly identify it, the generalized Hosios condition applies in their setting and it is the fact that this condition holds endogenously that ensures constrained efficiency.

Consider a simple version of their model with homogeneous sellers, each with reservation value $b = 0$, and buyers who are ex ante identical but heterogeneous ex post. Sellers pay a cost κ to enter and they attract buyers by posting second-price auctions with reserve prices. The buyer-seller ratio is $\theta \equiv N_B/N_S$. The meeting technology is Poisson and $P_n(\theta) = \frac{\theta^n e^{-\theta}}{n!}$ is the probability that n buyers approach a given seller. The matching probability for sellers is $m(\theta) = 1 - e^{-\theta}$.

Buyers' valuations y are private information and they are drawn *ex post* (i.e. after meetings) independently from a distribution with cdf G that is twice differentiable with density $g = G' > 0$, a finite mean, and support $[0, 1]$.

In Albrecht et al. (2014), the social planner maximizes the total social surplus

$$\Lambda(\theta)N_S - \kappa N_S \tag{54}$$

where $\Lambda(\theta)$ is the expected surplus per seller. The social surplus per *buyer* is

$$\Omega_B(\theta) = \frac{\Lambda(\theta)}{\theta} - \frac{\kappa}{\theta} \tag{55}$$

and the first-order condition for the planner's problem is

$$\Omega_B'(\theta) = \frac{\Lambda'(\theta)}{\theta} - \frac{\Lambda(\theta)}{\theta^2} + \frac{\kappa}{\theta^2} = 0. \tag{56}$$

Rearranging, the social optimum θ^P satisfies

$$1 - \frac{\Lambda'(\theta)\theta}{\Lambda(\theta)} = \frac{\kappa}{\Lambda(\theta)}. \tag{57}$$

Now, the surplus per seller $\Lambda(\theta)$ in Albrecht et al. (2014) is equal to $\Lambda(\theta) = m(\theta)s(\theta)$.[20] Therefore, the elasticity $\eta_\Lambda(\theta) \equiv \Lambda'(\theta)\theta/\Lambda(\theta)$ is given by $\eta_\Lambda(\theta) = \eta_m(\theta) + \eta_s(\theta)$ and equation (57) says that θ^P satisfies

$$1 - \eta_m(\theta) - \eta_s(\theta) = \frac{\kappa}{m(\theta)s(\theta)}. \tag{58}$$

We therefore have constrained efficiency if and only if θ^* satisfies (58), which is exactly the generalized Hosios condition for seller entry given by (18). It is the fact that this condition holds endogenously in this environment that ensures constrained efficiency.

The output externality that arises in Section 3.5 also appears in Albrecht et al. (2014) due to the selection channel. Through the auction mechanism, sellers choose to trade with the buyer who has the highest valuation. From Section 3.5, we know that $y'(\theta) > 0$ if G is well-behaved.[21] Importantly, this is a *negative* externality with regard to seller entry since $\theta = N_B/N_S$ and thus $y(.)$ is decreasing in the number of sellers. When there is a fixed number of buyers, more seller entry implies less buyers for each seller, thereby reducing the power of the selection channel. Applying Corollary 2, the standard Hosios condition

[20] Since $b = 0$ in this example, $\Lambda(\theta) = f(\theta)$ as given by equation (47) in the previous example, provided it is adjusted so that the support of G is $[0, 1]$ as in Albrecht et al. (2014).

[21] We have $s(\theta) = \Lambda(\theta)/m(\theta)$ and $s(\theta) = y(\theta)$ if $b = 0$, so $y(\theta) = \Lambda(\theta)/m(\theta)$.

would result in *over-entry* of sellers due to this negative output externality.

Albrecht et al. (2014) considers a negative externality from seller entry called the "business-stealing" externality. When an additional seller enters, the seller "steals" potential buyers from existing sellers, thereby reducing the expected surplus for those sellers. This is reflected in the fact that $\Lambda'(\theta) > 0$ and thus $\Lambda(.)$ is decreasing in N_S. As Albrecht et al. (2014) write, one might expect the "business-stealing" effect would lead to over-entry of sellers relative to the social optimum; however, this is exactly offset by the "informational rents" that buyers extract from sellers through the auction mechanism, thus delivering constrained efficiency. In fact, since $\Lambda(\theta) = m(\theta)s(\theta)$, the "business-stealing" effect can be decomposed into two effects: the effect on sellers' matching probability $m(\theta)$, and the effect on the expected match surplus $s(\theta)$. Both of these effects are clearly reflected in the generalized Hosios condition for seller entry (58) through the matching elasticity $\eta_m(\theta)$ and the surplus elasticity $\eta_s(\theta)$.

3.7 Applicant ranking and interviews

In environments with competing auctions, the expected match output $y(\theta)$ depends on the market tightness. However, private information is not necessary for this feature: what is essential is that agents on the "many" side of the market participate in many-on-one or multilateral meetings in which agents on the other side of the market can *choose* with whom to trade. It is the possibility of choice among trading partners that gives rise to the *selection channel*.

Consider an environment where multiple workers can apply for the same job. If there are many-on-one meetings and heterogeneous applicants for the same vacancies, the generalized Hosios condition will typically be required for constrained efficiency whenever there is a non-random selection mechanism. The method of selecting applicants may be *imperfect* (i.e. the "best" applicant may not always be chosen), provided that the expected match output is increasing in the number of applicants per vacancy (i.e. decreasing in θ).

Gavrel (2012) develops a model of applicant ranking. There is free entry of firms or vacancies. Workers apply to firms and firms rank applicants according to the degree of (match-specific) *mismatch* between the worker and the firm.

The worker with the least "mismatch" is hired by the firm. As in Marimon and Zilibotti (1999), the degree of mismatch x is measured by the distance on a circle between a worker and a firm. Let $y(x)$ be the match output given x where $y'(x) < 0$. The expected match output is

$$y(\theta) = \int_0^{1/2} y(x)\rho(x,\theta)dx \tag{59}$$

where $\theta \equiv V/U$, the ratio of firms to unemployed workers, and $\rho(x,\theta)$ is the density of mismatch among filled vacancies.

In this environment, the selection channel gives rise to a *negative* output externality from firm entry via the applicant ranking mechanism. Gavrel (2012) proves that $y'(\theta) < 0$ for all $\theta \in \mathbb{R}_+$. Intuitively, a greater number of firms per unemployed worker (higher θ) implies *fewer* applicants per vacancy (lower $1/\theta$), which increases the expected degree of mismatch between the best applicant and the firm, and thereby lowers output per match. As θ increases, firms are less selective and the greater resulting mismatch between workers and firms reduces the expected match output.

Gavrel's key result is that the presence of applicant ranking leads to an *over-entry* of vacancies (i.e. job creation is inefficiently high) when wages are determined by Nash bargaining and the standard Hosios condition is imposed.[22] That is, the unemployment rate is inefficiently *low* under the standard Hosios condition. Since $y'(\theta^*) < 0$, the fact that there is over-entry of firms under the standard Hosios condition is consistent with Corollary 1. To obtain constrained efficiency, what is needed is the generalized Hosios condition.

One example of an "imperfect" selection method is the use of *interviews.* Wolthoff (2017) develops a detailed search model of the labor market in which firms' recruitment intensity is endogenous and this determines the maximum number of interviews per vacancy. Here, we simply present the basic idea of interviews to show how they can give rise to the selection channel and therefore to dependence of the expected match output on the market tightness, i.e. $y(\theta)$.

[22] On the other hand, Gavrel shows that competitive search through wage posting à la Moen (1997) restores constrained efficiency.

Suppose that workers are ex ante heterogeneous with respect to productivity and firms are identical. Workers randomly apply for one job and firms can receive many applications from different workers. Firms cannot directly observe workers' productivities so interviews are necessary to reveal an applicant's productivity. A maximum of $n_R \geq 1$ applicants can be interviewed. If a firm receives $n \geq 1$ applications, they randomly choose $n_I = \min\{n, n_R\}$ applicants to interview and then select the best among those interviewed.

With interviews, the *selection channel* arises whenever $n_R \geq 2$, despite the fact that the method of selection is imperfect. Even if $n_R = 2$, a higher θ implies a lower probability that a firm receives two applications, i.e. $n = 2$, and therefore firms can be less selective. While the best applicant is not always hired (since they may not be interviewed at all), the expected match output $y(\theta)$ still depends on the market tightness θ. In particular, we have $y'(\theta) < 0$ since a greater number of firms per unemployed worker (higher θ) implies *fewer* applicants per vacancy and therefore firms can be less selective. The generalized Hosios condition is thus necessary for constrained efficiency.

3.8 Competitive search with referrals and an endogenous quality distribution

Campbell, Leister, and Zenou (2017) presents a dynamic model of consumer sales with word-of-mouth communication through social networks. We present a related but different model to illustrate how an endogenous quality distribution may arise through the possibility of "referrals". In our setting, the key variable θ is the ratio of *referrals* to consumers and the endogenous quality distribution is the probability that a traded good is low quality, i.e. the *market share* of low-quality firms. We use competitive search to model the market for referrals and consider whether the entry of sellers of referrals is constrained efficient. This environment generates a novel externality that is captured by the generalized Hosios condition but is not internalized by competitive search.

There is a fixed measure of consumers who seek to purchase one unit of a good. After purchasing the good, consumers exit the market and are replaced by new consumers. Goods are produced by a large number of competitive firms of two types: high quality and low quality. The share of firms that are low quality is $\mu \in (0,1)$.[23] The high-quality good has quality x_H and the low-quality good has quality x_L. The price of the good is p for both types of firm.

We are interested in the market for *referrals*. Consumers cannot directly observe firms' quality, but they can receive referrals. A single referral tells a consumer about the quality of a good purchased in the previous period. In each period $t \in \{0, 1, ...\}$, the expected number of referrals per consumer is θ_t (which is endogenous) and $P_n(\theta_t)$ is the probability a consumer receives n referrals at time t. This is a kind of "meeting technology" which matches referrals with consumers. If a consumer receives at least one referral, they pick the "best" referral and then choose whether to purchase from that firm or instead choose a firm randomly.[24] If a consumer receives no referrals, they purchase the good from a random firm, i.e. they buy a low-quality good with probability μ.

The presence of referrals influences the quality of traded goods. Let α_t be the *market share* of low-quality firms, i.e. the probability that a good traded in period t is low quality. More referrals per consumer θ_t implies a greater proportion of high-quality goods are purchased, which in turn increases the proportion of high-quality referrals made in the next period. In this way, the *selection channel* ensures the average quality of a traded good is increasing in θ_t and it also ensures that the quality distribution, reflected in the market share α_t, itself evolves over time as a result of this selection channel.

Low-quality goods are purchased only if *all* n of a consumer's referrals are to low-quality firms (which occurs with probability α_t^n) *and* the consumer picks a low-quality firm when choosing randomly (which occurs with probability μ). We therefore obtain the following law of motion for α_t:

$$\alpha_{t+1} = \mu \sum_{n=0}^{\infty} P_n(\theta_{t+1}) \alpha_t^n \tag{60}$$

where we assume that $\alpha_0 = \mu \in (0,1)$, the share of low-quality firms. If $P_n(\theta)$

[23] Since our focus is on the market for *referrals*, we do not endogenize the entry of low and high quality firms as in Campbell et al. (2017) but instead assume that μ is exogenous.

[24] If the consumer is indifferent between two referrals, they pick one at random.

is Poisson, i.e, $P_n(\theta) = \frac{\theta^n e^{-\theta}}{n!}$, we have

$$\alpha_{t+1} = \mu e^{-\theta_{t+1}(1-\alpha_t)}. \tag{61}$$

We now endogenize the steady state equilibrium number of referrals per consumer. There is a large number of potential entrants who can pay a cost $c > 0$ to acquire information about a random good purchased in the previous period. This information can be sold to consumers as a "referral". The mechanism for selling referrals is that consumers post *referral fees* and commit to paying a single fee for the "best" referral they receive.

Similarly to the competitive search environment in Section 3.4, consumers form a submarket i by choosing a referral fee r_i^* and a ratio of referrals to consumers θ_i^* to maximize their expected payoff:

$$m(\theta_i)(y(\theta_i, \alpha) - r_i - p) + (1 - m(\theta_i))(y_\mu - p) \tag{62}$$

subject to the following zero profit condition for sellers of referrals:

$$\frac{m(\theta_i)}{\theta_i} r_i = c \tag{63}$$

where $m(\theta_i) = 1 - e^{-\theta_i}$ is the probability a consumer receives a referral, $m(\theta_i)/\theta_i$ is the probability a seller is paid a referral fee, $y(\theta_i, \alpha)$ is the expected quality of a good purchased if the consumer receives a referral, and $y_\mu = \mu x_L + (1-\mu)x_H$ is the expected quality of a good purchased from a random firm.

Using (63), the choice of a consumer in submarket i is equivalent to

$$\theta_i^* = \arg\max_{\theta_i \in \mathbb{R}_+} (m(\theta_i)(y(\theta_i, \alpha) - y_\mu) + y_\mu - p - c\theta_i) \tag{64}$$

and θ_i^* satisfies the first-order condition

$$m'(\theta_i)s(\theta_i) + m(\theta_i)\frac{\partial y(\theta_i, \alpha)}{\partial \theta_i} = c \tag{65}$$

where the expected match surplus is $s(\theta_i) = y(\theta_i, \alpha) - y_\mu$, i.e. the difference between the expected quality in submarket i with and without receiving referrals. In symmetric equilibrium, $\theta_i^* = \theta^*$ for all submarkets i and θ^* satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\frac{\frac{\partial y(\theta,\alpha)}{\partial \theta}\theta}{s(\theta)}}_{\text{direct surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{surplus share of referral sellers}} \tag{66}$$

as well as the steady state condition

$$\alpha = \mu e^{-\theta(1-\alpha)}. \tag{67}$$

If $\mu < \frac{1}{2}$, there exists a unique steady state equilibrium (θ^*, α^*).[25]

Now consider a social planner who can directly choose the number of referrals per consumer θ but is constrained by the same "matching" technology $m(.)$ and "production" technology $y(., \alpha)$ as the decentralized economy.[26] While consumers take α *as given*, the social planner takes the effect of θ on α into account. The planner maximizes the steady state social surplus per consumer:

$$\Omega(\theta) = m(\theta)y(\theta, \alpha(\theta)) + (1 - m(\theta))y_\mu - c\theta \tag{68}$$

where $\alpha(\theta)$ is given by (67) and

$$y(\theta, \alpha(\theta)) = \frac{(1 - \mu e^{-\theta(1-\alpha(\theta))})x_H + \mu e^{-\theta(1-\alpha(\theta))}x_L - e^{-\theta}y_\mu}{1 - e^{-\theta}}. \tag{69}$$

Using $s(\theta) = y(\theta, \alpha(\theta)) - y_\mu$, the first-order condition is

$$\Omega'(\theta) = m'(\theta)s(\theta) + m(\theta)s'(\theta) - c = 0 \tag{70}$$

where

$$s'(\theta) = \underbrace{\frac{\partial y(\theta, \alpha(\theta))}{\partial \theta}}_{\text{direct output externality}} + \underbrace{\frac{\partial y(\theta, \alpha(\theta))}{\partial \alpha}\alpha'(\theta)}_{\text{indirect output externality}} \tag{71}$$

[25] A detailed derivation of the steady state equilibrium can be found in the Appendix.

[26] In the Appendix, we solve the dynamic planner's problem subject to the law of motion for α_t and derive a steady state condition that is identical to the one presented here.

and any social optimum θ^P must satisfy

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{surplus share of referral sellers}}. \tag{72}$$

Therefore, we have constrained efficiency only if the generalized Hosios condition holds. That is, only if θ^* satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\frac{\frac{\partial y(\theta,\alpha)}{\partial \theta}\theta}{s(\theta)} + \frac{\frac{\partial y(\theta,\alpha)}{\partial \alpha}\alpha'(\theta)\theta}{s(\theta)}}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{surplus share of referral sellers}}. \tag{73}$$

Comparing (73) with (66), it is clear the economy is not constrained efficient.

The decentralized market internalizes both the search externalities and the *direct* component of the "output externality", i.e. the direct effect of θ on the expected match surplus. However, there is an additional externality arising from the use of referrals. This is reflected in the term $\frac{\partial y(\theta,\alpha(\theta))}{\partial \alpha}\alpha'(\theta)$, which captures the *indirect* component of the "output externality" via the quality distribution. Since $\frac{\partial y(\theta,\alpha(\theta))}{\partial \alpha} < 0$ and the market share of low-quality firms is decreasing in the number of referrals per consumer at any equilibrium θ^*, i.e. $\alpha'(\theta^*) < 0$, this is a positive externality that is not internalized by the decentralized economy. The equilibrium number of referrals is therefore inefficiently *low*.

While the generalized Hosios condition does indeed *apply* in this environment, we do not have constrained efficiency because it does not hold. The static economy is constrained efficient since the probability α is exogenous, but the dynamic economy is not efficient. The novel externality that is the source of this inefficiency is similar in flavor to that found in Guerrieri (2008), which shows that competitive search is not always dynamically efficient. In Guerrieri (2008), the inefficiency arises because firms do not internalize the effect of their decisions on the outside options of workers hired in earlier periods. Here, consumers do not internalize the effect of their decisions on future consumers through the impact of referrals on the evolution of the quality distribution itself.

4 Conclusion

This paper presents a generalized version of the well-known Hosios rule that determines the conditions under which buyer (or seller) entry in search and matching models is constrained efficient. We extend this simple rule to environments where the expected match output depends on the market tightness. Such environments give rise to a novel externality that we call the *output externality.* This externality is not captured by the standard Hosios condition, which internalizes only the search externalities arising from the matching frictions.

To ensure constrained efficiency, decentralized markets must internalize the effect of entry on both the *number* of matches created and the average *value* created by each match. We show that this occurs precisely when buyers' surplus share equals the *matching elasticity* plus the *surplus elasticity.* We call this simple, intuitive condition the *generalized Hosios condition.* Like the standard Hosios condition, the simplicity of this general rule carries over directly to dynamic environments with enduring matches. When this condition holds, both the matching externalities and the output externality are internalized.

We focus attention exclusively on the efficiency of entry and assume that the social planner is constrained by both the matching technology and the *output technology,* i.e. the "technology" which transforms the market tightness into the expected match output. However, the nature of the output technology arises directly from specific features of the decentralized market – such as the underlying meeting technology and trading mechanism – which are taken as given. One possible direction for future research would be to integrate our general result regarding the efficiency of entry with the literature, such as Eeckhout and Kircher (2010b), that considers which trading mechanisms can emerge as an equilibrium outcome in various environments with search frictions.

Another potential direction for future research would be to consider search and matching environments with two-sided heterogeneity. In such environments, as Eeckhout and Kircher (2010a) point out, the social planner also cares about both the number of matches created and the average value of a match – which depends on the types of agents that form matches. In future research, it would be interesting to integrate results regarding the efficiency of frictional environments with two-sided heterogeneity with the general result presented here.

Appendix

Proof of Lemma 1

In steady state, we have the following Bellman equations:

$$rU_B = -c + \frac{m(\theta)}{\theta}(V_B - U_B), \tag{74}$$

$$rV_B = y(\theta) - w(\theta) + \delta(U_B - V_B), \tag{75}$$

$$rU_S = b + m(\theta)(V_S - U_S), \tag{76}$$

$$rV_S = w(\theta) + \delta(U_S - V_S), \tag{77}$$

where $w(\theta)$ is the expected transfer paid to sellers by buyers each period.

With free entry, $U_B = 0$ and $s(\theta) = V_B + V_S - U_S$, so we have

$$V_B + V_S = \frac{y(\theta) - \delta s(\theta)}{r}. \tag{78}$$

Substituting back into $s(\theta) = V_B + V_S - U_S$, and rearranging yields

$$s(\theta) = \frac{y(\theta) - rU_S}{r + \delta}. \tag{79}$$

Next, using (76) and (77), we find that

$$U_S = \frac{b(r+\delta) + m(\theta)w(\theta)}{r(r+\delta+m(\theta))}, \tag{80}$$

and, substituting into (79), we obtain

$$s(\theta) = \left(\frac{y(\theta) - b + m(\theta)\left(\frac{y(\theta)-w(\theta)}{r+\delta}\right)}{r+\delta+m(\theta)}\right). \tag{81}$$

Now (74) implies $V_B = c\theta/m(\theta)$ when $U_B = 0$. Substituting into (75), we have

$$\frac{y(\theta) - w(\theta)}{r + \delta} = \frac{c\theta}{m(\theta)}, \tag{82}$$

and, substituting (82) into $s(\theta)$, we obtain

$$s(\theta) = \frac{y(\theta) - b + c\theta}{r + \delta + m(\theta)}. \tag{83}$$

Proof of Proposition 1

In discrete time, the law of motion for the unemployment rate u_t is

$$u_{t+dt} - u_t = \delta dt(1 - u_t) - m(\theta_t)dt\, u_t \tag{84}$$

and the law of motion for average match output y_t is given by

$$y_{t+dt} = \frac{(1 - \delta dt)(1 - u_t)y_t + m(\theta_t)dt\, u_t y(\theta_t)}{1 - u_{t+dt}}. \tag{85}$$

Defining $x_t \equiv (1 - u_t)y_t$, we have

$$x_{t+dt} - x_t = -\delta dt x_t + m(\theta_t)dt\, u_t y(\theta_t). \tag{86}$$

In continuous time ($dt \to 0$), the laws of motion for u_t and x_t are

$$\dot{u}_t = \frac{du_t}{dt} = \lim_{dt \to 0} \left(\frac{u_{t+dt} - u_t}{dt} \right) = \delta(1 - u_t) - m(\theta_t)u_t \tag{87}$$

and

$$\dot{x}_t = \frac{dx_t}{dt} = \lim_{dt \to 0} \left(\frac{x_{t+dt} - x_t}{dt} \right) = -(\delta x_t - m(\theta_t)u_t y(\theta_t)). \tag{88}$$

Also, since $x_t \equiv (1 - u_t)y_t$, we have

$$\dot{x}_t = -\dot{u}_t y_t + (1 - u_t)\dot{y}_t \tag{89}$$

and, rearranging, we have

$$\dot{y}_t = \frac{\dot{x}_t + \dot{u}_t y_t}{1 - u_t}. \tag{90}$$

Substituting in $\dot{x}_t$ and $\dot{u}_t$ from (88) and (87) and the fact that $x_t \equiv (1 - u_t)y_t$ leads to:

$$\dot{y}_t = \frac{m(\theta_t)u_t(y(\theta_t) - y_t)}{1 - u_t}. \tag{91}$$

The social planner chooses θ_t for all $t \in \mathbb{R}_+$ to maximize the following:

$$\int_0^\infty e^{-rt}((1 - u_t)y_t + bu_t - c\theta_t u_t)dt \tag{92}$$

subject to

$$\dot{u}_t = \delta(1 - u_t) - m(\theta_t)u_t \tag{93}$$

and

$$\dot{y}_t = \frac{m(\theta_t)u_t\,(y(\theta_t) - y_t)}{1 - u_t}. \tag{94}$$

The current value Hamiltonian is

(95)

$$H = ((1-u_t)y_t + bu_t - c\theta_t u_t) + \lambda_t(\delta(1-u_t) - m(\theta_t)u_t) + \mu_t\left(\frac{m(\theta_t)u_t\,(y(\theta_t) - y_t)}{1 - u_t}\right).$$

The first-order necessary conditions are

$$\frac{\partial H}{\partial \theta_t} = -cu_t - \lambda_t m'(\theta_t)u_t + \mu_t\left(\frac{m'(\theta_t)u_t\,(y(\theta_t) - y_t) + m(\theta_t)u_t y'(\theta_t)}{1 - u_t}\right) = 0 \tag{96}$$

and

$$\begin{aligned}
(97) \quad \frac{dH}{du_t} &= -(y_t - b + c\theta_t) - \lambda_t(\delta + m(\theta_t)) \\
&\quad + \mu_t \left(\frac{(1-u_t)\, m(\theta_t)\,(y(\theta_t) - y_t) + u_t m(\theta_t)\,(y(\theta_t) - y_t)}{(1-u_t)^2} \right) \\
&= -\dot{\lambda}_t + r\lambda_t
\end{aligned}$$

and

$$\frac{\partial H}{\partial y_t} = 1 - u_t - \mu_t \left(\frac{m(\theta_t)u_t}{1-u_t} \right) = -\dot{\mu}_t + r\mu_t \tag{98}$$

and

$$\frac{\partial H}{\partial \lambda_t} = \delta(1-u_t) - m(\theta_t)u_t = \dot{u}_t \tag{99}$$

and

$$\frac{\partial H}{\partial \mu_t} = \frac{m(\theta_t)u_t\,(y(\theta_t) - y_t)}{1-u_t} = \dot{y}_t. \tag{100}$$

The transversality conditions are

$$\lim_{t\to\infty} e^{-rt}\lambda_t u_t = 0, \tag{101}$$

$$\lim_{t\to\infty} e^{-rt}\mu_t y_t = 0. \tag{102}$$

Now, in steady state, we have $\dot{u}_t = 0$ and $\dot{y}_t = 0$ and therefore $y(\theta_t) = y_t = y(\theta)$. Also, in steady state, $\dot{\mu}_t = 0$ and $\dot{\lambda}_t = 0$. Substituting into the above first-order conditions,

$$-cu - \lambda m'(\theta)u + \mu \left(\frac{m(\theta)y'(\theta)}{1-u} \right) = 0, \tag{103}$$

$$-(y(\theta) - b + c\theta) - \lambda(\delta + m(\theta)) = r\lambda, \tag{104}$$

$$1-u-\mu\left(\frac{m(\theta)u}{1-u}\right)=r\mu. \tag{105}$$

Using the fact that $\delta(1-u)=m(\theta)u$ in steady state, we have

$$-\lambda m'(\theta)u+\mu\delta y'(\theta)=cu, \tag{106}$$

$$\lambda=-\left(\frac{y(\theta)-b+c\theta}{r+\delta+m(\theta)}\right), \tag{107}$$

$$\mu=\frac{1-u}{r+\delta}. \tag{108}$$

It is clear that the transversality conditions are satisfied by λ and μ. Substituting λ and μ into the first equation, we have

$$\frac{y(\theta)-b+c\theta}{r+\delta+m(\theta)}m'(\theta)u+\frac{(1-u)\delta y'(\theta)}{r+\delta}=cu. \tag{109}$$

Again using $\delta(1-u)=m(\theta)u$ and simplifying,

$$\frac{y(\theta)-b+c\theta}{r+\delta+m(\theta)}m'(\theta)+\frac{m(\theta)y'(\theta)}{r+\delta}=c. \tag{110}$$

Defining $s(\theta)$ as in (12), and multiplying by $\theta/m(\theta)s(\theta)$,

$$\frac{m'(\theta)\theta}{m(\theta)}+\frac{y'(\theta)\theta}{(r+\delta)s(\theta)}=\frac{c\theta}{m(\theta)s(\theta)}. \tag{111}$$

That is,

$$\eta_m(\theta)+\frac{y'(\theta)\theta}{(r+\delta)s(\theta)}=\frac{c\theta}{m(\theta)s(\theta)}, \tag{112}$$

where $\eta_m(\theta)\equiv m'(\theta)\theta/m(\theta)$. Any social optimum must satisfy (112).

Proof of Lemma 2

It follows immediately from Assumption 2a, together with the intermedate value theorem, that there exists a unique $\theta^P > 0$ that satisfies the necessary condition (13). We now prove that the steady state solution θ^P given by (13) is indeed a global maximum using Arrow's Sufficiency Theorem. To show this, it is simpler to formulate the current value Hamiltonian in terms of the state variable x_t. Using (87) and (88), the current value Hamiltonian as a function of state and control variables is

$$(113)\quad H(x,u,\theta) = (x+bu-c\theta u)+\lambda_1(\delta(1-u)-m(\theta)u)+\mu_1(-(\delta x-m(\theta)uy(\theta))).$$

First, we define the maximized Hamiltonian as follows:

$$M_H(x,u) \equiv \max_{\theta\in\mathbb{R}_+}[(x+bu-c\theta u)+\lambda_1(\delta(1-u)-m(\theta)u)+\mu_1(-(\delta x-m(\theta)uy(\theta))].$$

We now apply Arrow's Sufficiency Theorem.[27] To prove that the solution θ^P to (112) is a global maximum, it is sufficient to show that *(i)* the maximized Hamiltonian $M_H(x,u)$ is jointly weakly concave in u and x; and *(ii)* there exists a unique solution θ^P that satisfies the necessary condition (112). Since we know that part *(ii)* holds, it remains only to prove *(i)*. To find $\theta^* \equiv \arg\max_{\theta\in\mathbb{R}_+} H(x,u,\theta)$, we set

$$(114)\qquad \frac{\partial H}{\partial \theta} = -cu - \lambda_1 m'(\theta)u + \mu_1 u(m'(\theta)y(\theta) + m(\theta)y'(\theta)) = 0.$$

Also, we have

$$(115)\quad \frac{\partial^2 H}{\partial \theta^2} = -\lambda_1 m''(\theta)u + \mu_1 u(m''(\theta)y(\theta) + 2m'(\theta)y'(\theta) + m(\theta)y''(\theta)) < 0,$$

provided that $m''(\theta)y(\theta) + 2m'(\theta)y'(\theta) + m(\theta)y''(\theta) < 0$ and $m''(\theta) < 0$ since $\lambda_1 < 0$ and $\mu_1 > 0$. Assumption 1a states that $m''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$ and Assumption 2a says that $\Lambda''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$ where $\Lambda(\theta) \equiv m(\theta)s(\theta)$. In

[27] Arrow's Sufficiency Theorem generalizes Mangasarian's sufficiency conditions. See Kamien and Schwartz (1991), p. 221-222.

the special case where $b = 0$ in the static economy, we have $\Lambda(\theta) \equiv m(\theta)y(\theta)$ and therefore $\Lambda''(\theta) < 0$ implies that $m''(\theta)y(\theta) + 2m'(\theta)y'(\theta) + m(\theta)y''(\theta) < 0$.

So θ^* is indeed a maximum and the maximized Hamiltonian is

(116)

$$M_H(x, u) = (x + bu - c\theta^* u) + \lambda_1(\delta(1-u) - m(\theta^*)u) + \mu_1(-(\delta x - m(\theta^*)uy(\theta^*))).$$

Since the u cancels out in (114) and x does not appear in that equation, θ^* does not depend directly on u or x. Also, it can be verified that neither λ_1 nor μ_1 depends on either u or x.[28] The function $M_H(x, u)$ is linear in both x and u and it is therefore weakly concave. Since there exists a unique solution θ^P that satisfies the necessary condition (112), this solution is the global maximum.

Proof of Corollary 1 and 2

Assume that the standard Hosios condition holds, namely

$$\frac{m'(\theta^*)\theta^*}{m(\theta^*)} = \frac{c\theta^*}{m(\theta^*)s(\theta^*)}. \tag{117}$$

We prove the result in two parts. First, we show that there is under-entry (over-entry) of buyers if and only if $s'(\theta^*) > (<) 0$. Second, we show that $s'(\theta^*) > 0$ if and only if $y'(\theta^*) > 0$. Letting $\Lambda(\theta) = m(\theta)s(\theta)$ and simplifying (117), we have $m'(\theta^*)s(\theta^*) = c$. We also have $\Lambda'(\theta^P) = c$ and therefore $\Lambda'(\theta^P) = m'(\theta^*)s(\theta^*)$. Now $m'(\theta^*)s(\theta^*) = \Lambda'(\theta^*) - m(\theta^*)s'(\theta^*)$ and thus

$$\Lambda'(\theta^P) = \Lambda'(\theta^*) - m(\theta^*)s'(\theta^*). \tag{118}$$

If $s'(\theta^*) > 0$, then $\Lambda'(\theta^P) < \Lambda'(\theta^*)$. If $\Lambda''(\theta) < 0$ for all θ then $\Lambda'(\theta^P) < \Lambda'(\theta^*)$ implies that $\theta^* < \theta^P$ and there is *under-entry* of buyers. Similarly, if $s'(\theta^*) < 0$, there is *over-entry* of buyers, $\theta^* > \theta^P$. Differentiating $s(\theta)$ using (12),

$$s'(\theta) = \frac{(y'(\theta) + c)(r + \delta + m(\theta)) - m'(\theta)(y(\theta) - b + c\theta)}{(r + \delta + m(\theta))^2}. \tag{119}$$

[28] Note that the co-state variables λ_1 and μ_1 for the current value Hamiltonian with state variables u_t and x_t are different to the co-state variables λ and μ for the current value Hamiltonian with state variables u_t and p_t.

Using expression (12) for $s(\theta)$ and rearranging, $s'(\theta^*) > 0$ if and only if

$$y'(\theta^*) > m'(\theta^*)s(\theta^*) - c, \tag{120}$$

and since $m'(\theta^*)s(\theta^*) = c$, we have $s'(\theta^*) > 0$ if and only if $y'(\theta^*) > 0$.

With seller entry, the direction of Corollary 1 is reversed to yield Corollary 2 since $\theta^* < \theta^P$ implies *over-entry* of sellers because $\theta = N_B/N_S$, and $\theta^* > \theta^P$ implies *under-entry* of sellers relative to the social optimum.

Proofs for Section 3.8

Equilibrium. In period t, the expected payoff for a seller of a referral in submarket i with referral fee $r_{i,t}$ and market tightness $\theta_{i,t}$ is

$$\Pi(\theta_{i,t}, r_{i,t}) = \frac{m(\theta_{i,t})}{\theta_{i,t}} r_{i,t} - c \tag{121}$$

and the expected payoff for consumers in submarket i is

$$V(\theta_{i,t}, r_{i,t}) = m(\theta_{i,t})(y(\theta_{i,t}, \alpha_{t-1}) - r_{i,t} - p) + (1 - m(\theta_{i,t}))(y_\mu - p)). \tag{122}$$

Consumers in submarket i choose a referral fee $r^*_{i,t}$ and market tightness $\theta^*_{i,t}$ that maximize $V(\theta_{i,t}, r_{i,t})$ subject to $\Pi(\theta_{i,t}, r_{i,t}) \leq c$ and $\theta_{i,t} \geq 0$ with complementary slackness. To induce participation by sellers in submarket i, i.e. $\theta_{i,t} > 0$, the constraint $\Pi(\theta_{i,t}, r_{i,t}) \leq c$ is binding and we have

$$\frac{m(\theta_{i,t})}{\theta_{i,t}} r_{i,t} = c. \tag{123}$$

Using (123) to replace $r_{i,t}$ in $V(\theta_{i,t}, r_{i,t})$, the choice of a consumer in submarket i is equivalent to

$$\theta^*_{i,t} = \arg\max_{\theta_{i,t} \in \mathbb{R}_+} (m(\theta_{i,t})(y(\theta_{i,t}, \alpha_{t-1}) - y_\mu) + y_\mu - p - c\theta_{i,t}). \tag{124}$$

Differentiating with respect to $\theta_{i,t}$, the first-order condition of this problem is

$$m'(\theta_{i,t})(y(\theta_{i,t},\alpha_{t-1})-y_\mu)+m(\theta_{i,t})\frac{\partial y(\theta_{i,t},\alpha_{t-1})}{\partial\theta_{i,t}}-c=0. \tag{125}$$

In symmetric equilibrium, $\theta^*_{i,t}=\theta^*_t$ for all submarkets i and θ^*_t satisfies

$$m'(\theta_t)(y(\theta_t,\alpha_{t-1})-y_\mu)+m(\theta_t)\frac{\partial y(\theta_t,\alpha_{t-1})}{\partial\theta_t}=c. \tag{126}$$

In steady state, $\theta_t=\theta_{t-1}=\theta$ and $\alpha_t=\alpha_{t-1}=\alpha$ and any equilibrium (θ^*,α^*) satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\frac{\frac{\partial y(\theta,\alpha)}{\partial\theta}\theta}{s(\theta)}}_{\text{direct surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{surplus share of referral sellers}} \tag{127}$$

where the expected match surplus is $s(\theta)=y(\theta,\alpha)-y_\mu$.

To solve for the equilibrium, we use the fact that the average quality of a traded good in period t is given by

$$(1-\alpha_t)x_H+\alpha_t x_L=f(\theta_t,\alpha_{t-1})+(1-m(\theta_t)y_\mu \tag{128}$$

where $f(\theta_t,\alpha_{t-1})=m(\theta_t)y(\theta_t,\alpha_{t-1})$. Using the fact that $\alpha_t=\mu e^{-\theta_t(1-\alpha_{t-1})}$,

$$f(\theta_t,\alpha_{t-1})=(1-\mu e^{-\theta_t(1-\alpha_{t-1})})x_H+\mu e^{-\theta_t(1-\alpha_{t-1})}x_L-e^{-\theta_t}y_\mu \tag{129}$$

or equivalently,

$$f(\theta_t,\alpha_{t-1})=x_H-\mu\Delta x e^{-\theta_t(1-\alpha_{t-1})}-e^{-\theta_t}y_\mu \tag{130}$$

where $\Delta x=x_H-x_L$. The first-order condition (126) is equivalent to

$$\frac{\partial f(\theta_t,\alpha_{t-1})}{\partial\theta_t}-m'(\theta_t)y_\mu-c=0. \tag{131}$$

Differentiating (130) with respect to θ_t, this is equivalent to

$$(1-\alpha_{t-1})\mu\Delta x e^{-\theta_t(1-\alpha_{t-1})} - c = 0 \tag{132}$$

and the second-order condition is

$$-(1-\alpha_{t-1})^2\mu\Delta x e^{-\theta_t(1-\alpha_{t-1})} < 0. \tag{133}$$

Using the fact that $\alpha_t = \mu e^{-\theta_t(1-\alpha_{t-1})}$, this is equivalent to

$$(1-\alpha_{t-1})\alpha_t = \frac{c}{\Delta x}. \tag{134}$$

In steady state, $\theta_t = \theta_{t-1} = \theta$ and $\alpha_t = \alpha_{t-1} = \alpha$ and any equilibrium α satisfies

$$-\alpha^2 + \alpha - \frac{c}{\Delta x} = 0 \tag{135}$$

as well as $\alpha = \mu e^{-\theta(1-\alpha)}$. Since $\mu \in (0,1)$, there are two solutions $\alpha \in (0,1)$ provided that $\frac{c}{\Delta x} < \frac{1}{4}$ and one solution if $\frac{c}{\Delta x} = \frac{1}{4}$. The two solutions are

$$\alpha = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{c}{\Delta x}}. \tag{136}$$

Since $\alpha < \mu$ for $\theta > 0$, if $\mu < \frac{1}{2}$ we obtain a unique steady state equilibrium

$$\alpha^* = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{c}{\Delta x}} \tag{137}$$

and

$$\theta^* = \frac{1}{1-\alpha}\ln\left(\frac{\mu}{\alpha}\right). \tag{138}$$

Social planner. Given $\alpha_0 = \mu \in (0,1)$, the social planner chooses $\{\theta_t\}_{t=1}^{\infty}$ to maximize the total discounted social surplus per consumer:

$$\Omega = \sum_{t=1}^{\infty} \beta^t \left(m(\theta_t)y(\theta_t, \alpha_{t-1}) + (1-m(\theta_t))y_\mu - c\theta_t\right) \tag{139}$$

subject to $\theta_t \geq 0$ and the law of motion for α_t :

$$\alpha_t = \mu e^{-\theta_t(1-\alpha_{t-1})} \tag{140}$$

The Lagrangian for this problem is

$$\mathcal{L} = \sum_{t=1}^{\infty} \beta^t \left(m(\theta_t)(y(\theta_t, \alpha_{t-1}) - y_\mu) + y_\mu - c\theta_t\right) + \lambda_t(\alpha_t - \mu e^{-\theta_t(1-\alpha_{t-1})}). \tag{141}$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial \theta_t} = \beta^t(m'(\theta_t)(y(\theta_t, \alpha_{t-1}) - y_\mu) + m(\theta_t)\frac{\partial y(\theta_t, \alpha_{t-1})}{\partial \theta_t} - c) + \lambda_t(1-\alpha_{t-1})\mu e^{-\theta_t(1-\alpha_{t-1})} = 0 \tag{142}$$

and

$$\frac{\partial \mathcal{L}}{\partial \alpha_{t-1}} = \lambda_{t-1} + \beta^t m(\theta_t)\frac{\partial y(\theta_t, \alpha_{t-1})}{\partial \alpha_{t-1}} - \lambda_t\theta_t\mu e^{-\theta_t(1-\alpha_{t-1})} = 0. \tag{143}$$

and

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = \alpha_t - \mu e^{-\theta_t(1-\alpha_{t-1})} = 0. \tag{144}$$

Now, in steady state we have $\theta_{t+1} = \theta_t = \theta$, $\alpha_{t+1} = \alpha_t = \alpha$, and we have

$$\beta^t(m'(\theta)(y(\theta, \alpha) - y_\mu) + m(\theta)\frac{\partial y(\theta, \alpha)}{\partial \theta} - c) = -\lambda(1-\alpha)\mu e^{-\theta(1-\alpha)} \tag{145}$$

and

$$\lambda + \beta^t m(\theta)\frac{\partial y(\theta, \alpha)}{\partial \alpha} = \lambda\theta\mu e^{-\theta(1-\alpha)} \tag{146}$$

and

$$\alpha = \mu e^{-\theta(1-\alpha)}. \tag{147}$$

Rearranging (145), we obtain

$$\lambda = \frac{-\beta^t (m'(\theta)(y(\theta,\alpha) - y_\mu) + m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \theta} - c)}{(1-\alpha)\mu e^{-\theta(1-\alpha)}}, \tag{148}$$

and rearranging (146) delivers

$$\lambda = \frac{-\beta^t m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \alpha}}{1 - \theta\mu e^{-\theta(1-\alpha)}}. \tag{149}$$

Equating (148) and (149) yields

$$\frac{-\beta^t (m'(\theta)(y(\theta,\alpha) - y_\mu) + m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \theta} - c)}{(1-\alpha)\mu e^{-\theta(1-\alpha)}} = \frac{-\beta^t m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \alpha}}{1 - \theta\mu e^{-\theta(1-\alpha)}} \tag{150}$$

and rearranging, and substituting in (147), we obtain

$$m'(\theta)(y(\theta,\alpha) - y_\mu) + m(\theta)\left(\frac{\partial y(\theta,\alpha)}{\partial \theta} - \frac{\partial y(\theta,\alpha)}{\partial \alpha}\frac{(1-\alpha)\alpha}{1-\theta\alpha}\right) = c. \tag{151}$$

Implicitly differentiating $\alpha = \mu e^{-\theta(1-\alpha)}$, we have

$$\alpha'(\theta) = \frac{-(1-\alpha)\alpha}{1-\theta\alpha} \tag{152}$$

and substituting (152) into (151) yields

$$m'(\theta)s(\theta) + m(\theta)\left(\frac{\partial y(\theta,\alpha)}{\partial \theta} + \frac{\partial y(\theta,\alpha)}{\partial \alpha}\alpha'(\theta)\right) = c. \tag{153}$$

Rearranging (153), we obtain:

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\frac{\frac{\partial y(\theta,\alpha)}{\partial \theta}\theta}{s(\theta)} + \frac{\frac{\partial y(\theta,\alpha)}{\partial \alpha}\alpha'(\theta)\theta}{s(\theta)}}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{surplus share of referral sellers}}, \tag{154}$$

which is identical to (73) in Section 3.8. This is a necessary condition that any solution to the planner's problem must satisfy.

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