Efficiency in Search and Matching Models: A Generalized Hosios Condition*

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21 December 2017

Abstract

When is the level of entry of buyers or sellers *efficient* in markets with search and matching frictions? This paper generalizes the well-known Hosios condition for constrained efficiency to a wide range of dynamic search and matching environments where the expected match output depends on the market tightness. The generalized Hosios condition is simple and intuitive: entry is constrained efficient when buyers' surplus share equals the *matching elasticity* plus the *surplus elasticity* (i.e. the elasticity of the expected joint match surplus with respect to buyers). This condition ensures that agents are paid for their contribution to both *match creation* and *surplus creation*. In search models of the labor market, for example, the equilibrium levels of vacancy entry and unemployment are not constrained efficient unless firms are compensated for the effect of firm entry on both employment and average labor productivity.

JEL Codes: C78, D83, E24, J64

Keywords: constrained efficiency, search and matching, directed search, competitive search, Nash bargaining, Hosios condition

*We would like to thank seminar and conference participants at the Australian National University, the National University of Singapore, the Madison Search and Matching meeting at the University of Wisconsin, the Dale T. Mortensen Centre Conference on Markets with Search Frictions at Aarhus University, and the Workshop of the Australasian Macroeconomics Society. We also thank Arthur Campbell, Mohammad Davoodalhosseini, Frédéric Gavrel, Ian King, Adrian Masters, Rob Shimer, Ronald Wolthoff, Randy Wright, and Yves Zenou for particularly valuable comments and discussions.

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1 Introduction

The well-known "Hosios condition" specifies a precise condition under which entry is *constrained efficient* in markets featuring search and matching frictions.[1] We say that entry is "constrained efficient" when the equilibrium level of buyer (or seller) entry is the same as that which would be chosen by a social planner who is constrained by the same frictions as the decentralized market. The rule introduced in Hosios (1990) states that buyer entry is constrained efficient only when buyers' share of the total joint surplus equals the elasticity of the matching function with respect to buyers. This condition has proven to be widely applicable across a broad range of search and matching models. For example, in search-theoretic models of the labor market, the Hosios condition tells us when the equilibrium level of vacancy creation – and therefore the unemployment rate – is constrained efficient.[2]

While it is widely known that the Hosios condition applies to a broad range of search models, it is not well-known that this condition does not always apply in settings where the expected match output depends on the market tightness.[3] In such environments, the standard Hosios condition may imply either inefficiently high or inefficiently low levels of entry. This is particularly important in search-and-matching models featuring Nash bargaining because such markets are generically inefficient. Unlike directed or competitive search models, which are typically constrained efficient, the Hosios condition does not hold endogenously in models with Nash bargaining. Instead, this condition is often *imposed* as a way of determining the bargaining parameter: firms' bargaining power is simply set equal to the "efficient" level given by the matching elasticity, e.g. Shimer (2005a).

This paper makes two contributions. First, we generalize the Hosios rule to environments where the expected match output depends on the market tightness. Second, we present a number of examples of such environments to show how this one simple condition provides a unifying lens for understanding the efficiency of entry in search and matching models.

Consider an environment with buyer entry. An equilibrium allocation is constrained efficient when buyers are paid their marginal contribution to the social surplus. If the expected match output is exogenous, buyers need only be paid for their effect on *match creation*, i.e. on the total *number* of matches. In such environments, the standard Hosios condition applies: entry is constrained efficient only when buyers' surplus share equals the matching elasticity. If the expected match output is endogenous, however, buyers must also

[1] Early versions of the rule were discussed in Mortensen (1982b), Mortensen (1982a), and Pissarides (1986).

[2] There is a vast literature on search-theoretic models of the labor market. The classic survey is Rogerson, Shimer, and Wright (2005). See also the recent survey on directed and competitive search by Wright, Kircher, Julien, and Guerrieri (2017).

[3] We use the term "match output" because our examples focus on labor markets, but the term *output* can be interpreted more broadly to cover any trade or productive activity.

be compensated for their effect on *surplus creation*, i.e. on the expected *value* of the joint surplus created by each match. We show that entry is constrained efficient only when buyers' surplus share equals the *matching elasticity* plus the *surplus elasticity* (i.e. the elasticity of the expected match surplus with respect to buyers). We call this condition the *generalized Hosios condition*. Like the original version, it is simple and highly intuitive.

Dependence of the expected match output on the market tightness can arise naturally in many different environments. In particular, it may arise in markets where either buyers or sellers are heterogeneous prior to matching.[4] For example, with many-on-one or multilateral meetings – where many sellers can meet one buyer, or many buyers can meet one seller – the expected match output depends on market tightness because agents face a *choice* regarding potential trading partners. We call this the *selection channel*. The expected match output may also depend on the market tightness via alternative channels; for example, when there is an endogenous job acceptance or market participation decision.

When the Hosios condition obtains, decentralized markets internalize the search externalities that arise through the frictional matching process. However, when the expected match output depends on the market tightness, a novel externality arises: the *output externality*. Depending on the specific environment, the expected match output may be either increasing or decreasing in the market tightness and therefore the externality may be either positive or negative. When the generalized Hosios condition holds, both the standard search externalities and the output externality are fully internalized by a decentralized market. When the standard Hosios condition holds, the output externality is not internalized and there may be either over-entry or under-entry relative to the social optimum. For example, in labor markets featuring Nash bargaining, imposing the Hosios condition may entail setting workers' bargaining parameter too high, leading to inefficiently high unemployment because firms are not compensated for their effect on average labor productivity.

As a guiding principle, Hosios (1990) suggests that when we want to determine the efficiency properties of a particular model, the question we must ask is "whether the unattached agents who participate in the corresponding matching process receive more or less than their social marginal product" (p. 296). This guiding principle remains true. However, Hosios states that *all* we need to do to answer this question is determine the equilibrium surplus sharing rule and the matching technology, and then apply what is now known as the "Hosios rule". In many environments, however, this rule must be generalized. In addition to considering the surplus shares and the matching technology, we must also consider the *output*

[4] Buyers or sellers need not be ex ante heterogeneous: they can be identical prior to *meetings* provided there is some heterogeneity prior to *matching*, i.e. choice of trading partner. We focus on one-sided heterogeneity and do not consider search and matching environments with two-sided heterogeneity and assortative matching such as Shimer and Smith (2000, 2001), Shi (2001), Shimer (2005b), and Eeckhout and Kircher (2010a).

technology, which determines how changes in the market tightness affect the expected match output. That is, we need the generalized Hosios condition because it ensures that agents are paid for their contribution to both *match creation* and *surplus creation.*

Outline. In Section 2, we present our key result: the generalized Hosios condition. We first discuss a static economy and then derive the main result for a dynamic economy with enduring matches, as in the labor market. Section 3 provides a number of different examples of search and matching models to which the generalized Hosios condition applies. We discuss the relevant literature throughout the paper. All proofs are in the Appendix.

2 Generalized Hosios Condition

To build intuition, we first consider a static environment and then consider a dynamic economy with enduring matches.

2.1 Static economy

There is a fixed measure N_S of risk-neutral sellers and a large number of risk-neutral potential buyers. If N_B is the measure of risk-neutral buyers who enter, the market tightness, or buyer/seller ratio, is defined by $\theta \equiv N_B/N_S$. Buyers and sellers are matched according to a constant-returns-to-scale matching function. The matching probabilities for sellers and buyers are denoted respectively by $m(\theta)$ and $m(\theta)/\theta$. We call the function $m(.)$ the *matching technology* and assume that it satisfies the following standard properties.

Assumption 1. *The function* $m(.)$ *has the following properties: (i)* $m'(\theta) > 0$ *and* $m''(\theta) < 0$ *for all* $\theta \in \mathbb{R}_+$*, (ii)* $\lim_{\theta\to 0} m(\theta) = 0$*, (iii)* $\lim_{\theta\to 0} m'(\theta) = 1$*, (iv)* $\lim_{\theta\to\infty} m(\theta) = 1$*, (v)* $\lim_{\theta\to\infty} m'(\theta) = 0$*, and (vi)* $m(\theta)/\theta$ *is strictly decreasing in* θ *for all* $\theta \in \mathbb{R}_+$*.*

Let $y(\theta)$ denote the *expected match output.* When the expected match output is exogenous, we have $y(\theta) = \bar{y}$ for all $\theta \in \mathbb{R}_+$. In general, we allow the expected match output $y(\theta)$ to be *endogenous* in the sense that it depends directly on the market tightness. We call the function $y(.)$ the *output technology.*

There is free entry of buyers, each paying a cost $c > 0$ to enter, and $z \geq 0$ is the outside option of sellers. We assume that $y(\theta) > z$ for all $\theta \in \mathbb{R}_+$ and all matches are accepted by both buyers and sellers.[5]

Suppose the social planner is constrained by both the matching technology $m(.)$ *and* the output technology $y(.)$. Taking both functions $m(.)$ and $y(.)$ as given, the social planner

[5] In Example 3.5, we relax the assumption that all matches are accepted.

chooses a level of buyer entry N_B, or equivalently a market tightness θ, that maximizes the total social surplus. If there exists a unique solution, we denote it by θ^P and call it the social optimum. We say that a decentralized equilibrium allocation is *constrained efficient* if and only if $\theta^* = \theta^P$ where θ^* is the equilibrium market tightness and θ^P is the social optimum.

The social surplus per seller $\Omega(\theta)$ is

$$\Omega(\theta) = m(\theta)y(\theta) + (1 - m(\theta)z - c\theta. \tag{1}$$

The first-order condition for the social planner's problem is

$$\Omega'(\theta) = m'(\theta)y(\theta) + m(\theta)y'(\theta) - m'(\theta)z - c = 0. \tag{2}$$

We can define $f(\theta) \equiv m(\theta)y(\theta)$, the expected market output of a seller. This function is a natural extension of the matching function to environments with endogenous match output.[6] In terms of the function $f(.)$, the first-order condition is

$$\Omega'(\theta) = f'(\theta) - m'(\theta)z - c = 0. \tag{3}$$

To ensure the existence of a unique solution to the social planner's problem, we make the following assumption.

Assumption 2. *The function $f(.)$ has the following properties: (i) $\lim_{\theta\to 0} f(\theta) = 0$; (ii) $\lim_{\theta\to 0} f'(\theta) - z > c$; (iii) $\lim_{\theta\to\infty} f'(\theta) \leq 0$; and (iv) $f''(\theta) - m''(\theta)z < 0$ for all $\theta \in \mathbb{R}_+$.*

Assumptions 1 and 2 imply that $\lim_{\theta\to 0} \Omega'(\theta) > 0$, $\lim_{\theta\to\infty} \Omega'(\theta) < 0$, and $\Omega''(\theta) < 0$. Therefore, there exists a unique social optimum $\theta^P > 0$ where $\Omega'(\theta^P) = 0$. Note that if all of the conditions except (iv) are satisfied, we have existence but not uniqueness.

The expected joint match surplus created by each match is $s(\theta) \equiv y(\theta) - z$.[7] In terms of $s(\theta)$, the first-order condition for the social planner's problem is

$$\Omega'(\theta) = m'(\theta)s(\theta) + m(\theta)s'(\theta) - c = 0. \tag{4}$$

Rearranging (4), the social planner's solution θ^P satisfies the following:

$$\frac{m'(\theta)\theta}{m(\theta)} + \frac{s'(\theta)\theta}{s(\theta)} = \frac{c\theta}{m(\theta)s(\theta)}. \tag{5}$$

Now let $\eta_m(\theta) \equiv m'(\theta)\theta/m(\theta)$, the elasticity of the matching probability $m(\theta)$ with

[6] In many of our examples, $f(.)$ is actually the primitive function and we define $y(\theta) \equiv f(\theta)/m(\theta)$.
[7] Ljungqvist and Sargent (2017) call this the *fundamental surplus*.

respect to θ. We call this the *matching elasticity.* Let $\eta_s(\theta) \equiv s'(\theta)\theta/s(\theta)$, the elasticity of the expected joint match surplus, $s(\theta)$. We call this the *surplus elasticity.* Substituting into (5), the social optimum θ^P solves

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{buyers' surplus share}}. \tag{6}$$

Since there is free entry of buyers, the expected payoff per buyer equals the cost of entry c and the term on the right of (6) equals buyers' total *surplus share,* $cN_B/m(\theta)s(\theta)N_S$. Condition (6) says that the social planner chooses the market tightness θ^P that sets the buyers' surplus share equal to the matching elasticity *plus* the surplus elasticity. Since θ^P is unique, we have constrained efficiency, i.e. $\theta^* = \theta^P$, if and only if θ^* also satisfies condition (6).

We call this the *generalized Hosios condition* because it generalizes the standard Hosios condition to static search and matching environments where the expected match output depends directly on the market tightness. In special cases where the surplus elasticity $\eta_s(\theta)$ equals zero – for example, when the expected match output is *exogenous* – we recover the standard Hosios condition: the matching elasticity with respect to buyers must equal their surplus share. In general, however, buyers' surplus share must equal the matching elasticity *plus* the surplus elasticity. This condition ensures that agents are paid for their contribution to both *match creation* and *surplus creation.*

Since the buyers' surplus share and the sellers' surplus share add to one, the social optimum θ^P must also satisfy

$$1 - \underbrace{\eta_m(\theta)}_{\text{matching elasticity}} - \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{\pi(\theta)}{m(\theta)s(\theta)}}_{\text{sellers' surplus share}} \tag{7}$$

where $\pi(\theta)$ is the expected payoff for sellers. If there is free entry of *sellers* at cost κ instead of free entry of buyers, then substituting $\pi(\theta) = \kappa$ into the above equation delivers the generalized Hosios condition for seller entry.

Output externality. In search and matching models with free entry of buyers, there are two standard externalities related to the frictional matching process: the congestion and thick market externalities.[8] In general, these search externalities are fully internalized when the Hosios condition holds. In environments where the expected match output depends on

[8] The congestion externality is a negative externality that arises because a higher buyer/seller ratio reduces the matching probability of each buyer. The thick market externality is a positive externality that arises because a higher buyer/seller ratio increases the matching probability of each seller.

market tightness, however, a novel externality arises. Depending on the specific environment, a higher buyer/seller ratio may either increase or decrease the expected match output. We call this the *output externality* and it may be either positive or negative. When the standard Hosios condition holds, buyers' entry decisions fail to internalize the output externality and entry may not be constrained efficient. To ensure that entry is efficient, we need the generalized Hosios condition. When this condition is satisfied, buyers' entry decisions internalize both the search externalities *and* the output externality.

2.2 Dynamic economy

Consider a continuous-time dynamic environment. In period t, there is measure one of risk-neutral sellers and a large number of risk-neutral potential buyers. The measure of risk-neutral buyers who enter is denoted by v_t. There is a measure u_t of unmatched sellers in period t and the market tightness is defined by $\theta_t \equiv v_t/u_t$. We have in mind a labor market interpretation where buyers are vacancies and sellers are unemployed workers.

There is free entry of buyers who pay a cost $c > 0$ each period when unmatched. At the start of each period, buyer-seller matches are destroyed at an exogenous rate $\delta \in (0,1]$.[9] Future payoffs are discounted at a rate $r > 0$.

In continuous time, $m(\theta_t)$ and $m(\theta_t)/\theta_t$ are now arrival rates rather than matching probabilities for buyers and sellers respectively, and thus Assumption 1 needs to be amended.

Assumption 1a. *The function $m(.)$ has the following properties: (i) $m'(\theta) > 0$ and $m''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$, (ii) $\lim_{\theta\to 0} m(\theta) = 0$, (iii) $\lim_{\theta\to 0} m'(\theta) = +\infty$, (iv) $\lim_{\theta\to +\infty} m(\theta) = +\infty$, (v) $\lim_{\theta\to\infty} m'(\theta) = 0$, and (vi) $m(\theta)/\theta$ is strictly decreasing in θ for all $\theta \in \mathbb{R}_+$.*

The expected output for a new match *created* in period t is $y(\theta_t)$. The flow payoff for unmatched sellers is $z \geq 0$ for all t. We assume that $y(\theta_t) > z$ for all $\theta_t \in \mathbb{R}_+$ and all matches are accepted. Let y_t denote the average match output across all matches during period t. Note that y_t is not equal to $y(\theta_t)$, since y_t is a weighted average across *all* active matches, i.e. both newly formed matches and existing matches that have survived from previous periods.

In the Appendix, we derive the following law of motion for y_t:

$$\dot{y}_t = \underbrace{(y(\theta_t) - y_t)}_{\text{difference in expected output}} \underbrace{\frac{m(\theta_t)u_t}{1-u_t}}_{\text{share of } new \text{ matches}} \tag{8}$$

[9] In the special case $\delta = 1$, the economy features non-enduring matches, i.e. all matches are destroyed at the end of each period. All of the following results therefore apply to economies with non-enduring matches.

Intuitively, $\dot{y}_t$ is equal to the difference in expected output between new matches and *all* active matches, weighted by the share of active matches that are new.

Now let Ω denote the social surplus, given by

$$\Omega = \int_0^\infty e^{-rt}((1-u_t)y_t + zu_t - c\theta_t u_t)dt. \tag{9}$$

Given initial conditions u_0 and y_0, the social planner chooses θ_t for all $t \in \mathbb{R}_+$ to maximize (9) subject to the following constraints:

$$\dot{u}_t = \delta(1-u_t) - m(\theta_t)u_t \tag{10}$$

$$\dot{y}_t = (y(\theta_t) - y_t)\frac{m(\theta_t)u_t}{1-u_t}. \tag{11}$$

In the proof of Proposition 1, we solve the current value Hamiltonian for this problem. We focus on steady state solutions where $\dot{u}_t = \dot{y}_t = 0$ and $\dot{\theta}_t = 0$. Before presenting Proposition 1, we first determine the steady state expected joint match surplus $s(\theta)$ since the solution will be stated in terms of this.

Let V_S and V_B denote the steady state asset values for matched sellers and buyers respectively, and let U_S and U_B denote the steady state asset values for unmatched sellers and buyers respectively. In steady state, the expected joint match surplus is $s(\theta) \equiv V_B + V_S - U_B - U_S$. Using the Bellman equations, and the fact that $U_B = 0$ with free entry of buyers, Lemma 1 provides a useful expression for the expected match surplus $s(\theta)$ in the dynamic economy.[10]

Lemma 1. *With free entry of buyers, the steady state expected match surplus is*

$$s(\theta) = \frac{y(\theta) - z + c\theta}{r + \delta + m(\theta)}. \tag{12}$$

We are now in a position to present a necessary condition for efficiency.

Proposition 1. *Any steady state social optimum θ^P must satisfy*

$$\eta_m(\theta) + \frac{y'(\theta)\theta}{(r+\delta)s(\theta)} = \frac{c\theta}{m(\theta)s(\theta)}. \tag{13}$$

Lemma 2 states that the above condition (13) is equivalent to the generalized Hosios condition (6) derived in the static environment, except that $s(\theta)$ is now the steady state expected match surplus given by (12).

[10] In the static economy, $V_B + V_S = y(\theta)$ and $U_S = z$, so we have $s(\theta) = y(\theta) - z$.

Lemma 2. *Condition (13) is equivalent to*

$$\eta_m(\theta) + \eta_s(\theta) = \frac{c\theta}{m(\theta)s(\theta)}. \tag{14}$$

Define the function $\Lambda(.)$ by $\Lambda(\theta) \equiv m(\theta)s(\theta)$. Condition (14) is equivalent to $\Lambda'(\theta) = c$. To ensure existence and uniqueness of a social optimum, we make the following assumption. Note that if all the conditions except (iv) are satisfied, we have existence but not uniqueness.

Assumption 2a. *The function* $\Lambda(.)$ *has the following properties: (i)* $\lim_{\theta\to 0}\Lambda(\theta) = 0$; *(ii)* $\lim_{\theta\to 0}\Lambda'(\theta) > c$; *(iii)* $\lim_{\theta\to\infty}\Lambda'(\theta) \leq 0$*; and (iv)* $\Lambda''(\theta) < 0$ *for all* $\theta \in \mathbb{R}_+$.

If Assumption 2a holds, there exists a unique $\theta^P > 0$ that satisfies $\Lambda'(\theta) = c$ and therefore also the necessary condition (13). Lemma 3 follows by using Arrow's sufficiency theorem to prove that θ^P is indeed a global maximum for Ω.

Lemma 3. *There exists a unique social optimum* $\theta^P > 0$*.*

Proposition 2 generalizes the standard Hosios condition to dynamic search and matching environments where the expected match output depends on the market tightness. While the equivalent condition (13) may be easier to apply in practise, condition (15) is simpler and more intuitive. To achieve constrained efficiency, buyers' surplus share must equal the *matching elasticity* plus the *surplus elasticity.* When this condition holds, buyers' entry decisions fully internalize both the search externalities and the output externality.

Proposition 2 (**Generalized Hosios Condition**)**.** *A steady state equilibrium allocation is constrained efficient if and only if* θ^* *satisfies*

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{buyers' surplus share}}. \tag{15}$$

In the special case where match output is constant, i.e. $y(\theta) = \bar{y} \in \mathbb{R}_+$, it is well-known that the standard Hosios condition applies. That is, we have constrained efficiency if the matching elasticity equals buyers' surplus share. In a dynamic setting, where $s(\theta)$ is given by (12), it is not immediately clear that $\eta_s(\theta^*) = 0$ in this case. However, it is clear from the equivalent condition (13) that the standard Hosios condition applies whenever $y'(\theta^*) = 0$.

2.2.1 Direction of inefficiency

Depending on the specific environment, the surplus elasticity may be either positive or negative. This means that simply applying the standard Hosios condition may result in either over-entry or under-entry of buyers relative to the social optimum. Corollary 1 tells us that the direction of the inefficiency depends only on the output technology $y(\theta)$. In particular, the direction of the inefficiency depends on whether the expected match output $y(\theta)$ is increasing or decreasing in the buyer/seller ratio at the equilibrium θ^*.

Corollary 1. *There is under-entry (over-entry) of buyers under the standard Hosios condition if and only if* $y'(\theta^*) > (<)\ 0$.

When $y'(\theta^*) > 0$, the output externality arising from buyer entry is *positive* and the standard Hosios condition results in under-entry. Alternatively, if $y'(\theta^*) < 0$, the output externality from buyer entry is *negative* and the standard Hosios condition results in over-entry. If $y'(\theta^*) = 0$, there is no output externality and entry is constrained efficient under the standard Hosios condition.

When there is seller entry instead of buyer entry, the direction of the effect of entry is reversed. If there is free entry of *sellers* at cost κ, the analogue of (15) is

$$1 - \underbrace{\eta_m(\theta)}_{\text{matching elasticity}} - \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{\kappa}{m(\theta)s(\theta)}}_{\text{sellers' surplus share}} . \tag{16}$$

Corollary 2. *There is over-entry (under-entry) of sellers under the standard Hosios condition if and only if* $y'(\theta^*) > (<)\ 0$.

When $y'(\theta^*) > 0$, the output externality arising from seller entry is *negative* since $\theta = N_B/N_S$ and therefore $y(.)$ is decreasing in the measure of sellers N_S. In this case, the standard Hosios condition results in over-entry. If $y'(\theta^*) < 0$, the output externality from seller entry is *positive* and the standard Hosios condition results in under-entry. If $y'(\theta^*) = 0$, there is no output externality and entry is constrained efficient under the standard Hosios condition.

3 Examples

This section presents a number of different examples of search and matching environments in order to demonstrate the wide applicability of the generalized Hosios condition. For concreteness, we focus mainly on labor market environments in which sellers and buyers are unemployed workers and firms (or vacancies), but the results apply more generally. For simplicity, we focus on static environments except when a dynamic setting is necessary to illustrate a particular feature.

We first consider some examples in which prices are determined by Nash bargaining. In these examples, the generalized Hosios condition applies but it holds only in a knife-edge special case and we do not generally have constrained efficiency. Next, we consider some examples in which prices are determined through directed or competitive search. In these examples, the generalized Hosios condition holds endogenously and we have constrained efficiency. We also provide one example in which the generalized Hosios condition applies but it does not hold and therefore the economy is *not* constrained efficient – even though it features competitive search.

In the Appendix, we consider an example where the expected match output is endogenous in the sense that it depends on both ex ante capital investment *and* the market tightness.

3.1 Endogenous match output

Consider a simple, static Diamond-Mortensen-Pissarides (DMP) style environment where meetings are bilateral and wages are determined by Nash bargaining.[11] The measure of vacancies or firms is V, the measure of unemployed workers is U, and the labor market tightness is $\theta \equiv V/U$. While the environment is otherwise standard, we assume for now that the expected match output depends directly on the market tightness θ. In the following examples, we will see how this dependence can arise naturally.

There is free entry of firms (or vacancies) at a cost $c > 0$. The matching probabilities for workers and firms are $m(\theta)$ and $m(\theta)/\theta$ respectively where $m(.)$ satisfies Assumption 1. Workers' bargaining parameter is $\beta \in (0, 1)$ and the value of non-market activity is z where $y(\theta) > z$ for all $\theta \in \mathbb{R}_+$ and all matches are accepted.[12]

The expected match surplus is $s(\theta) = y(\theta) - z$. Under certain conditions,[13] there exists a unique equilibrium $\theta^* > 0$ that satisfies

$$\frac{m(\theta)}{\theta}(1-\beta)(y(\theta)-z) = c. \tag{17}$$

Equivalently, the equilibrium market tightness $\theta^* > 0$ satisfies

$$\underbrace{1-\beta}_{\text{firms' bargaining power}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{firms' surplus share}}. \tag{18}$$

If Assumption 2 is satisfied, there exists a unique social optimum $\theta^P > 0$. Applying the generalized Hosios condition in Proposition 2, and using (18), the economy is constrained

[11] The classic references are Mortensen and Pissarides (1994) and Pissarides (2000).

[12] In Example 3.5, we relax the assumption that all matches are accepted.

[13] Let $\Lambda(\theta) = m(\theta)s(\theta)$. If $\eta_\Lambda(\theta) < 1$ for all $\theta \in \mathbb{R}_+$, $\lim_{\theta\to\infty} \Lambda(\theta)/\theta = 0$, and $c < (1-\beta)\lim_{\theta\to 0} \Lambda(\theta)/\theta$, then there exists a unique equilibrium $\theta^* > 0$. Note that $\eta_\Lambda(\theta) < 1$ if and only if $\Lambda(\theta)/\theta$ is strictly decreasing.

efficient if and only if θ^* satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{1-\beta}_{\text{firms' bargaining power}}. \tag{19}$$

Intuitively, the economy is efficient only when firms are paid for their contribution to both the *number* of matches and the *value* of the expected match surplus. Corollary 1 says that the standard Hosios condition may result in either under-entry or over-entry of firms, depending on whether the output externality is positive or negative, i.e. $y'(\theta^*) > 0$ or $y'(\theta^*) < 0$.

Example 3.1.1

Suppose the expected match output is *exogenous*, i.e. $y(\theta) = \bar{y} > z$ for all $\theta \in \mathbb{R}_+$. Output per match may be either constant or stochastic provided that the *expected* match output does not depend on the market tightness θ. For example, we could have match-specific productivities y drawn from an exogenous distribution G with $E_G(y) = \bar{y}$. According to (19), we have constrained efficiency only when the equilibrium θ^* satisfies

$$\eta_m(\theta) = 1 - \beta. \tag{20}$$

This well-known version of the Hosios condition is a special case of (19). When the expected match output is exogenous, we have constrained efficiency only in the knife-edge case where firms' bargaining power equals the matching elasticity $\eta_m(\theta)$ at the equilibrium θ^*.

3.2 Many-on-one meetings and worker choice

In this example, both the fact that the expected match output $y(\theta)$ depends directly on the market tightness θ, and the specific properties of the function $y(.)$, are not assumptions: the function $y(.)$ and its properties arise *endogenously*.

When there are multilateral or many-on-one meetings (i.e. many buyers meet one seller, or many sellers meet one buyer), dependence of the expected match output on market tightness can arise when there is *ex post* heterogeneity (i.e. after meetings but before trade) on the "many" side of the market. This is because such meetings give rise to the possibility of *choice* among potential trading partners. For example, when a seller can choose one buyer among many heterogeneous buyers in a meeting, a greater number of buyers per seller means that sellers can be more *selective*, thereby increasing the expected match output. We call this the *selection channel* and it features in many of our examples.

Consider a simple model with identical workers and ex ante identical firms. There is free entry of firms at a cost $c > 0$. The measure of vacancies or firms is V, the measure of unemployed workers is U, and the labor market tightness is $\theta \equiv V/U$. Unemployed workers and firms are matched according to a Poisson meeting technology where $P_n(\theta) = \frac{\theta^n e^{-\theta}}{n!}$ is the probability that n firms approach a given worker. The matching probabilities for workers and firms are $m(\theta) = 1 - e^{-\theta}$ and $m(\theta)/\theta$ respectively, where $m(.)$ satisfies Assumption 1. Wages are determined by Nash bargaining with workers' bargaining power $\beta \in (0,1)$ and the value of non-market activity is $z = 0$.

After meetings occur, firms draw a productivity. The probability that a firm is low productivity is α and the probability that a firm is high productivity is $1-\alpha$. Low productivity firms produce output x_L and high productivity firms produce output $x_H > x_L$. Workers observe the productivity of each firm that approaches and choose to work for one of them.

Wages equal $w_H = \beta x_H$ for high productivity firms and $w_L = \beta x_L$ for low productivity firms. We assume that $z < \beta x_L$, so workers always choose to work for the highest productivity firm that approaches. Since workers are only hired by low productivity firms if *all* of the n firms approaching are low productivity, the probability of a worker being hired by a low productivity firm, conditional on being hired, is

$$\Pr(x = x_L) = \frac{\sum_{n=1}^{\infty} P_n(\theta)\alpha^n}{1 - P_0(\theta)} = \frac{e^{-\theta(1-\alpha)} - e^{-\theta}}{1 - e^{-\theta}} \tag{21}$$

and the probability of being hired by a high productivity firm is

$$\Pr(x = x_H) = \frac{1 - e^{-\theta(1-\alpha)}}{1 - e^{-\theta}}. \tag{22}$$

The expected market output per capita $f(\theta)$ is given by

$$f(\theta) = (1 - e^{-\theta(1-\alpha)})x_H + (e^{-\theta(1-\alpha)} - e^{-\theta})x_L \tag{23}$$

and the expected match output, or average labor productivity, is $y(\theta) = f(\theta)/m(\theta)$, i.e.

$$y(\theta) = \frac{(1 - e^{-\theta(1-\alpha)})x_H + (e^{-\theta(1-\alpha)} - e^{-\theta})x_L}{1 - e^{-\theta}}. \tag{24}$$

The expected match surplus is $s(\theta) = y(\theta)$ since $z = 0$. It is straightforward to show that the expected match output is increasing in the market tightness, i.e. $y'(\theta) > 0$. Intuitively, this is because a greater number of firms per worker allows workers to be more *selective*.

The equilibrium market tightness θ^* is determined by the zero profit condition (17) where $z = 0$ and $y(\theta)$ is given by (24). There exists a unique social optimum $\theta^P > 0$ if Assumption

2 is satisfied. Applying the generalized Hosios condition in Proposition 2, the economy is constrained efficient if and only if θ^* satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{1-\beta}_{\text{firms' bargaining power}} . \tag{25}$$

Since $y'(\theta^*) > 0$, Corollary 1 says the standard Hosios rule would result in *under-entry* of firms or, equivalently, inefficiently high unemployment. The standard Hosios condition sets workers' bargaining power β too high: it does not incorporate the fact that greater entry of firms leads not only to lower unemployment for workers, but also a higher average labor productivity. Under the generalized Hosios condition, however, firms' entry decisions internalize both the effect on unemployment and the positive output externality. Since wages are determined by Nash bargaining, this condition holds only in a knife-edge special case.

3.3 Effect of labor market outcomes on goods market

Another way in which the expected match output can depend on the labor market tightness is when there is an *additional market*, such as a goods market, and the possibility of trade in that market depends on the matching outcomes in the labor market. A classic example can be found in Berentsen, Menzio, and Wright (2011), which features both a labor market and a goods market. We present a static, highly simplified version of that model.[14]

Workers first sell their labor to firms in the labor market and then purchase goods from firms in the goods market. Importantly, while all workers can search in the goods market, only *active* firms (i.e. filled vacancies) can produce and trade in the goods market. In this way, the labor market tightness affects the goods market tightness by affecting the measure of firms who search in the goods market. In turn, the goods market tightness determines the probability of trade for both workers and firms. This implies that the labor market tightness affects the expected match "output" since it includes both the direct match output in the labor market *and* the expected gains from trade in the goods market.

The labor market is identical to that in Example 3.1, where the labor market tightness is $\theta = V/U$ and the matching probabilities for workers and firms respectively are $m(\theta)$ and $m(\theta)/\theta$. All matches produce direct output $y > 0$ and vacancies enter at a cost $\kappa > 0$. Wages are determined by Nash bargaining with workers' bargaining power $\beta \in (0,1)$. Similarly to

[14] In particular, we simplify the model in Berentsen et al. (2011) by eliminating the third market, the Arrow-Debreu market, since it is unnecessary in the static model considered here.

(17), the free entry condition is

$$\frac{m(\theta)}{\theta}(1-\beta)s(\theta)=\kappa, \tag{26}$$

where $s(\theta)$ is the expected match surplus *including the expected gains from trade in the goods market*. This will be determined below.

In the goods market, the probabilities of trade for workers and firms respectively are $m^G(\phi)$ and $m^G(\phi)/\phi$, where ϕ is the goods market tightness. Since all workers (including the unemployed) search but only active firms search (i.e. those that have successfully hired a worker), we have $\phi=(m(\theta)/\theta)V/U=m(\theta)$. Since the unemployment rate is $u(\theta)=1-m(\theta)$, we have $\phi(\theta)=1-u(\theta)$.

Active firms can produce a single unit of an indivisible good at a production cost $c>0$. For simplicity, we assume the good is sold at a fixed price $p>c$.[15] Unemployed workers value the good at $v_u>0$ and employed workers value the good at $v_e>v_u$. If we assume, for simplicity, that $p=v_u$ (i.e. unemployed workers do not gain any surplus), the expected match surplus in the labor market is $s(\theta)=y(\theta)-z$ where

$$y(\theta)=\underbrace{y}_{\text{direct output}}+\underbrace{\frac{m^G(\phi(\theta))}{\phi(\theta)}(p-c)}_{\text{firms' expected gains from trade}}+\underbrace{m^G(\phi(\theta))(v_e-p)}_{\text{workers' expected gains from trade}}. \tag{27}$$

Applying the generalized Hosios condition found in Proposition 2, and using (26), we have constrained efficiency if and only if the equilibrium labor market tightness θ^* satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}}+\underbrace{\eta_s(\theta)}_{\text{surplus elasticity}}=\underbrace{1-\beta}_{\text{firms' bargaining power}}. \tag{28}$$

Now consider whether the output externality is positive or negative, i.e. $y'(\theta^*)>0$ or $y'(\theta^*)<0$. Since $\phi'(\theta)>0$, $\frac{dm^G(\phi)}{d\phi}>0$, and $\frac{d(m^G(\phi)/\phi)}{d\phi}<0$, the effect of an increase in the labor market tightness θ on the expected match output $y(\theta)$ can be broken into two components: a *negative* effect on firms' expected gains from trade, and a *positive* effect on workers' expected gains from trade in the goods market. If $p=c$, we have $y'(\theta)>0$, while if $v_e=p$, we have $y'(\theta)<0$. In general, the output externality may be either positive or negative depending on the terms of trade in the goods market. This is because the price in the goods market determines whether firms' or workers' expected gains from trade receive a higher weighting in expression (27).[16]

[15] Alternatively, the price p could be determined by Nash bargaining in the goods market.

[16] Note that when unemployed and employed workers have the same valuation, i.e. $v_e=v_u$, we have

Differentiating (27), it is straightforward to show that $y'(\theta) < 0$ if and only if

$$\frac{\eta_m^G(\phi)}{1-\eta_m^G(\phi)} < \left(\frac{p-c}{v_e - p}\right)\frac{1}{\phi} \tag{29}$$

where $\eta_m^G(\phi) \equiv \frac{dm^G(\phi)}{d\phi}\frac{\phi}{m^G(\phi)}$, the matching elasticity in the goods market. Since $\phi(\theta) = 1 - u(\theta)$, we have $\phi < 1$ for $\theta > 0$ and thus a sufficient condition for $y'(\theta) < 0$ is

$$\frac{\eta_m^G(\phi)}{1-\eta_m^G(\phi)} \leq \frac{p-c}{v_e - v_u}. \tag{30}$$

Note that if the price p of the good were instead determined by Nash bargaining in the goods market, with workers' bargaining power β_G, this condition is equivalent to

$$\eta_m^G(\phi) \leq 1 - \beta_G. \tag{31}$$

In particular, this means that if the Hosios condition held in the *goods* market, i.e. if $\eta_m^G(\phi) = 1 - \beta_G$, the above condition would clearly hold and therefore $y'(\theta) < 0$.

When $y'(\theta^*) < 0$, there is a *negative* output externality from firm entry in the labor market. The standard Hosios condition would result in *over-entry* of vacancies and the equilibrium unemployment rate would be inefficiently low. Instead, what is needed for constrained efficiency is the generalized Hosios condition. Since Berentsen et al. (2011) impose the standard Hosios condition to calibrate their model, this may be quantitatively important.

3.4 Applicant ranking and interviews

Consider an environment where multiple workers can apply for the same job. If there are many-on-one meetings and heterogeneous applicants for the same vacancies, the generalized Hosios condition will typically be required for constrained efficiency whenever there is a non-random selection mechanism. The method of selecting applicants may be *imperfect* (i.e. the "best" applicant may not always be chosen), provided that the expected match output is increasing in the number of applicants per vacancy due to the *selection channel*.

One example can be found in Gavrel (2012), which develops a model of applicant ranking. There is free entry of firms or vacancies. Workers apply to firms and firms rank applicants according to the degree of (match-specific) *mismatch* between the worker and the firm. The worker with the least "mismatch" is hired by the firm. As in Marimon and Zilibotti (1999),

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$s(\theta) = y - z + \frac{m^G(\phi)}{\phi}(p-c)$. That is, only firms' expected gains from trade in the goods market appears in the match surplus. This is because the expected gains from trade for employed and unemployed are equal.

the degree of mismatch x is measured by the distance on a circle between a worker and a firm. Let $y(x)$ be the match output given x where $y'(x) < 0$. The expected match output is

$$y(\theta) = \int_0^{1/2} y(x)\rho(x, \theta)dx \tag{32}$$

where $\rho(x, \theta)$ is the density of mismatch among filled vacancies.

The selection channel gives rise to a *negative* output externality from firm entry via the applicant ranking mechanism. Gavrel (2012) proves that $y'(\theta) < 0$ for all $\theta \in \mathbb{R}_+$. Intuitively, a greater number of vacancies per unemployed worker (higher θ) implies *fewer* applicants per vacancy (lower $1/\theta$), which increases the expected degree of mismatch between the best applicant and the firm. As θ increases, firms are less selective and the greater resulting mismatch between workers and firms reduces the expected match output $y(\theta)$. Gavrel shows that applicant ranking leads to an *over-entry* of vacancies (i.e. job creation is inefficiently high) when wages are determined by Nash bargaining and the standard Hosios condition is imposed. That is, the unemployment rate is inefficiently *low* under the standard Hosios condition. Since $y'(\theta^*) < 0$, this is consistent with Corollary 1. To obtain constrained efficiency, what is needed is the generalized Hosios condition found in Proposition 2.

Another example, which features an "imperfect" selection method, is the use of *interviews*. Wolthoff (2017) develops a detailed search model of the labor market in which firms' recruitment intensity is endogenous and this determines the maximum number of interviews per vacancy. Consider a highly simplified environment where the maximum number of interviews is exogenous. Workers are ex ante heterogeneous with respect to productivity and firms are identical. Workers randomly apply for one job and firms can receive many applications from different workers. Firms cannot directly observe workers' productivities so interviews are necessary to reveal an applicant's productivity. A maximum of $n_R \geq 1$ applicants can be interviewed. If a firm receives $n \geq 1$ applications, they randomly choose $n_I = \min\{n, n_R\}$ applicants to interview and then select the best applicant among those interviewed.

With interviews, the *selection channel* arises whenever $n_R \geq 2$, despite the fact that the method of selection is imperfect. Even if $n_R = 2$, a higher θ implies a lower probability that a firm receives two applications, i.e. $n = 2$, and therefore firms can be less selective. While the best applicant is not always hired (since they may not be interviewed at all), the expected match output $y(\theta)$ still depends on the market tightness θ. In particular, we have $y'(\theta) < 0$ since a greater number of firms per unemployed worker (higher θ) implies *fewer* applicants per vacancy and therefore firms can be less selective. The generalized Hosios condition is thus necessary for constrained efficiency of entry in such environments.

3.5 Endogenous job acceptance

The expected match output can also depend on the market tightness when agents make a decision about whether or not to enter the market, or whether to accept or reject a match, and that decision depends on the market tightness. In the next two examples, we consider such environments. Importantly, we assume that in choosing the market tightness θ, the social planner is constrained not only by the matching frictions but also by the entry or acceptance decision rules that agents would choose in the decentralized equilibrium. This is because the function $y(.)$ arises as a consequence of these entry or acceptance decisions. Since the planner takes both the matching technology $m(.)$ *and* the output technology $y(.)$ as given, the planner is constrained by these decision rules.[17]

Consider the steady state of a continuous-time dynamic DMP style model with random, bilateral meetings. Workers and firms discount future payoffs at a rate $r > 0$. The market tightness is $\theta = V/U$ and workers' arrival rate for meetings is $m(\theta)$. After workers and firms meet, a match-specific productivity y is drawn from a distribution with cdf G and density $g = G'$ where $g(y) > 0$ for all $y \in [0, 1]$. After observing the productivity y, workers and firms decide whether to accept the match. There is free entry of vacancies at cost $c > 0$ and matches are destroyed at an exogenous rate $\delta > 0$.

A job with match-specific productivity y is acceptable to both worker and firm if and only if the match surplus $S(y) \geq 0$.[18] There is a cut-off productivity y^* such that all jobs with productivity $y \geq y^*$ are acceptable to both workers and firms. We write $y^*(\theta)$ since the equilibrium cut-off productivity will depend on the value of unemployment U_S and therefore on the market tightness. The probability a match is *acceptable* is $a(\theta) = 1 - G(y^*(\theta))$.

The expected match output $y(\theta)$ across *all* acceptable matches is

$$y(\theta) = \int_{y^*(\theta)}^{1} \frac{ydG(y)}{1 - G(y^*(\theta))}, \tag{33}$$

where the equilibrium cut-off productivity $y^*(\theta)$ is given by equating $S(y^*) = 0$. With free entry of firms, it can be shown that $S(y) = \frac{y - rU_S}{r+\delta}$, so we have $y^*(\theta) = rU_S$. Clearly, since the cut-off productivity $y^*(\theta)$ depends on the market tightness, the expected match output $y(\theta)$ also depends on the market tightness.

Suppose that wages are determined by Nash bargaining where workers' bargaining power is β and the flow value of non-market activity is z. Similarly to (17), the steady state

[17] In these two examples, the constrained efficiency is "doubly constrained" since the planner's problem is solved subject to an additional constraint which is one of the equilibrium conditions.

[18] The productivity-specific match surplus $S(y)$ is defined by $S(y) \equiv V_S(y) + V_B(y) - U_S - U_B$ where the Bellman equations for V_S and V_B found in the Appendix are adjusted to be productivity-specific.

equilibrium market tightness θ^* is given by the free entry condition

$$\frac{\hat{m}(\theta)}{\theta}(1-\beta)\hat{s}(\theta) = c, \tag{34}$$

where $\hat{m}(\theta) \equiv a(\theta)m(\theta)$, the matching probability for workers *adjusted by the probability of acceptance*, and $\hat{s}(\theta)$ is the expected match surplus for acceptable matches.[19]

The social planner chooses θ^P to maximize the total social surplus net of entry costs. Importantly, we assume the planner uses the same cut-off productivity rule as in the equilibrium, i.e. $y^*(\theta) = rU_S$.[20] Solving the planner's problem yields an adjusted version of the generalized Hosios condition.[21] If Assumption 2a holds for $\Lambda(\theta) \equiv \hat{m}(\theta)\hat{s}(\theta)$, we have constrained efficiency if and only if θ^* satisfies

$$\underbrace{\eta_{\hat{m}}(\theta)}_{\text{adjusted matching elasticity}} + \underbrace{\eta_{\hat{s}}(\theta)}_{\text{adjusted surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)\hat{s}(\theta)}}_{\text{firms' surplus share}} \tag{35}$$

where $\eta_{\hat{s}}(\theta) \equiv \hat{s}'(\theta)\theta/\hat{s}(\theta)$. This is consistent with Proposition 2, except that $\hat{m}(\theta)$ and $\hat{s}(\theta)$ have been adjusted to incorporate the acceptance probability $a(\theta)$.

Since $\hat{m}(\theta) = a(\theta)m(\theta)$, condition (35) is equivalent to

$$\eta_m(\theta) + \eta_a(\theta) + \eta_{\hat{s}}(\theta) = \frac{c\theta}{m(\theta)\hat{s}(\theta)} \tag{36}$$

where $\eta_a(\theta) \equiv a'(\theta)\theta/a(\theta)$. Using a modified version of Lemma 2, we have

$$\eta_{\hat{s}}(\theta) = \frac{y'(\theta)\theta}{(r+\delta)\hat{s}(\theta)}, \tag{37}$$

and differentiating $a(\theta) = 1 - G(y^*(\theta))$ yields

$$\eta_a(\theta) = -\frac{g(y^*)\frac{dy^*}{d\theta}\theta}{1-G(y^*)}. \tag{38}$$

Using (38) and (37), and the fact that

$$y'(\theta) = (y(\theta) - y^*(\theta))\frac{g(y^*)\frac{dy^*}{d\theta}\theta}{1-G(y^*)}, \tag{39}$$

[19] The adjusted steady state expected match surplus is given by $\hat{s}(\theta) = \frac{y(\theta)-z+c\theta}{r+\delta+\hat{m}(\theta)}$.

[20] Alternatively, we could allow the planner to choose *both* the cut-off productivity and the market tightness independently, but in that case the standard Hosios condition clearly applies.

[21] All of the results for this example can be easily obtained by modifying the proofs of Lemmas 1, 2, and 3, as well as Propositions 1 and 2, so that $m(\theta)$ is replaced by $\hat{m}(\theta) = a(\theta)m(\theta)$ throughout.

we obtain the following:

$$\eta_a(\theta) + \eta_{\hat{s}}(\theta) = \left(\frac{y(\theta) - y^*(\theta)}{(r+\delta)\hat{s}(\theta)} - 1 \right) \frac{g(y^*)\frac{dy^*}{d\theta}\theta}{1 - G(y^*)}. \tag{40}$$

Combining $y^*(\theta) = rU_S$ with the fact that $(r+\delta)\hat{s}(\theta) = y(\theta) - rU_S$, we have $(r+\delta)\hat{s}(\theta) = y(\theta) - y^*(\theta)$. Substituting into (40), we obtain $\eta_a(\theta) + \eta_{\hat{s}}(\theta) = 0$. Therefore, using condition (34), entry is constrained efficient if and only if θ^* satisfies

$$\eta_m(\theta) = 1 - \beta. \tag{41}$$

This is an example where the generalized Hosios condition (35) applies, since the expected match output $y(\theta)$ depends on the market tightness, but the standard Hosios condition is sufficient for constrained efficiency.[22] This is because there are two offsetting effects of an increase in the market tightness θ. First, there is an increase in the cut-off productivity y^*, which *decreases* the job acceptance probability $a(\theta)$ since workers are more selective. Second, the increase in y^* leads to an *increase* in the expected match surplus $\hat{s}(\theta)$ for acceptable matches, since these matches have higher productivity. The fact that these two effects *exactly offset* each other is reflected in the fact that $\eta_a(\theta) + \eta_{\hat{s}}(\theta) = 0$.

3.6 Ex ante heterogeneity and market composition

When there is *ex ante* heterogeneity among buyers or sellers, dependence of the expected match output on market tightness can arise naturally through market composition. If the market tightness influences the individual entry decisions of buyers or sellers that are ex ante heterogeneous with respect to characteristics that affect match output, then average output per match will depend on market tightness. We call this the *composition channel*.

Albrecht, Navarro, and Vroman (2010) consider an environment where workers are ex ante heterogeneous with respect to their market productivity and there is both firm entry and a labor force participation decision.[23] The authors show that such an environment can violate the standard Hosios rule: when workers' bargaining parameter satisfies the standard Hosios condition, there is *over-entry* of firms relative to the social optimum.[24] To illustrate the use of the generalized Hosios condition, we consider a related but simpler environment that features ex ante *firm* heterogeneity instead of worker heterogeneity.

[22] Note that Corollary 1 does not directly apply here since it assumes that all matches are accepted.

[23] Related literature following Albrecht et al. (2010) includes Gavrel (2011), Charlot, Malherbet, and Ulus (2013), and Masters (2015). See also Albrecht, Navarro, and Vroman (2009).

[24] Julien and Mangin (2017) show that the environment in Albrecht et al. (2010) with labor force participation features both an output externality and a *participation externality.*

Suppose there is a measure U of unemployed workers and a measure M of firms that may choose to search. Firms' productivities y are distributed according to a twice differentiable distribution with cdf G and density g where $G(0) = 0$ and $g(y) > 0$ for all $y \in [0, 1]$. Firms learn their own productivity before deciding whether to pay the entry cost $c > 0$ and search. Wages are determined by generalized Nash bargaining where workers' bargaining parameter is β and the value of non-market activity is zero. We assume that $c < 1 - \beta$.

Let V be the measure of *searching* firms and define $\theta = V/U$. Meetings are bilateral and the probabilities of matching for workers and firms are $m(\theta)$ and $m(\theta)/\theta$ respectively. A firm with productivity y chooses to pay the cost c to search for a worker if and only if

$$\frac{m(\theta)}{\theta}(1-\beta)y > c \tag{42}$$

and therefore the cut-off productivity for firm entry is

$$y^*(\theta) = \frac{c\theta}{(1-\beta)m(\theta)} \tag{43}$$

and average labor productivity is given by

$$y(\theta) = \int_{y^*(\theta)}^{1} \frac{ydG(y)}{1 - G(y^*(\theta))}. \tag{44}$$

The cut-off productivity y^* is increasing in θ since $m(\theta)/\theta$ is decreasing. This is intuitive: as the market tightness increases, the probability of finding a worker is lower so only high productivity firms choose to pay the cost c and search. At the same time, the average match output $y(\theta)$ is increasing in the cut-off productivity y^* and therefore $y'(\theta) > 0$ for all $\theta \in \mathbb{R}_+$.

The equilibrium market tightness θ^* satisfies

$$\theta = (1 - G(y^*(\theta)))\frac{M}{U} \tag{45}$$

where y^* is given by (43). Defining $R(\theta) \equiv 1 - G(y^*(\theta))$, the proportion of firms that choose to search, the equilibrium condition (45) is equivalent to $R(\theta)/\theta = U/M$.[25]

Suppose the social planner chooses a market tightness θ to maximize the total social surplus minus the total entry costs. As in the previous example, we assume the planner uses the same cut-off productivity rule $y^*(\theta)$ as in the decentralized economy. If Assumption 2 is satisfied, there exists a unique social optimum θ^P and we can apply the generalized Hosios

[25] Using (43) and Assumption 1, we have $\lim_{\theta \to 0} R(\theta)/\theta = \infty$ and $\lim_{\theta \to \infty} R(\theta)/\theta = 0$. Also, $R'(\theta) < 0$ and therefore there exists a unique equilibrium $\theta^* > 0$.

condition in Proposition 2. We have constrained efficiency if and only if θ^* satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{firms' surplus share}} . \tag{46}$$

In this example, the *composition channel* ensures that the expected match output depends on the market tightness. The threshold y^* is increasing in θ, leading to a positive output externality from firm entry, i.e. $y'(\theta) > 0$. Since $y'(\theta^*) > 0$, Corollary 1 implies that there is under-entry of firms under the standard Hosios condition.

3.7 Competitive search with endogenous match output

Unlike DMP style models with generalized Nash bargaining, models with directed or competitive search are typically constrained efficient (Shimer (1996); Moen (1997)). In such models, firms internalize the search externalities arising from the matching process and the standard Hosios condition typically holds *endogenously*.[26] Early papers on directed or competitive search include Montgomery (1991), Peters (1991), Acemoglu and Shimer (1999b,a), Julien, Kennes, and King (2000), Burdett, Shi, and Wright (2001), Shi (2001, 2002).[27] For a detailed survey, see Wright et al. (2017).

Consider a simple competitive search model in the spirit of Moen (1997). There is a continuum of submarkets indexed by $i \in [0, 1]$ and free entry of firms at cost $c > 0$. Workers in submarket i post the same wage w_i and face the same market tightness θ_i, the ratio of firms to workers in that submarket. Firms' search is *directed* by observing the posted wages and deciding which submarkets to enter. Within submarket i, the matching probabilities for workers and firms are $m(\theta_i)$ and $m(\theta_i)/\theta_i$ respectively, where $m(.)$ satisfies Assumption 1.

Suppose that the expected match output $y(\theta_i)$ in submarket i depends on the market tightness θ_i in that submarket. The value of non-market activity is z where $y(\theta_i) > z$ for all $\theta_i \in \mathbb{R}_+$. The expected match surplus in submarket i is $s(\theta_i) = y(\theta_i) - z$ and $f(\theta_i) = m(\theta_i)y(\theta_i)$ where $f(.)$ satisfies Assumption 2.

In competitive search models where the match surplus is exogenous, agents simply trade off prices against the probability of matching. Here, agents trade off prices against both the probability of matching *and* the expected match surplus if trade occurs. The fact that

[26] Guerrieri (2008) develops a dynamic competitive search model with informational asymmetries and identifies a new externality that means the decentralized equilibrium is not always constrained efficient. Guerrieri, Shimer, and Wright (2010), Moen and Rosen (2011), and Julien and Roger (2015) also consider competitive search with informational frictions.

[27] Hosios (1990) also considers an example based on Peters (1984) that is similar to directed search and is constrained efficient.

agents can do so is what delivers constrained efficiency.

The expected payoff for firms in submarket i with wage w_i and market tightness θ_i is

$$\Pi(\theta_i, w_i) = \frac{m(\theta_i)}{\theta_i}(y(\theta_i) - w_i), \tag{47}$$

and the expected payoff for workers in submarket i with market tightness θ_i is

$$V(\theta_i, w_i) = m(\theta_i)w_i + (1 - m(\theta_i))z. \tag{48}$$

Workers in submarket i choose a wage w_i^* and market tightness θ_i^* that solve

$$\max_{w_i, \theta_i \in \mathbb{R}_+} (m(\theta_i)w_i + (1 - m(\theta_i))z) \tag{49}$$

subject to $\Pi(\theta_i, w_i) \leq c$ and $\theta_i \geq 0$ with complementary slackness. To induce participation by firms in submarket i, i.e. $\theta_i > 0$, the constraint $\Pi(\theta_i, w_i) \leq c$ is binding:

$$\frac{m(\theta_i)}{\theta_i}(y(\theta_i) - w_i) = c. \tag{50}$$

Solving for w_i as a function of θ_i using (50), we obtain

$$w(\theta_i) = y(\theta_i) - \frac{c\theta_i}{m(\theta_i)}. \tag{51}$$

Choosing a wage w_i^* is thus equivalent to choosing a market tightness θ_i^* where

$$\theta_i^* = \arg\max_{\theta_i \in \mathbb{R}_+} (m(\theta_i)w(\theta_i) + (1 - m(\theta_i))z) \tag{52}$$

and using (51), this is equivalent to

$$\theta_i^* = \arg\max_{\theta_i \in \mathbb{R}_+} (m(\theta_i)y(\theta_i) + (1 - m(\theta_i))z - c\theta_i) \tag{53}$$

and θ_i^* is the unique solution to the first-order condition

$$m'(\theta_i)s(\theta_i) + m(\theta_i)s'(\theta_i) = c, \tag{54}$$

or equivalently, θ_i^* solves

$$\underbrace{\eta_m(\theta_i)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta_i)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta_i}{m(\theta_i)s(\theta_i)}}_{\text{firms' surplus share}} . \tag{55}$$

Clearly, the generalized Hosios condition holds endogenously *within each active submarket* i. If we consider a symmetric equilibrium in which firms are indifferent across submarkets and all workers post the same wage, then $\theta_i^* = \theta^*$ for all submarkets i and Proposition 2 tells us that firm entry is constrained efficient.[28]

In this example, as in Example 3.1, we have simply assumed an arbitrary output technology $y(.)$. Next, we present a competitive search model in which the output technology $y(.)$ and its properties arise *endogenously* through the selection channel.

3.8 Competing auctions with ex post firm heterogeneity

In a competing auctions environment, a large number of sellers compete to attract buyers by posting auctions. Following the seminal work of Peters and Severinov (1997), recent papers that use competing auctions include Albrecht, Gautier, and Vroman (2012, 2014, 2016); Kim and Kircher (2015); Lester, Visschers, and Wolthoff (2015); and Mangin (2017). Competing auctions models with buyer heterogeneity are essentially competitive search models with private information. Such models feature the *selection channel* because the auction mechanism enables sellers to "select" the buyer with the highest valuation. This endogenizes the expected match output $y(\theta)$ because a greater number of buyers per seller implies a greater expected value of the highest valuation.

Consider the labor market environment in Mangin (2017). Workers are identical sellers who auction their labor using second-price auctions and post reservation wages to attract firms. Firms are ex ante identical buyers who pay a cost $c > 0$ to enter and search for workers. The labor market tightness is $\theta \equiv V/U$, the ratio of vacancies or firms to unemployed workers. The meeting technology is Poisson and $P_n(\theta) = \frac{\theta^n e^{-\theta}}{n!}$ is the probability that n firms approach a given worker. The matching probability for workers is $m(\theta) = 1 - e^{-\theta}$.

Firms' valuations y of workers' labor are match-specific productivity draws that are private information. Valuations are drawn *ex post* (i.e. after meetings) independently from a distribution with cdf G that is twice differentiable with density $g = G' > 0$, a finite mean, and support $[y_0, \infty)$ where $y_0 \geq 0$.

[28] While we consider only a static model in this example, the generalized Hosios condition also holds endogenously if we consider a dynamic competitive search model with endogenous match output. Proof available on request.

Let $w(\theta; r)$ be the expected wage when a worker posts reservation wage r. For any given reservation wage $r \in \mathbb{R}^+$, the market tightness $\theta^*(r)$ must satisfy

$$\frac{m(\theta)}{\theta}(y(\theta) - w(\theta; r)) \leq c \tag{56}$$

and $\theta^*(r) \geq 0$, with complementary slackness. Workers' reservation wage r^* maximizes their expected payoff, anticipating the effect on firm entry:

$$r^* = \arg \max_{r \in [0,\infty)} \left(m(\theta^*(r))w(\theta^*(r); r) + (1 - m(\theta^*(r)))z\right). \tag{57}$$

We consider symmetric equilibria where workers post the same reservation wage. If $c < E_G(y) - z$, there exists a unique equilibrium function $\theta^*(.)$ where $\theta^* \equiv \theta^*(r^*) > 0$ and workers' reservation wage r^* equals the value of non-market activity, $z \in [0, y_0]$.[29] The equilibrium market tightness $\theta^* \in \mathbb{R}_+$ satisfies

$$\int_{y_0}^{\infty} e^{-\theta(1-G(y))}(1 - G(y))dy + e^{-\theta}(y_0 - z) = c \tag{58}$$

The expected *output per capita* $f(\theta)$ is given by

$$f(\theta) = \int_{y_0}^{\infty} \theta e^{-\theta(1-G(y))} y dG(y). \tag{59}$$

and the properties of $f(.)$ are summarized in Proposition 1 of Mangin (2017). The expected output per match $y(\theta)$ is defined as $y(\theta) \equiv f(\theta)/m(\theta)$. Mangin (2017) proves that $y'(\theta) > 0$ for all $\theta \in \mathbb{R}_+$ if G is *well-behaved,* i.e. if it satisfies a mild regularity condition that is satisfied by almost all standard distributions. Therefore, the output externality that arises from the selection channel is positive here.

To establish constrained efficiency, it is easier to work directly with the function $f(.)$. Let $\eta_f(\theta) \equiv f'(\theta)\theta/f(\theta)$, the elasticity of $f(\theta)$ with respect to θ. It is straightforward to show that the generalized Hosios condition in Proposition 2 is equivalent to[30]

$$\eta_f(\theta) = \frac{c\theta}{f(\theta)} + \frac{\eta_m(\theta)z}{y(\theta)}. \tag{60}$$

[29] In this example, $r^* = z < y_0$, but in a dynamic setting we may have $r^*(\theta) > y_0$, in which case the expected match output depends on the market tightness through both a selection channel *and* a composition channel, since not all matches will be acceptable to workers. See e.g. Mangin and Sedlacek (2017).

[30] Using the fact that $s(\theta) = y(\theta) - z$, we can express the generalized Hosios condition in terms of $\eta_y(\theta) \equiv y'(\theta)\theta/y(\theta)$. Since $y(\theta) = f(\theta)/m(\theta)$, we have $\eta_y(\theta) = \eta_f(\theta) - \eta_m(\theta)$. Substituting into (15) yields (60).

Mangin (2017) shows that condition (60) holds at θ^* and we therefore have constrained efficiency. That is, the generalized Hosios condition holds *endogenously* in this example.

Example 3.8.1

In the special case where match output is constant, i.e. $y(\theta) = \bar{y} \in \mathbb{R}_+$, we recover the large economy version of the directed search model found in Julien et al. (2000). The model is closely related to Burdett et al. (2001), but it considers a labor market in which workers post second-price auctions with reservation wages. If $c < \bar{y} - z$, there exists a unique equilibrium market tightness $\theta^* > 0$ that satisfies

$$e^{-\theta}(\bar{y} - z) = c. \tag{61}$$

We can easily recover the constrained efficiency of directed search models such as Julien et al. (2000) by applying Proposition 2. In this special case, it is just the standard Hosios condition: entry is efficient if and only if θ^* satisfies

$$\eta_m(\theta) = \frac{c\theta}{m(\theta)s(\theta)}. \tag{62}$$

Substituting $\eta_m(\theta) = \theta e^{-\theta}/(1-e^{-\theta})$ and $s(\theta) = \bar{y} - z$ into (62) and rearranging, this condition holds *endogenously* and we have constrained efficiency since θ^* satisfies (61).

3.9 Business-stealing: competing auctions with seller entry

Albrecht et al. (2014) examines the efficiency of *seller* entry in a competing auctions environment. The authors consider both ex ante and ex post buyer heterogeneity, as well as seller heterogeneity, and they prove that seller entry is always constrained efficient. Although they do not explicitly identify it, the generalized Hosios condition applies in their setting and it is the fact that this condition holds endogenously that ensures constrained efficiency.

Consider a simple version of their model with homogeneous sellers, each with reservation value $z = 0$, and buyers who are ex ante identical but heterogeneous ex post. Sellers pay a cost κ to enter and they attract buyers by posting second-price auctions with reserve prices. The buyer-seller ratio is $\theta \equiv N_B/N_S$. The meeting technology is Poisson and $P_n(\theta) = \frac{\theta^n e^{-\theta}}{n!}$ is the probability that n buyers approach a given seller. The matching probability for sellers is $m(\theta) = 1 - e^{-\theta}$.

Buyers' valuations y are private information and they are drawn *ex post* (i.e. after meetings) independently from a distribution with cdf G that is twice differentiable with density $g = G' > 0$ and support $[0, 1]$.

In Albrecht et al. (2014), the social planner maximizes the total social surplus

$$\Lambda(\theta)N_S - \kappa N_S \tag{63}$$

where $\Lambda(\theta)$ is the expected surplus per seller. The social surplus per *buyer* is

$$\Omega_B(\theta) = \frac{\Lambda(\theta)}{\theta} - \frac{\kappa}{\theta} \tag{64}$$

and the first-order condition for the planner's problem is

$$\Omega_B'(\theta) = \frac{\Lambda'(\theta)}{\theta} - \frac{\Lambda(\theta)}{\theta^2} + \frac{\kappa}{\theta^2} = 0. \tag{65}$$

Rearranging, the social optimum θ^P satisfies

$$1 - \frac{\Lambda'(\theta)\theta}{\Lambda(\theta)} = \frac{\kappa}{\Lambda(\theta)}. \tag{66}$$

Now, the surplus per seller $\Lambda(\theta)$ in Albrecht et al. (2014) is $\Lambda(\theta) = m(\theta)s(\theta)$.[31] Therefore, the elasticity $\eta_\Lambda(\theta) \equiv \Lambda'(\theta)\theta/\Lambda(\theta)$ is given by $\eta_\Lambda(\theta) = \eta_m(\theta) + \eta_s(\theta)$ and equation (66) says

$$1 - \underbrace{\eta_m(\theta)}_{\text{matching elasticity}} - \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{\kappa}{m(\theta)s(\theta)}}_{\text{sellers' surplus share}}. \tag{67}$$

We therefore have constrained efficiency if and only if θ^* satisfies (67), which is exactly the generalized Hosios condition for seller entry given by (16). It is the fact that this condition holds endogenously here that ensures constrained efficiency.

The output externality that arises in Example 3.8 also appears in Albrecht et al. (2014) due to the selection channel. Through the auction mechanism, sellers choose to trade with the buyer who has the highest valuation. From Example 3.8, we know that $y'(\theta) > 0$ if the distribution G is well-behaved.[32] Importantly, this is a *negative* externality with regard to seller entry since $\theta = N_B/N_S$ and thus $y(.)$ is decreasing in the number of sellers. When there is a fixed number of buyers, more seller entry implies less buyers for each seller, thereby reducing the power of the selection channel.

Albrecht et al. (2014) considers a negative externality from seller entry called the *business-*

[31]Since $z = 0$ in this example, $\Lambda(\theta) = f(\theta)$ as given by equation (59) in Example 3.8, provided it is adjusted so that the support of G is $[0, 1]$ as in Albrecht et al. (2014).

[32]Note that $y(\theta) = \Lambda(\theta)/m(\theta)$ since $s(\theta) = \Lambda(\theta)/m(\theta)$ and $s(\theta) = y(\theta)$ if $z = 0$.

stealing externality. When an additional seller enters, the seller "steals" potential buyers from existing sellers, thereby reducing the expected surplus for those sellers. This is reflected in the fact that $\Lambda'(\theta) > 0$ and thus $\Lambda(.)$ is decreasing in N_S. As Albrecht et al. (2014) write, one might expect the "business-stealing" effect would lead to over-entry of sellers relative to the social optimum; however, this is exactly offset by the "informational rents" that buyers extract from sellers through the auction mechanism, thus delivering constrained efficiency. In fact, since $\Lambda(\theta) = m(\theta)s(\theta)$, the "business-stealing" effect can be decomposed into two effects: the effect on sellers' matching probability $m(\theta)$, and the effect on the expected match surplus $s(\theta)$. Both of these effects are clearly reflected in the generalized Hosios condition for seller entry (67) through the matching elasticity $\eta_m(\theta)$ and the surplus elasticity $\eta_s(\theta)$.

3.10 Competitive search with endogenous quality dynamics

We now present an example that illustrates how an endogenous quality distribution may arise through the possibility of "referrals". The model is closely related to – but different from – Campbell, Leister, and Zenou (2017), which presents a dynamic model of consumer sales with word-of-mouth communication through social networks. In our setting, the key variable θ is the ratio of *referrals* to consumers and the endogenous quality distribution is the probability that a traded good is low quality, i.e. the market share of low-quality firms. We use competitive search to model the *market for referrals* (not goods) and consider whether the entry of *sellers of referrals* (not firms) is constrained efficient.

There is a fixed measure of consumers who seek to purchase one unit of a durable good. After purchasing the good, consumers exit the market and are replaced by new consumers. Goods are produced by a large number of competitive firms of two types: high quality and low quality. The share of firms that produce low quality goods is $\mu \in (0, 1)$.[33] The low-quality good has quality x_L and the high-quality good has quality $x_H > x_L$. The price of the good is p for both types of firm.

Consumers cannot directly observe firms' quality, but they can receive referrals. A single referral tells a consumer about the quality of a good purchased in the previous period. In each period $t \in \{0, 1, ...\}$, the expected number of referrals per consumer is θ_t (which is endogenous) and $P_n(\theta_t)$ is the probability a consumer receives n referrals at time t. This is a kind of "meeting technology" which matches referrals with consumers. If a consumer receives at least one referral, they pick the "best" referral and then choose whether to purchase from that firm or instead choose a firm randomly.[34] If a consumer receives no referrals, they

[33] Since our focus is on the market for *referrals*, we do not endogenize the entry of low and high quality firms as in Campbell et al. (2017) but instead assume that μ is exogenous.

[34] If the consumer is indifferent between two referrals, they pick one at random.

purchase the good from a random firm, i.e. they buy a low-quality good with probability μ.

Let α_t be the *market share* of low-quality firms, i.e. the probability that a good traded in period t is low quality. Low-quality goods are purchased only if *all* n of a consumer's referrals are to low-quality firms (which occurs with probability α_t^n) *and* the consumer picks a low-quality firm when choosing randomly (which occurs with probability μ). We therefore obtain the following law of motion for α_t:

$$\alpha_{t+1} = \mu \sum_{n=0}^{\infty} P_n(\theta_{t+1}) \alpha_t^n \tag{68}$$

where $\alpha_0 = \mu \in (0, 1)$. If $P_n(\theta)$ is Poisson, i.e, $P_n(\theta) = \frac{\theta^n e^{-\theta}}{n!}$, we have

$$\alpha_{t+1} = \mu e^{-\theta_{t+1}(1-\alpha_t)}. \tag{69}$$

Both the selection channel and an additional channel are present. The *selection channel* implies that the average quality of a traded good is increasing in the number of referrals per consumer θ_t since consumers can be more selective. An additional channel, similar to the *composition channel,* ensures that the quality distribution α_t itself evolves over time. This is because the market composition, i.e. the composition of the pool of referrals, depends on the previous period's θ_t since referrals are only drawn from traded goods.

We now endogenize the steady state equilibrium number of referrals per consumer. Suppose there is a large number of potential entrants who can pay a cost $c > 0$ to acquire information about a random good purchased in the previous period. This information can be sold to consumers as a "referral". The mechanism for selling referrals is that consumers post *referral fees* and commit to paying a single fee for the best referral they receive.

Similarly to the competitive search environment in Example 3.7, consumers form a submarket i by choosing a referral fee r_i^* and a ratio of referrals to consumers θ_i^* to maximize their expected payoff:

$$m(\theta_i)(y(\theta_i, \alpha) - r_i - p) + (1 - m(\theta_i))(y_\mu - p) \tag{70}$$

subject to the following zero profit condition for sellers of referrals:

$$\frac{m(\theta_i)}{\theta_i} r_i = c \tag{71}$$

where $m(\theta_i) = 1 - e^{-\theta_i}$ is the probability a consumer receives a referral, $m(\theta_i)/\theta_i$ is the probability a seller is paid a referral fee, $y(\theta_i, \alpha)$ is the expected quality of a good purchased *if the consumer receives a referral*, and $y_\mu = \mu x_L + (1-\mu)x_H$ is the expected quality of a good purchased from a random firm.

Using (71), the choice of a consumer in submarket i is equivalent to

$$\theta_i^* = \arg\max_{\theta_i \in \mathbb{R}_+} (m(\theta_i)(y(\theta_i, \alpha) - y_\mu) + y_\mu - p - c\theta_i) \tag{72}$$

and θ_i^* satisfies the first-order condition

$$m'(\theta_i)s(\theta_i) + m(\theta_i)\frac{\partial y(\theta_i, \alpha)}{\partial \theta_i} = c \tag{73}$$

where the expected match surplus is $s(\theta_i) = y(\theta_i, \alpha) - y_\mu$, i.e. the difference between the expected quality in submarket i with and without receiving referrals. In symmetric equilibrium, $\theta_i^* = \theta^*$ for all submarkets i and θ^* satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\frac{\frac{\partial y(\theta,\alpha)}{\partial \theta}\theta}{s(\theta)}}_{\text{direct surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{surplus share of referral sellers}} \tag{74}$$

as well as the steady state condition

$$\alpha = \mu e^{-\theta(1-\alpha)}. \tag{75}$$

If $\mu < \frac{1}{2}$, there exists a unique steady state equilibrium (θ^*, α^*) where $\alpha^* \in (0,1)$.[35]

Now consider a social planner who can directly choose the number of referrals per consumer θ but is constrained by the same "matching" technology $m(.)$ and "production" technology $y(., \alpha)$ as the decentralized economy.[36] While consumers take α *as given*, the social planner takes the effect of θ on α into account. The planner maximizes the steady state social surplus per consumer:

$$\Omega(\theta) = m(\theta)y(\theta, \alpha(\theta)) + (1 - m(\theta))y_\mu - c\theta \tag{76}$$

where $\alpha(\theta)$ is given by (75) and

$$y(\theta, \alpha(\theta)) = \frac{(1 - \mu e^{-\theta(1-\alpha(\theta))})x_H + \mu e^{-\theta(1-\alpha(\theta))}x_L - e^{-\theta}y_\mu}{1 - e^{-\theta}}. \tag{77}$$

[35] A detailed derivation of the steady state equilibrium can be found in the Appendix.

[36] In the Appendix, we solve the dynamic planner's problem subject to the law of motion for α_t and derive a steady state condition that is identical to the one presented here.

Using $s(\theta) = y(\theta, \alpha(\theta)) - y_\mu$, the planner's first-order condition is equivalent to

(78)
$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{surplus share of referral sellers}},$$

which is just the generalized Hosios condition. Differentiating $s(\theta)$, we obtain

(79)
$$s'(\theta) = \underbrace{\frac{\partial y(\theta, \alpha(\theta))}{\partial \theta}}_{\text{direct output externality}} + \underbrace{\frac{\partial y(\theta, \alpha(\theta))}{\partial \alpha}\alpha'(\theta)}_{\text{indirect output externality}}$$

and thus we have constrained efficiency only if θ^* satisfies

(80)
$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\frac{\frac{\partial y(\theta,\alpha)}{\partial \theta}\theta}{s(\theta)} + \frac{\frac{\partial y(\theta,\alpha)}{\partial \alpha}\alpha'(\theta)\theta}{s(\theta)}}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{surplus share of referral sellers}}.$$

Comparing (80) with (74), it is clear the economy is not constrained efficient.

The decentralized market internalizes both the search externalities and the *direct* component of the "output externality", i.e. the direct effect of θ on the expected match surplus. However, there is an additional externality arising from the use of referrals. This is reflected in the term $\frac{\partial y(\theta,\alpha(\theta))}{\partial \alpha}\alpha'(\theta)$, which captures the *indirect* component of the "output externality" via the quality distribution. Since $\frac{\partial y(\theta,\alpha(\theta))}{\partial \alpha} < 0$ and the market share of low-quality firms is decreasing in the number of referrals per consumer at any equilibrium θ^*, i.e. $\alpha'(\theta^*) < 0$, this is a positive externality that is not internalized by the decentralized economy. The equilibrium number of referrals is therefore inefficiently *low*.

While the generalized Hosios condition does indeed *apply* in this environment, we do not have constrained efficiency because it does not *hold*. The static economy is always constrained efficient since the probability α is exogenous, but the dynamic economy is not efficient. The novel externality that is the source of this inefficiency is similar in flavor to that found in Guerrieri (2008), which shows that competitive search is not always dynamically efficient. In Guerrieri (2008), the inefficiency arises because firms do not internalize the effect of their decisions on the outside options of workers hired in earlier periods. In this example, consumers do not internalize the effect of their decisions on *future consumers* through the impact of referrals on the evolution of the quality distribution itself.

4 Conclusion

This paper presents a generalized version of the well-known Hosios rule that determines the conditions under which buyer (or seller) entry in search and matching models is constrained efficient. We extend this simple rule to environments where the expected match output depends on the market tightness. Such environments give rise to a novel externality that we call the *output externality.* This externality is not captured by the standard Hosios condition, which internalizes only the search externalities arising from the matching frictions.

To ensure constrained efficiency, decentralized markets must internalize the effect of entry on both the *number* of matches created and the average *value* created by each match. That is, agents must be compensated for their effect on both *match creation* and *surplus creation.* We show that this occurs precisely when buyers' surplus share equals the *matching elasticity* plus the *surplus elasticity.* We call this simple, intuitive condition the "generalized Hosios condition". When this condition holds, both the search externalities and the output externality are internalized by a decentralized market.

We focus attention exclusively on the efficiency of entry and assume that the social planner is constrained by both the matching technology and the *output technology,* i.e. the "technology" which transforms the market tightness into the expected match output. However, the nature of the output technology may arise endogenously from specific features of the decentralized market – such as the underlying meeting technology and trading mechanism – which the planner takes as given. One possible direction for future research would be to integrate our general result regarding the efficiency of entry with the literature, such as Eeckhout and Kircher (2010b), that considers which trading mechanisms can emerge as an equilibrium outcome in various environments with search and matching frictions.

Another potential direction for future research would be to consider search and matching environments with two-sided heterogeneity. In such environments, as Eeckhout and Kircher (2010a) point out, the social planner also cares about both the number of matches created and the average value of a match – which depends on the types of agents that form matches. In future research, it would be interesting to integrate results regarding the efficiency of frictional environments with two-sided heterogeneity with the generalized Hosios condition.

Appendix A: Proofs

Proof of Lemma 1

In steady state, we have the following Bellman equations:

$$rU_B = -c + \frac{m(\theta)}{\theta}(V_B - U_B), \tag{81}$$

$$rV_B = y(\theta) - w(\theta) + \delta(U_B - V_B), \tag{82}$$

$$rU_S = z + m(\theta)(V_S - U_S), \tag{83}$$

$$rV_S = w(\theta) + \delta(U_S - V_S), \tag{84}$$

where $w(\theta)$ is the expected transfer paid to sellers by buyers each period.

With free entry, $U_B = 0$ and $s(\theta) = V_B + V_S - U_S$, so we have

$$V_B + V_S = \frac{y(\theta) - \delta s(\theta)}{r}. \tag{85}$$

Substituting back into $s(\theta) = V_B + V_S - U_S$, and rearranging yields

$$s(\theta) = \frac{y(\theta) - rU_S}{r + \delta}. \tag{86}$$

Next, using (83) and (84), we find that

$$U_S = \frac{z(r + \delta) + m(\theta)w(\theta)}{r(r + \delta + m(\theta))}, \tag{87}$$

and, substituting into (86), we obtain

$$s(\theta) = \left(\frac{y(\theta) - z + m(\theta)\left(\frac{y(\theta) - w(\theta)}{r+\delta}\right)}{r + \delta + m(\theta)} \right). \tag{88}$$

Now (81) implies $V_B = c\theta/m(\theta)$ when $U_B = 0$. Substituting into (82), we have

$$\frac{y(\theta) - w(\theta)}{r + \delta} = \frac{c\theta}{m(\theta)}, \tag{89}$$

and, substituting (89) into (88), we obtain

$$s(\theta) = \frac{y(\theta) - z + c\theta}{r + \delta + m(\theta)}. \tag{90}$$

Proof of Proposition 1

In discrete time, the law of motion for the unemployment rate u_t is

$$u_{t+dt} - u_t = \delta dt(1 - u_t) - m(\theta_t)dt\, u_t \tag{91}$$

and the law of motion for average match output y_t is given by

$$y_{t+dt} = \frac{(1 - \delta dt)(1 - u_t)y_t + m(\theta_t)dt\, u_t y(\theta_t)}{1 - u_{t+dt}}. \tag{92}$$

Defining $x_t \equiv (1 - u_t)y_t$, we have

$$x_{t+dt} - x_t = -\delta dt x_t + m(\theta_t)dt\, u_t y(\theta_t). \tag{93}$$

In continuous time ($dt \to 0$), the laws of motion for u_t and x_t are

$$\dot{u}_t = \frac{du_t}{dt} = \lim_{dt\to 0}\left(\frac{u_{t+dt} - u_t}{dt}\right) = \delta(1 - u_t) - m(\theta_t)u_t \tag{94}$$

and

$$\dot{x}_t = \frac{dx_t}{dt} = \lim_{dt\to 0}\left(\frac{x_{t+dt} - x_t}{dt}\right) = -(\delta x_t - m(\theta_t)u_t y(\theta_t)). \tag{95}$$

Also, since $x_t \equiv (1 - u_t)y_t$, we have

$$\dot{x}_t = -\dot{u}_t y_t + (1 - u_t)\dot{y}_t \tag{96}$$

and, rearranging, we have

$$\dot{y}_t = \frac{\dot{x}_t + \dot{u}_t y_t}{1 - u_t}. \tag{97}$$

Substituting in $\dot{x}_t$ and $\dot{u}_t$ from (95) and (94) and the fact that $x_t \equiv (1-u_t)y_t$ leads to:

$$\dot{y}_t = \frac{m(\theta_t)u_t(y(\theta_t) - y_t)}{1-u_t}. \tag{98}$$

The social planner chooses θ_t for all $t \in \mathbb{R}_+$ to maximize the following:

$$\int_0^\infty e^{-rt}((1-u_t)y_t + zu_t - c\theta_t u_t)dt \tag{99}$$

subject to

$$\dot{u}_t = \delta(1-u_t) - m(\theta_t)u_t \tag{100}$$

and

$$\dot{y}_t = \frac{m(\theta_t)u_t\left(y(\theta_t) - y_t\right)}{1-u_t}. \tag{101}$$

The current value Hamiltonian is

$$H = ((1-u_t)y_t + zu_t - c\theta_t u_t) + \lambda_t(\delta(1-u_t) - m(\theta_t)u_t) + \mu_t\left(\frac{m(\theta_t)u_t\left(y(\theta_t) - y_t\right)}{1-u_t}\right). \tag{102}$$

The first-order necessary conditions are

$$\frac{\partial H}{\partial \theta_t} = -cu_t - \lambda_t m'(\theta_t)u_t + \mu_t\left(\frac{m'(\theta_t)u_t\left(y(\theta_t) - y_t\right) + m(\theta_t)u_t y'(\theta_t)}{1-u_t}\right) = 0 \tag{103}$$

and

$$\begin{aligned} \frac{dH}{du_t} &= -(y_t - z + c\theta_t) - \lambda_t(\delta + m(\theta_t)) \\ &\quad + \mu_t\left(\frac{(1-u_t)\, m(\theta_t)\left(y(\theta_t) - y_t\right) + u_t m(\theta_t)\left(y(\theta_t) - y_t\right)}{(1-u_t)^2}\right) \\ &= -\dot{\lambda}_t + r\lambda_t \end{aligned} \tag{104}$$

and

$$\frac{\partial H}{\partial y_t} = 1 - u_t - \mu_t\left(\frac{m(\theta_t)u_t}{1-u_t}\right) = -\dot{\mu}_t + r\mu_t \tag{105}$$

and

$$\frac{\partial H}{\partial \lambda_t} = \delta(1-u_t) - m(\theta_t)u_t = \dot{u}_t \tag{106}$$

and

$$\frac{\partial H}{\partial \mu_t} = \frac{m(\theta_t)u_t\,(y(\theta_t) - y_t)}{1-u_t} = \dot{y}_t. \tag{107}$$

The transversality conditions are

$$\lim_{t\to\infty} e^{-rt}\lambda_t u_t = 0, \tag{108}$$

$$\lim_{t\to\infty} e^{-rt}\mu_t y_t = 0. \tag{109}$$

Now, in steady state, we have $\dot{u}_t = 0$ and $\dot{y}_t = 0$ and therefore $y(\theta_t) = y_t = y(\theta)$. Also, in steady state, $\dot{\mu}_t = 0$ and $\dot{\lambda}_t = 0$. Substituting into the above first-order conditions,

$$-cu - \lambda m'(\theta)u + \mu\left(\frac{m(\theta)y'(\theta)}{1-u}\right) = 0, \tag{110}$$

$$-(y(\theta) - z + c\theta) - \lambda(\delta + m(\theta)) = r\lambda, \tag{111}$$

$$1 - u - \mu\left(\frac{m(\theta)u}{1-u}\right) = r\mu. \tag{112}$$

Using the fact that $\delta(1-u) = m(\theta)u$ in steady state, we have

$$-\lambda m'(\theta)u + \mu\delta y'(\theta) = cu, \tag{113}$$

$$\lambda = -\left(\frac{y(\theta) - z + c\theta}{r + \delta + m(\theta)}\right), \tag{114}$$

$$\mu = \frac{1-u}{r+\delta}. \tag{115}$$

It is clear that the transversality conditions are satisfied by λ and μ.

Substituting λ and μ into the first equation, we have

$$\frac{y(\theta) - z + c\theta}{r + \delta + m(\theta)} m'(\theta) u + \frac{(1-u)\delta y'(\theta)}{r+\delta} = cu. \tag{116}$$

Again using $\delta(1-u) = m(\theta)u$ and simplifying,

$$\frac{y(\theta) - z + c\theta}{r + \delta + m(\theta)} m'(\theta) + \frac{m(\theta) y'(\theta)}{r+\delta} = c. \tag{117}$$

Defining $s(\theta)$ as in (12), and multiplying by $\theta/m(\theta)s(\theta)$,

$$\frac{m'(\theta)\theta}{m(\theta)} + \frac{y'(\theta)\theta}{(r+\delta)s(\theta)} = \frac{c\theta}{m(\theta)s(\theta)}. \tag{118}$$

That is,

$$\eta_m(\theta) + \frac{y'(\theta)\theta}{(r+\delta)s(\theta)} = \frac{c\theta}{m(\theta)s(\theta)}, \tag{119}$$

where $\eta_m(\theta) \equiv m'(\theta)\theta/m(\theta)$. Any social optimum must satisfy (119).

Proof of Lemma 2

Starting with condition (119) and multiplying both sides by $r + \delta$, we obtain

$$(r+\delta)\eta_m(\theta) + \frac{y'(\theta)\theta}{s(\theta)} = \frac{c\theta(r+\delta)}{m(\theta)s(\theta)}. \tag{120}$$

Rearranging, this is equivalent to

$$(r + \delta + m(\theta))\eta_m(\theta) + \frac{y'(\theta)\theta}{s(\theta)} + \frac{c\theta}{s(\theta)} - \eta_m(\theta)m(\theta) = \frac{c\theta(r+\delta+m(\theta))}{m(\theta)s(\theta)}. \tag{121}$$

Dividing by $r + \delta + m(\theta)$, and using expression (12) for $s(\theta)$, we obtain

$$\eta_m(\theta) + \frac{(y'(\theta)+c)\theta}{y(\theta) - z + c\theta} - \frac{m'(\theta)\theta}{r+\delta+m(\theta)} = \frac{c\theta}{m(\theta)s(\theta)}. \tag{122}$$

Now, using expression (12) for $s(\theta)$, we can write $\eta_s(\theta) \equiv s'(\theta)\theta/s(\theta)$ as the elasticity of the numerator minus the elasticity of the denominator:

$$\eta_s(\theta) = \frac{(y'(\theta)+c)\theta}{y(\theta) - z + c\theta} - \frac{m'(\theta)\theta}{r+\delta+m(\theta)}. \tag{123}$$

Therefore, condition (119) is equivalent to:

$$\eta_m(\theta) + \eta_s(\theta) = \frac{c\theta}{m(\theta)s(\theta)}. \tag{124}$$

Proof of Lemma 3

It follows immediately from Assumption 2a that there exists a unique $\theta^P > 0$ that satisfies the necessary condition (13). We now prove that the steady state solution θ^P given by (13) is indeed a global maximum using Arrow's Sufficiency Theorem. To show this, it is simpler to formulate the current value Hamiltonian in terms of the state variable x_t. Using (94) and (95), the current value Hamiltonian as a function of state and control variables is

$$H(x,u,\theta) = (x + zu - c\theta u) + \lambda_1(\delta(1-u) - m(\theta)u) + \mu_1(-(\delta x - m(\theta)uy(\theta))). \tag{125}$$

First, we define the maximized Hamiltonian as follows:

$$M_H(x,u) \equiv \max_{\theta\in\mathbb{R}_+}[(x + zu - c\theta u) + \lambda_1(\delta(1-u) - m(\theta)u) + \mu_1(-(\delta x - m(\theta)uy(\theta))].$$

We now apply Arrow's Sufficiency Theorem.[37] To prove that the solution θ^P to (119) is a global maximum, it is sufficient to show that *(i)* the maximized Hamiltonian $M_H(x,u)$ is jointly weakly concave in u and x; and *(ii)* there exists a unique solution θ^P that satisfies the necessary condition (119). Since we know that part *(ii)* holds, it remains only to prove *(i)*. To find $\theta^* \equiv \arg\max_{\theta\in\mathbb{R}_+} H(x,u,\theta)$, we set

$$\frac{\partial H}{\partial \theta} = -cu - \lambda_1 m'(\theta)u + \mu_1 u(m'(\theta)y(\theta) + m(\theta)y'(\theta)) = 0. \tag{126}$$

Also, we have

$$\frac{\partial^2 H}{\partial \theta^2} = -\lambda_1 m''(\theta)u + \mu_1 u(m''(\theta)y(\theta) + 2m'(\theta)y'(\theta) + m(\theta)y''(\theta)) < 0, \tag{127}$$

provided that $m''(\theta)y(\theta) + 2m'(\theta)y'(\theta) + m(\theta)y''(\theta) < 0$ and $m''(\theta) < 0$ since $\lambda_1 < 0$ and $\mu_1 > 0$. Assumption 1a states that $m''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$ and Assumption 2a says that $\Lambda''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$ where $\Lambda(\theta) \equiv m(\theta)s(\theta)$. In the special case where $z = 0$ in the static economy, we have $\Lambda(\theta) \equiv m(\theta)y(\theta)$ and therefore $\Lambda''(\theta) < 0$ implies that $m''(\theta)y(\theta) + 2m'(\theta)y'(\theta) + m(\theta)y''(\theta) < 0$.

[37] Arrow's Sufficiency Theorem generalizes Mangasarian's sufficiency conditions. See Kamien and Schwartz (1991), p. 221-222.

So θ^* is indeed a maximum and the maximized Hamiltonian is

$$M_H(x,u) = (x + zu - c\theta^* u) + \lambda_1(\delta(1-u) - m(\theta^*)u) + \mu_1(-(\delta x - m(\theta^*)uy(\theta^*))). \quad (128)$$

Since the u cancels out in (126) and x does not appear in that equation, θ^* does not depend directly on u or x. Also, it can be verified that neither λ_1 nor μ_1 depends on either u or x.[38] The function $M_H(x,u)$ is linear in both x and u and it is therefore weakly concave. Since there exists a unique solution θ^P that satisfies the necessary condition (119), this solution is the global maximum.

Proof of Corollary 1 and 2

Assume that the standard Hosios condition holds, namely

$$\frac{m'(\theta^*)\theta^*}{m(\theta^*)} = \frac{c\theta^*}{m(\theta^*)s(\theta^*)}. \quad (129)$$

We prove the result in two parts. First, we show that there is under-entry (over-entry) of buyers if and only if $s'(\theta^*) > (<) 0$. Second, we show that $s'(\theta^*) > 0$ if and only if $y'(\theta^*) > 0$. Letting $\Lambda(\theta) = m(\theta)s(\theta)$ and simplifying (129), we have $m'(\theta^*)s(\theta^*) = c$. We also have $\Lambda'(\theta^P) = c$ and therefore $\Lambda'(\theta^P) = m'(\theta^*)s(\theta^*)$. Now $m'(\theta^*)s(\theta^*) = \Lambda'(\theta^*) - m(\theta^*)s'(\theta^*)$ and thus

$$\Lambda'(\theta^P) = \Lambda'(\theta^*) - m(\theta^*)s'(\theta^*). \quad (130)$$

If $s'(\theta^*) > 0$, then $\Lambda'(\theta^P) < \Lambda'(\theta^*)$. If $\Lambda''(\theta) < 0$ for all θ then $\Lambda'(\theta^P) < \Lambda'(\theta^*)$ implies that $\theta^* < \theta^P$ and there is *under-entry* of buyers. Similarly, if $s'(\theta^*) < 0$, there is *over-entry* of buyers, $\theta^* > \theta^P$. Differentiating $s(\theta)$ using (12),

$$s'(\theta) = \frac{(y'(\theta)+c)(r+\delta+m(\theta)) - m'(\theta)(y(\theta)-z+c\theta)}{(r+\delta+m(\theta))^2}. \quad (131)$$

Using expression (12) for $s(\theta)$ and rearranging, $s'(\theta^*) > 0$ if and only if

$$y'(\theta^*) > m'(\theta^*)s(\theta^*) - c, \quad (132)$$

and since $m'(\theta^*)s(\theta^*) = c$, we have $s'(\theta^*) > 0$ if and only if $y'(\theta^*) > 0$.

[38] Note that the co-state variables λ_1 and μ_1 for the current value Hamiltonian with state variables u_t and x_t are different to the co-state variables λ and μ for the current value Hamiltonian with state variables u_t and p_t.

With seller entry, the direction of Corollary 1 is reversed to yield Corollary 2 since $\theta^* < \theta^P$ implies *over-entry* of sellers because $\theta = N_B/N_S$, and $\theta^* > \theta^P$ implies *under-entry* of sellers relative to the social optimum.

Proofs for Example 3.10

Equilibrium. In period t, the expected payoff for a seller of a referral in submarket i with referral fee $r_{i,t}$ and market tightness $\theta_{i,t}$ is

$$\Pi(\theta_{i,t}, r_{i,t}) = \frac{m(\theta_{i,t})}{\theta_{i,t}} r_{i,t} - c \tag{133}$$

and the expected payoff for consumers in submarket i is

$$V(\theta_{i,t}, r_{i,t}) = m(\theta_{i,t})(y(\theta_{i,t}, \alpha_{t-1}) - r_{i,t} - p) + (1 - m(\theta_{i,t}))(y_\mu - p)). \tag{134}$$

Consumers in submarket i choose a referral fee $r^*_{i,t}$ and market tightness $\theta^*_{i,t}$ that maximize $V(\theta_{i,t}, r_{i,t})$ subject to $\Pi(\theta_{i,t}, r_{i,t}) \leq c$ and $\theta_{i,t} \geq 0$ with complementary slackness. To induce participation by sellers in submarket i, i.e. $\theta_{i,t} > 0$, the constraint $\Pi(\theta_{i,t}, r_{i,t}) \leq c$ is binding and we have

$$\frac{m(\theta_{i,t})}{\theta_{i,t}} r_{i,t} = c. \tag{135}$$

Using (135) to replace $r_{i,t}$ in $V(\theta_{i,t}, r_{i,t})$, the choice of a consumer in submarket i is equivalent to

$$\theta^*_{i,t} = \arg \max_{\theta_{i,t} \in \mathbb{R}_+} (m(\theta_{i,t})(y(\theta_{i,t}, \alpha_{t-1}) - y_\mu) + y_\mu - p - c\theta_{i,t}). \tag{136}$$

Differentiating with respect to $\theta_{i,t}$, the first-order condition of this problem is

$$m'(\theta_{i,t})(y(\theta_{i,t}, \alpha_{t-1}) - y_\mu) + m(\theta_{i,t}) \frac{\partial y(\theta_{i,t}, \alpha_{t-1})}{\partial \theta_{i,t}} - c = 0. \tag{137}$$

In symmetric equilibrium, $\theta^*_{i,t} = \theta^*_t$ for all submarkets i and θ^*_t satisfies

$$m'(\theta_t)(y(\theta_t, \alpha_{t-1}) - y_\mu) + m(\theta_t) \frac{\partial y(\theta_t, \alpha_{t-1})}{\partial \theta_t} = c. \tag{138}$$

In steady state, $\theta_t = \theta_{t-1} = \theta$ and $\alpha_t = \alpha_{t-1} = \alpha$ and any equilibrium (θ^*, α^*) satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} \quad + \quad \underbrace{\frac{\frac{\partial y(\theta,\alpha)}{\partial \theta}\theta}{s(\theta)}}_{\text{direct surplus elasticity}} \quad = \quad \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{surplus share of referral sellers}} \tag{139}$$

where the expected match surplus is $s(\theta) = y(\theta, \alpha) - y_\mu$.

To solve for the equilibrium, we use the fact that the average quality of a traded good in period t is given by

$$(1 - \alpha_t)x_H + \alpha_t x_L = f(\theta_t, \alpha_{t-1}) + (1 - m(\theta_t)y_\mu \tag{140}$$

where $f(\theta_t, \alpha_{t-1}) = m(\theta_t)y(\theta_t, \alpha_{t-1})$. Using the fact that $\alpha_t = \mu e^{-\theta_t(1-\alpha_{t-1})}$,

$$f(\theta_t, \alpha_{t-1}) = (1 - \mu e^{-\theta_t(1-\alpha_{t-1})})x_H + \mu e^{-\theta_t(1-\alpha_{t-1})}x_L - e^{-\theta_t}y_\mu \tag{141}$$

or equivalently,

$$f(\theta_t, \alpha_{t-1}) = x_H - \mu\Delta x e^{-\theta_t(1-\alpha_{t-1})} - e^{-\theta_t}y_\mu \tag{142}$$

where $\Delta x = x_H - x_L$. The first-order condition (138) is equivalent to

$$\frac{\partial f(\theta_t, \alpha_{t-1})}{\partial \theta_t} - m'(\theta_t)y_\mu - c = 0. \tag{143}$$

Differentiating (142) with respect to θ_t, this is equivalent to

$$(1 - \alpha_{t-1})\mu\Delta x e^{-\theta_t(1-\alpha_{t-1})} - c = 0 \tag{144}$$

and the second-order condition is

$$-(1 - \alpha_{t-1})^2\mu\Delta x e^{-\theta_t(1-\alpha_{t-1})} < 0. \tag{145}$$

Using the fact that $\alpha_t = \mu e^{-\theta_t(1-\alpha_{t-1})}$, this is equivalent to

$$(1 - \alpha_{t-1})\alpha_t = \frac{c}{\Delta x}. \tag{146}$$

In steady state, $\theta_t = \theta_{t-1} = \theta$ and $\alpha_t = \alpha_{t-1} = \alpha$ and any equilibrium α satisfies

$$-\alpha^2 + \alpha - \frac{c}{\Delta x} = 0 \tag{147}$$

as well as $\alpha = \mu e^{-\theta(1-\alpha)}$. Since $\mu \in (0,1)$, there are two solutions $\alpha \in (0,1)$ provided that $\frac{c}{\Delta x} < \frac{1}{4}$ and one solution if $\frac{c}{\Delta x} = \frac{1}{4}$. The two solutions are

$$\alpha = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{c}{\Delta x}}. \tag{148}$$

Since $\alpha < \mu$ for $\theta > 0$, if $\mu < \frac{1}{2}$ we obtain a unique steady state equilibrium

$$\alpha^* = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{c}{\Delta x}} \tag{149}$$

and

$$\theta^* = \frac{1}{1-\alpha} \ln\left(\frac{\mu}{\alpha}\right). \tag{150}$$

Social planner. Given $\alpha_0 = \mu \in (0,1)$, the social planner chooses $\{\theta_t\}_{t=1}^{\infty}$ to maximize the total discounted social surplus per consumer:

$$\Omega = \sum_{t=1}^{\infty} \beta^t \left(m(\theta_t) y(\theta_t, \alpha_{t-1}) + (1 - m(\theta_t)) y_\mu - c\theta_t\right) \tag{151}$$

subject to $\theta_t \geq 0$ and the law of motion for α_t :

$$\alpha_t = \mu e^{-\theta_t(1-\alpha_{t-1})} \tag{152}$$

The Lagrangian for this problem is

$$\mathcal{L} = \sum_{t=1}^{\infty} \beta^t \left(m(\theta_t)(y(\theta_t, \alpha_{t-1}) - y_\mu) + y_\mu - c\theta_t\right) + \lambda_t(\alpha_t - \mu e^{-\theta_t(1-\alpha_{t-1})}). \tag{153}$$

The first-order conditions are:

(154)
$$\frac{\partial \mathcal{L}}{\partial \theta_t} = \beta^t (m'(\theta_t)(y(\theta_t, \alpha_{t-1}) - y_\mu) + m(\theta_t) \frac{\partial y(\theta_t, \alpha_{t-1})}{\partial \theta_t} - c) + \lambda_t (1 - \alpha_{t-1}) \mu e^{-\theta_t(1-\alpha_{t-1})} = 0$$

and

$$\frac{\partial \mathcal{L}}{\partial \alpha_{t-1}} = \lambda_{t-1} + \beta^t m(\theta_t) \frac{\partial y(\theta_t, \alpha_{t-1})}{\partial \alpha_{t-1}} - \lambda_t \theta_t \mu e^{-\theta_t(1-\alpha_{t-1})} = 0. \tag{155}$$

and

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = \alpha_t - \mu e^{-\theta_t(1-\alpha_{t-1})} = 0. \tag{156}$$

Now, in steady state we have $\theta_{t+1} = \theta_t = \theta$, $\alpha_{t+1} = \alpha_t = \alpha$, and we have

$$\beta^t(m'(\theta)(y(\theta,\alpha) - y_\mu) + m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \theta} - c) = -\lambda(1-\alpha)\mu e^{-\theta(1-\alpha)} \tag{157}$$

and

$$\lambda + \beta^t m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \alpha} = \lambda\theta\mu e^{-\theta(1-\alpha)} \tag{158}$$

and

$$\alpha = \mu e^{-\theta(1-\alpha)}. \tag{159}$$

Rearranging (157), we obtain

$$\lambda = \frac{-\beta^t(m'(\theta)(y(\theta,\alpha) - y_\mu) + m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \theta} - c)}{(1-\alpha)\mu e^{-\theta(1-\alpha)}}, \tag{160}$$

and rearranging (158) delivers

$$\lambda = \frac{-\beta^t m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \alpha}}{1 - \theta\mu e^{-\theta(1-\alpha)}}. \tag{161}$$

Equating (160) and (161) yields

$$\frac{-\beta^t(m'(\theta)(y(\theta,\alpha) - y_\mu) + m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \theta} - c)}{(1-\alpha)\mu e^{-\theta(1-\alpha)}} = \frac{-\beta^t m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \alpha}}{1 - \theta\mu e^{-\theta(1-\alpha)}} \tag{162}$$

and rearranging, and substituting in (159), we obtain

$$m'(\theta)(y(\theta,\alpha) - y_\mu) + m(\theta)\left(\frac{\partial y(\theta,\alpha)}{\partial \theta} - \frac{\partial y(\theta,\alpha)}{\partial \alpha}\frac{(1-\alpha)\alpha}{1-\theta\alpha}\right) = c. \tag{163}$$

Implicitly differentiating $\alpha = \mu e^{-\theta(1-\alpha)}$, we have

$$\alpha'(\theta) = \frac{-(1-\alpha)\alpha}{1-\theta\alpha} \tag{164}$$

and substituting (164) into (163) yields

$$m'(\theta)s(\theta) + m(\theta)\left(\frac{\partial y(\theta,\alpha)}{\partial \theta} + \frac{\partial y(\theta,\alpha)}{\partial \alpha}\alpha'(\theta)\right) = c. \tag{165}$$

Rearranging (165), we obtain:

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\frac{\frac{\partial y(\theta,\alpha)}{\partial \theta}\theta}{s(\theta)} + \frac{\frac{\partial y(\theta,\alpha)}{\partial \alpha}\alpha'(\theta)\theta}{s(\theta)}}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{surplus share of referral sellers}}, \tag{166}$$

which is identical to (80) in Example 3.10. Note that this is only a necessary condition.

Appendix B: Ex ante capital investment

Consider a model with *ex ante* capital investment and post-match bargaining based on Acemoglu and Shimer (1999b).[39] To illustrate the generalized Hosios condition, we incorporate a novel feature: the endogenous match output depends directly on *both* capital and the labor market tightness. The reason why we present this example is to highlight the *differences* between these two channels. If the expected match output is endogenous only in the sense that it depends on capital, as in Acemoglu and Shimer (1999b), the standard Hosios condition applies. However, if the expected match output is endogenous in the sense that it depends on the labor market tightness, the generalized Hosios condition applies.

To align the results with the previous examples, we incorporate a cost $c > 0$ of vacancy creation. There exists a perfect capital market where capital can be rented at price r. Importantly, capital k is an ex ante investment that is made by firms *prior* to the matching process. Wages are determined ex post by generalized Nash bargaining where workers have bargaining power $\beta \in [0, 1]$ and the value of non-market activity z is zero.

Let the output from a worker-firm match be $g(k, \theta)$ where $\theta \equiv V/U$, the ratio of vacancies to unemployed workers. We assume that $g(k, \theta) = A(\theta)y(k)$ where $A(\theta)$ is total factor productivity (TFP) and $y(k)$ is a standard neoclassical production function with capital-output elasticity $\varepsilon_y(k) \equiv y'(k)k/y(k)$ where $\varepsilon_y(k) < 1$. Since endogenizing $A(\theta)$ is not the focus of this example, we abstract from details here and simply assume $A(\theta)$ is given.[40]

The bargaining problem takes k as given and the wage is

$$w(k, \theta) = \arg\max_{w \in \mathbb{R}_+} (g(k, \theta) - w)^{1-\beta} w^{\beta}, \tag{167}$$

with well-known solution $w(k, \theta) = \beta g(k, \theta)$. Firms take the wage $w(k, \theta)$ as given and choose capital intensity k by solving

$$k(\theta) = \arg\max_{k \in \mathbb{R}_+} \left(\frac{m(\theta)}{\theta} (g(k, \theta) - w(k, \theta)) - rk \right). \tag{168}$$

Substituting in the bargained wage,

$$k(\theta) = \arg\max_{k \in \mathbb{R}_+} \left(\frac{m(\theta)}{\theta} (1 - \beta) g(k, \theta) - rk \right). \tag{169}$$

[39] Masters (1998, 2011) examines a frictional labor market model with two-sided ex ante investment in both physical and human capital. Mailath, Postlewaite, and Samuelson (2013) considers two-sided ex ante investment in a matching model with two-sided heterogeneity.

[40] It would be straightforward to endogenize $A(\theta)$ by using multilateral meetings and a distribution of match-specific productivities similar to Example 3.2 or 3.8.

The first-order condition is

$$\frac{m(\theta)}{\theta}(1-\beta)g_k(k,\theta) = r. \tag{170}$$

Under free entry of vacancies, we also have

$$\frac{m(\theta)}{\theta}(1-\beta)g(k,\theta) - rk = c. \tag{171}$$

An equilibrium (k^*, θ^*) solves equations (170) and (171).

The social planner chooses capital k for each firm opening a vacancy, and the labor market tightness θ, to maximize the social surplus per worker,

$$\Omega(k,\theta) = m(\theta)g(k,\theta) - rk\theta - c\theta. \tag{172}$$

The first-order conditions are:

$$\Omega_k(k,\theta) = m(\theta)g_k(k,\theta) - r\theta = 0 \tag{173}$$

$$\Omega_\theta(k,\theta) = \; m'(\theta)g(k,\theta) + m(\theta)g_\theta(k,\theta) - rk - c = 0. \tag{174}$$

Now let $g(k,\theta) = A(\theta)y(k)$. To prove the following two propositions, we assume the function $f(.)$ defined by $f(\theta) \equiv m(\theta)A(\theta)$ has the following properties: *(i)* $f'(\theta) > 0$ and $f''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$, *(ii)* $\lim_{\theta\to 0} f(\theta) = 0$, *(iii)* $\lim_{\theta\to 0} f'(\theta) \geq 1$, *(iv)* $\lim_{\theta\to\infty} f(\theta) = +\infty$, *(v)* $\lim_{\theta\to\infty} f'(\theta) = 0$, and *(vi)* $f(\theta)/\theta$ is strictly decreasing in θ for all $\theta \in \mathbb{R}_+$.[41]

Proposition 3. *If $\varepsilon_y'(k) \leq 0$ for all $k \in \mathbb{R}_+$, there exists a unique equilibrium (k^*, θ^*).*

Let $\sigma_f(\theta)$ be the elasticity of substitution between vacancies and unemployed workers for the function $f(.)$. The condition that $\sigma_f(\theta) \leq 1$ is equivalent to $\eta_f'(\theta) \leq 0$.[42]

Proposition 4. *If $\varepsilon_y'(k) \leq 0$ for all $k \in \mathbb{R}_+$ and $\sigma_f(\theta) \leq 1$ for all $\theta \in \mathbb{R}_+$, there exists a unique social optimum (k^P, θ^P).*

Using (173), we have efficiency of capital intensity k only when (k^*, θ^*) satisfies

$$\frac{m(\theta)}{\theta}g_k(k,\theta) = r. \tag{175}$$

[41] Mangin (2017) shows that these properties arise *endogenously* in an environment with multilateral meetings and a distribution of match-specific productivities, as described in Example 3.8. As shown in Mangin (2017), the property that $\sigma_f(\theta) \leq 1$ (used in Proposition 4) also arises endogenously in such an environment.

[42] In the special case where $A(\theta) = A$, we recover the standard regularity condition that requires the elasticity of the matching function to be weakly decreasing, i.e. $\eta_m'(\theta) \leq 0$.

The joint match surplus is $s(k,\theta) = g(k,\theta)$ and $\eta_s(k,\theta) \equiv g_\theta(k,\theta)\theta/g(k,\theta)$. Rearranging (174), we have constrained efficiency of *entry* only when the equilibrium (k^*,θ^*) satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(k,\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{(rk+c)\theta}{m(\theta)s(k,\theta)}}_{\text{surplus share of firms + capital}}. \tag{176}$$

This is essentially just the generalized Hosios condition found in Proposition 2, where the *effective* cost of entry is $rk + c$.

Using both equilibrium conditions (170) and (171) above, (k^*,θ^*) satisfies

$$1-\beta = \frac{(rk+c)\theta}{m(\theta)s(k,\theta)}. \tag{177}$$

Using $g(k,\theta) = A(\theta)y(k)$ and comparing (177) above with condition (176), we have constrained efficiency of entry if and only if θ^* satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_A(\theta)}_{\text{TFP elasticity}} = \underbrace{1-\beta}_{\text{firms' bargaining power}} \tag{178}$$

where $\eta_A(\theta) \equiv A'(\theta)\theta/A(\theta)$. Efficient entry requires that firms are compensated for their effect on both the *number* of matches created and the *value* of the expected match output, i.e. for their contribution to both *match creation* and *surplus creation*. The value of the expected match output is directly influenced by firm entry through the endogenous TFP term $A(\theta)$. Consistent with Corollary 1, if $A'(\theta) > 0$ there is a positive output externality that would result in under-entry of vacancies under the standard Hosios condition.

For efficiency of capital intensity k, the equilibrium (k^*,θ^*) must satisfy (175). Comparing with (170), this implies $\beta = 0$. Due to the "hold up" problem created by their ex ante investment in capital, firms require all the bargaining power. As discussed in detail in Acemoglu and Shimer (1999b), this means that efficiency of both entry and capital investment is not possible with ex post bargaining.[43] This example extends the result of Acemoglu and Shimer (1999b) to an environment where the expected match output depends directly on the market tightness and efficiency of entry requires the generalized Hosios condition.[44]

[43] In the special case where $g(k,\theta) = Ay(k)$ and $c = 0$, we recover a static version of the results in Section 4 of Acemoglu and Shimer (1999b). Constrained efficiency of both entry and capital intensity would require both $\eta_m(\theta^*) = 1-\beta$ and $\beta = 0$, which is impossible.

[44] Acemoglu and Shimer (1999b) show that efficiency of both entry and capital investment is indeed possible in a competitive search environment where firms post capital and workers direct their search. We expect the generalized Hosios condition to hold *endogenously* in this environment if wages are determined by competitive search rather than Nash bargaining.

Proof of Proposition 3

An equilibrium (k^*, θ^*) solves the following two equations:

$$\frac{m(\theta)}{\theta}(1-\beta)g_k(k,\theta) = r \tag{179}$$

and

$$\frac{m(\theta)}{\theta}(1-\beta)g(k,\theta) - rk = c. \tag{180}$$

Letting $g(k,\theta) = A(\theta)y(k)$ and $f(\theta) = m(\theta)A(\theta)$, these are equivalent to

$$\frac{f(\theta)}{\theta}(1-\beta)y'(k) = r \tag{181}$$

and

$$\frac{f(\theta)}{\theta}(1-\beta)y(k) - rk = c. \tag{182}$$

Combining (181) and (182), an equilibrium k satisfies

$$\frac{r}{y'(k)} = \frac{c+rk}{y(k)}. \tag{183}$$

Rearranging and simplifying, (183) is equivalent to

$$k\left(\frac{y(k)}{y'(k)k} - 1\right) = \frac{c}{r}. \tag{184}$$

Substituting $\varepsilon_y(k) \equiv y'(k)k/y(k)$ into (184), an equilibrium k satisfies

$$k\left(\frac{1}{\varepsilon_y(k)} - 1\right) = \frac{c}{r}. \tag{185}$$

Now suppose that $\varepsilon_y'(k) \leq 0$ for all k. Then $\frac{d}{dk}\left(\frac{1}{\varepsilon_y(k)}\right) \geq 0$, so $k\left(\frac{1}{\varepsilon_y(k)} - 1\right)$ is strictly increasing in k. Also, we have

$$\lim_{k\to 0} k\left(\frac{1}{\varepsilon_y(k)} - 1\right) = \lim_{k\to 0}\left(\frac{y(k)}{y'(k)} - k\right) = 0 \tag{186}$$

since $\lim_{k\to 0} y(k) = 0$ and $\lim_{k\to 0} y'(k) = \infty$. Finally, since $\varepsilon_y'(k) \leq 0$ and $\varepsilon_y(k) \in [0,1)$ by assumption, we have $\lim_{k\to\infty} \varepsilon_y(k) \in [0,1)$ and $\lim_{k\to\infty} k\left(\frac{1}{\varepsilon_y(k)} - 1\right) = +\infty$. So there exists a unique solution $k^* > 0$ to (185).

Given k^*, an equilibrium θ must satisfy (181), which is equivalent to

$$\frac{f(\theta)}{\theta} = \frac{r}{(1-\beta)y'(k^*)}. \tag{187}$$

By assumption, $f(\theta)/\theta$ is strictly decreasing in θ, so any solution θ^* is unique. Also, we have $\lim_{\theta\to 0} f(\theta)/\theta = \lim_{\theta\to 0} f'(\theta)$ by L'Hopital's rule, and $\lim_{\theta\to 0} f'(\theta) \geq 1$ by assumption. Finally, $\lim_{\theta\to\infty} f(\theta)/\theta = \lim_{\theta\to\infty} f'(\theta)$ by L'Hopital's rule, and $\lim_{\theta\to\infty} f'(\theta) = 0$ by assumption. So there exists a unique equilibrium (k^*, θ^*) where $\theta^* > 0$ if $r < (1-\beta)y'(k^*)$ where k^* is the unique solution to (185). If $r \geq (1-\beta)y'(k^*)$, then $\theta^* = 0$.

Proof of Proposition 4

Suppose that $\sigma_f(\theta) \leq 1$ for all $\theta \in \mathbb{R}_+$. We show that there exists a unique social optimum (k^P, θ^P). The first-order conditions for the social planner's problem are:

$$\Omega_k(k,\theta) = m(\theta)g_k(k,\theta) - r\theta = 0 \tag{188}$$

and

$$\Omega_\theta(k,\theta) = m'(\theta)g(k,\theta) + m(\theta)g_\theta(k,\theta) - rk - c = 0. \tag{189}$$

Letting $g(k,\theta) = A(\theta)y(k)$ and $f(\theta) = m(\theta)A(\theta)$, these are equivalent to

$$\Omega_k(k,\theta) = f(\theta)y'(k) - r\theta = 0 \tag{190}$$

and

$$\Omega_\theta(k,\theta) = \ f'(\theta)y(k) - rk - c = 0. \tag{191}$$

We can rewrite (190) as

$$y'(k) = \frac{r\theta}{f(\theta)} \tag{192}$$

Since $f(\theta)/\theta$ is strictly decreasing by assumption, the right-hand side of (192) is strictly increasing in θ. So for any given $\theta \in \mathbb{R}_+$, there is a unique $y'(k)$ that satisfies (192). Since $y''(k) < 0$ for all $k \in \mathbb{R}_+$ by assumption, k must also be unique and we can therefore write $k(\theta)$.

Using (192), we have

$$\lim_{\theta\to 0} y'(k(\theta)) = \lim_{\theta\to 0} \frac{r\theta}{f(\theta)} = \lim_{\theta\to 0} \frac{r}{f'(\theta)} \leq r \tag{193}$$

since $f'(0) \geq 1$. By assumption, $\lim_{k\to 0} y'(k) = \infty$, so $\lim_{\theta\to 0} k(\theta) > 0$. Also,

$$\lim_{\theta\to\infty} y'(k(\theta)) = \lim_{\theta\to\infty} \frac{r\theta}{f(\theta)} = \lim_{\theta\to\infty} \frac{r}{f'(\theta)} = \infty \tag{194}$$

since $f'(\infty) = \infty$. Since $\lim_{k\to 0} y'(k) = \infty$ and $y''(k) < 0$, this implies $\lim_{\theta\to\infty} k(\theta) = 0$. So we have $k(0) > 0$ and $k(\infty) = 0$.

We can explicitly derive $k'(\theta)$ using the implicit function theorem:

$$k'(\theta) = \frac{-(1-\eta_f(\theta))y'(k)}{\theta y''(k)} \tag{195}$$

where $\eta_f(\theta) \equiv f'(\theta)\theta/f(\theta)$. We have $k'(\theta) < 0$ for all $\theta \in (0,\infty)$ because $f(\theta)/\theta$ is strictly decreasing by assumption and therefore $\eta_f(\theta) < 1$.

Uniqueness. First, we prove the uniqueness of any θ^P that satisfies the social planner's first-order conditions. Rewriting (190) as $r = f(\theta)y'(k)/\theta$ and substituting this and $k(\theta)$ into (191), we obtain

$$L(\theta) \equiv \; f'(\theta)y(k(\theta)) - \frac{f(\theta)}{\theta} y'(k(\theta))k(\theta) = c. \tag{196}$$

To prove uniqueness, we show that $L'(\theta) < 0$. Differentiating (196), we obtain

$$\begin{aligned} L'(\theta) \;&=\; f''(\theta)y(k) + f'(\theta)y'(k)k'(\theta) - \left(\frac{f'(\theta)\theta - f(\theta)}{\theta^2}\right) y'(k)k \\ &\quad - \frac{f(\theta)}{\theta}\left(y''(k)k'(\theta)k + y'(k)k'(\theta)\right). \end{aligned} \tag{197}$$

Substituting in $\eta_f(\theta) \equiv f'(\theta)\theta/f(\theta)$ and $k'(\theta)$ using (195) yields

$$-\left(\frac{f'(\theta)\theta - f(\theta)}{\theta^2}\right) y'(k)k - \frac{f(\theta)}{\theta} y''(k)k'(\theta)k = 0 \tag{198}$$

so (197) is equivalent to

$$L'(\theta) = f''(\theta)y(k) + \left(f'(\theta) - \frac{f(\theta)}{\theta}\right) y'(k)k'(\theta). \tag{199}$$

Now, the elasticity of substitution $\sigma_f(\theta)$ is given by

$$\sigma_f(\theta) = \frac{-f'(\theta)\left(f(\theta) - f'(\theta)\theta\right)}{f''(\theta)f(\theta)\theta}. \tag{200}$$

Substituting the definitions of $\sigma_f(\theta)$ and $\eta_f(\theta)$ into (199), we find

$$L'(\theta) = \frac{f''(\theta)(y'(k))^2}{y''(k)} \frac{1 - \eta_f(\theta)}{\eta_f(\theta)} \left(\frac{y(k)y''(k)}{(y'(k))^2} \frac{\eta_f(\theta)}{1 - \eta_f(\theta)} + \sigma_f(\theta) \right). \tag{201}$$

Since $f''(\theta) < 0$ and $y''(k) < 0$ by assumption, and $\eta_f(\theta) < 1$, we have $L'(\theta) < 0$ if and only if

$$\sigma_f(\theta) < \frac{-y(k)y''(k)}{(y'(k))^2} \frac{\eta_f(\theta)}{1 - \eta_f(\theta)}. \tag{202}$$

Using both first-order conditions (190) and (191), we have

$$\frac{rk + c}{r\theta} = \frac{f'(\theta)y(k)}{f(\theta)y'(k)} \tag{203}$$

and, rearranging, this is equivalent to

$$\eta_f(\theta) = \frac{y'(k)k}{y(k)} \left(\frac{rk + c}{rk} \right). \tag{204}$$

Since $c > 0$, we have

$$\eta_f(\theta) > \varepsilon_y(k) \tag{205}$$

at any point satisfying the first-order conditions. Since $\eta_f(\theta) > \varepsilon_y(k)$,

$$\frac{\eta_f(\theta)}{1 - \eta_f(\theta)} = \frac{1}{\frac{1}{\eta_f(\theta)} - 1} > \frac{1}{\frac{y(k)}{y'(k)k} - 1}, \tag{206}$$

or, equivalently,

$$\frac{\eta_f(\theta)}{1 - \eta_f(\theta)} > \frac{y'(k)k}{y(k) - y'(k)k} \tag{207}$$

To prove (202), it therefore suffices to show that

$$\sigma_f(\theta) \leq \frac{-y''(k)y(k)k}{y'(k)(y(k)-y'(k)k)}. \tag{208}$$

But the right-hand side is just the reciprocal of the elasticity of substitution, $\sigma_y(k)$, of the function $y(.)$. So the condition we require is

$$\sigma_f(\theta)\sigma_y(k) \leq 1, \tag{209}$$

which is true. We have $\sigma_f(\theta) \leq 1$ by assumption and $\sigma_y(k) \leq 1$ since we assume that $\varepsilon_y'(k) \leq 0$ and $\sigma_y(k) \leq 1$ is equivalent to $\varepsilon_y'(k) \leq 0$. We therefore have uniqueness of the social planner's solution (k^P, θ^P) where $k^P = k(\theta^P)$.

Existence. Next, we establish existence of (k^P, θ^P). Rearranging (196),

$$L(\theta) = \frac{y(k(\theta))f(\theta)}{\theta}\left(\frac{f'(\theta)\theta}{f(\theta)} - \frac{y'(k(\theta))k(\theta)}{y(k(\theta))}\right) \tag{210}$$

Or, equivalently,

$$L(\theta) = \frac{y(k(\theta))f(\theta)}{\theta}\big(\eta_f(\theta) - \varepsilon_y(k)\big) \tag{211}$$

From (205), we know that $\eta_f(\theta) > \varepsilon_y(k)$. We also know that $\eta_f(\theta) \in (0,1)$ and $\varepsilon_y(k) \in (0,1)$, so $\eta_f(\theta) - \varepsilon_y(k) \in (0,1)$. We therefore have

$$\lim_{\theta \mapsto \infty} L(\theta) = \lim_{\theta \mapsto \infty} \frac{y(k(\theta))f(\theta)}{\theta}\big(\eta_f(\theta) - \varepsilon_y(k)\big) = 0 \tag{212}$$

using the fact that $\lim_{\theta \mapsto \infty} \frac{y(k(\theta))f(\theta)}{\theta} = \lim_{\theta \mapsto \infty} y(k(\theta))f'(\theta) = 0$ since $\lim_{\theta \mapsto \infty} f'(\theta) = 0$ and $\lim_{\theta \mapsto \infty} y(k(\theta)) = \lim_{k \mapsto 0} y(k) = 0$ since $k(\infty) = 0$. We also have

$$\lim_{\theta \mapsto 0} L(\theta) = \lim_{\theta \mapsto 0} \frac{y(k(\theta))f(\theta)}{\theta}\big(\eta_f(\theta) - \varepsilon_y(k)\big) > \big(\eta_f(\theta) - \varepsilon_y(k)\big)\, y(k(0)) \tag{213}$$

where $k(0) > 0$, using the fact that $\lim_{\theta \mapsto 0} \frac{f(\theta)}{\theta} = \lim_{\theta \mapsto 0} f'(\theta)$ by L'Hopital's rule and $\lim_{\theta \mapsto 0} f'(\theta) \geq 1$ by assumption. Therefore, if $c < \lim_{\theta \mapsto 0} \big(\eta_f(\theta) - \varepsilon_y(k)\big)\, y(k(\theta))$, there exists a unique $\theta^P > 0$ and $k^P = k(\theta^P) > 0$. Otherwise, $\theta^P = 0$ and $k^P = k(\theta^P) = k(0) > 0$.

It remains only to prove that the unique solution (k^P, θ^P) is a global maximizer for $\Omega(k, \theta)$. Consider the Hessian matrix H of partial derivatives of $\Omega(k, \theta)$. If $\det H > 0$ and $\Omega_{\theta\theta}(k, \theta) < 0$ at (k^P, θ^P), then it is a unique local minimum and we need only check the boundaries to ensure it is also a global maximum. Substituting into (172), we find that $\Omega(k,0) = 0$ and $\Omega(0,\theta) = -c\theta < 0$. Since $\Omega(k^P, \theta^P) > 0$ when $\theta^P > 0$ and $k^P > 0$, (k^P, θ^P) is the unique global maximizer for $\Omega(.)$ provided that it is the unique local minimum.

Using (190) and (191), the partial derivatives of $\Omega(k,\theta)$ are

$$\begin{aligned} \Omega_{kk}(k,\theta) &= f(\theta)y''(k) \\ \Omega_{\theta\theta}(k,\theta) &= f''(\theta)y(k) \\ \Omega_{k\theta}(k,\theta) &= f'(\theta)y'(k) - r \end{aligned} \tag{214}$$

Clearly, $\Omega_{\theta\theta}(k,\theta) < 0$ if $k > 0$ since $f''(\theta) < 0$ by assumption. We have $\det H > 0$ if and only if

$$\Omega_{\theta\theta}(k,\theta)\Omega_{kk}(k,\theta) - (\Omega_{k\theta}(k,\theta))^2 > 0. \tag{215}$$

Substituting in the partial derivatives (214), $\det H > 0$ if and only if

$$f''(\theta)y(k)f(\theta)y''(k) > (f'(\theta)y'(k) - r)^2. \tag{216}$$

Using the first-order condition (190), we have $r = f(\theta)y'(k)/\theta$, so we require

$$f''(\theta)y(k)f(\theta)y''(k) > \left(f'(\theta)y'(k) - \frac{f(\theta)y'(k)}{\theta}\right)^2. \tag{217}$$

With some algebra, this is equivalent to

$$\sigma_f(\theta) < \frac{-y(k)y''(k)}{(y'(k))^2}\frac{\eta_f(\theta)}{1-\eta_f(\theta)} \tag{218}$$

which is identical to inequality (202) that is proven above.

References

D. Acemoglu and R. Shimer. Efficient unemployment insurance. *Journal of Political Economy*, 107(5):pp. 893–928, 1999a.

D. Acemoglu and R. Shimer. Holdups and Efficiency with Search Frictions. *International Economic Review*, 40(4):827–849, 1999b.

J. Albrecht, L. Navarro, and S. Vroman. The effects of labour market policies in an economy with an informal sector. *The Economic Journal*, 119(539):1105–1129, 2009.

J. Albrecht, L. Navarro, and S. Vroman. Efficiency in a search and matching model with endogenous participation. *Economics Letters*, 106(1):48–50, 2010.

J. Albrecht, P. Gautier, and S. Vroman. A Note on Peters and Severinov, "Competition Among Sellers Who Offer Auctions Instead of Prices". *Journal of Economic Theory*, 147: 389–392, 2012.

J. Albrecht, P. Gautier, and S. Vroman. Efficient Entry in Competing Auctions. *American Economic Review*, 104(10):3288–96, 2014.

J. Albrecht, P. A. Gautier, and S. Vroman. Directed search in the housing market. *Review of Economic Dynamics*, 2016.

A. Berentsen, G. Menzio, and R. Wright. Inflation and unemployment in the long run. *American Economic Review*, 101(1):371–98, February 2011.

K. Burdett, S. Shi, and R. Wright. Pricing and Matching with Frictions. *Journal of Political Economy*, 109(5):1060–1085, 2001.

A. Campbell, M. Leister, and Y. Zenou. Word of mouth communication and search. *CEPR Discussion Paper No. 12326*, 2017.

O. Charlot, F. Malherbet, and M. Ulus. Efficiency in a search and matching economy with a competitive informal sector. *Economics Letters*, 118(1):192 – 194, 2013.

J. Eeckhout and P. Kircher. Sorting and decentralized price competition. *Econometrica*, 78 (2):539–574, 2010a.

J. Eeckhout and P. Kircher. Sorting versus screening: Search frictions and competing mechanisms. *Journal of Economic Theory*, 145(4):1354–1385, 2010b.

F. Gavrel. On the efficiency of participation with vertically differentiated workers. *Economics Letters*, 112(1):100 – 102, 2011.

F. Gavrel. On the inefficiency of matching models of unemployment with heterogeneous workers and jobs when firms rank their applicants. *European Economic Review*, 56(8): 1746–1758, 2012.

V. Guerrieri. Inefficient unemployment dynamics under asymmetric information. *Journal of Political Economy*, 116(4):667–708, 2008.

V. Guerrieri, R. Shimer, and R. Wright. Adverse selection in competitive search equilibrium. *Econometrica*, 78(6):1823–1862, 2010.

A. J. Hosios. On the Efficiency of Matching and Related Models of Search and Unemployment. *Review of Economic Studies*, 57(2):279–298, 1990.

B. Julien and S. Mangin. Efficiency of job creation in a search and matching model with labor force participation. *Economics Letters*, 150:149–151, 2017.

B. Julien and G. Roger. Competitive Search Equilibria and Moral Hazard. *Working paper*, 2015.

B. Julien, J. Kennes, and I. King. Bidding for Labor. *Review of Economic Dynamics*, 3(4): 619–649, 2000.

M. Kamien and N. Schwartz. *Dynamic Optimization, 2nd edn.* New York: Elsevier Science Publishing, 1991.

K. Kim and P. Kircher. Efficient competition through cheap talk: The case of competing auctions. *Econometrica*, 83(5):1849–1875, 2015.

L. Ljungqvist and T. J. Sargent. The fundamental surplus. *American Economic Review*, 107 (9):2630–65, 2017.

G. J. Mailath, A. Postlewaite, and L. Samuelson. Pricing and investments in matching markets. *Theoretical Economics*, 8(2):535–590, 2013.

S. Mangin. A theory of production, matching, and distribution. *Journal of Economic Theory*, 172:376–409, 2017.

S. Mangin and P. Sedlacek. Unemployment and the Labor Share. *Journal of Monetary Economics (forthcoming)*, 2017.

R. Marimon and F. Zilibotti. Unemployment vs. mismatch of talents: Reconsidering unemployment benefits. *The Economic Journal*, 109(455):266–291, 1999.

A. Masters. Efficiency of investment in human and physical capital in a model of bilateral search and bargaining. *International Economic Review*, 39(2):477–94, 1998.

A. Masters. Commitment, advertising and efficiency of two-sided investment in competitive search equilibrium. *Journal of Economic Dynamics and Control*, 35(7):1017 – 1031, 2011.

A. Masters. Efficiency in a search and matching model with participation policy. *Economics Letters*, 134:111 – 113, 2015.

E. Moen. Competitive Search Equilibrium. *Journal of Political Economy*, 105(2):385–411, 1997.

E. R. Moen and A. Rosen. Incentives in competitive search equilibrium. *The Review of Economic Studies*, 78(2):733–761, 2011.

J. Montgomery. Equilibrium Wage Dispersion and Interindustry Wage Differentials. *Quarterly Journal of Economics*, 106(1):163–179, 1991.

D. Mortensen and C. Pissarides. Job Creation and Job Destruction in the Theory of Unemployment. *Review of Economic Studies*, 61(3):397–415, 1994.

D. T. Mortensen. Property rights and efficiency in mating, racing, and related games. *The American Economic Review*, 72(5):968–979, 1982a.

D. T. Mortensen. The matching process as a noncooperative bargaining game. *The Economics of Information and Uncertainty*, pages 233–258, 1982b.

M. Peters. Bertrand equilibrium with capacity constraints and restricted mobility. *Econometrica*, pages 1117–1127, 1984.

M. Peters. Ex Ante Price Offers in Matching Games: Non-Steady States. *Econometrica*, 59 (5):1425–1454, 1991.

M. Peters and S. Severinov. Competition among Sellers Who Offer Auctions Instead of Prices. *Journal of Economic Theory*, 75(1):141–179, 1997.

C. Pissarides. Unemployment and vacancies in britain. *Economic Policy*, 1(3):500–559, 1986.

C. Pissarides. *Equilibrium Unemployment Theory*. MIT Press, Cambridge, MA, 2000.

R. Rogerson, R. Shimer, and R. Wright. Search-Theoretic Models of the Labor Market: A Survey. *Journal of Economic Literature*, 43(4):959–988, 2005.

S. Shi. Frictional Assignment I: Efficiency. *Journal of Economic Theory*, 98(2):232–260, 2001.

S. Shi. A Directed Search Model of Inequality with Heterogeneous Skills and Skill-Biased Technology. *Review of Economic Studies*, 69:467–491, 2002.

R. Shimer. PhD Thesis. *M.I.T.*, 1996.

R. Shimer. The Cyclical Behavior of Equilibrium Unemployment and Vacancies. *American Economic Review*, 95(1):25–49, 2005a.

R. Shimer. The Assignment of Workers to Jobs in an Economy with Coordination Frictions. *Journal of Political Economy*, 113(5):996–1025, 2005b.

R. Shimer and L. Smith. Assortative matching and search. *Econometrica*, 68(2):343–369, 2000.

R. Shimer and L. Smith. Matching, search, and heterogeneity. *Advances in Macroeconomics*, 1(1), 2001.

R. Wolthoff. Applications and interviews: Firms' recruiting decisions in a frictional labor market. *The Review of Economic Studies, forthcoming*, 2017.

R. Wright, P. Kircher, B. Julien, and V. Guerrieri. Directed search: A guided tour. *CEPR Discussion Paper No. 12315*, 2017.