Efficiency in Search and Matching Models: A Generalized Hosios Condition*

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Abstract

When is entry efficient in markets with search and matching frictions? This paper generalizes the well-known Hosios condition to dynamic environments where the expected match output depends on the market tightness. Entry is efficient when buyers' surplus share equals the *matching elasticity* plus the *surplus elasticity* (i.e. the elasticity of the expected match surplus with respect to buyers). This ensures agents are paid for their contribution to both *match creation* and *surplus creation*. For example, vacancy entry in the labor market is efficient only when firms are compensated for the effect of job creation on both employment and labor productivity. *JEL Codes:* C78, D83, E24, J64

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1 Introduction

Consider a search-and-matching model in which buyers and sellers are matched according to a frictional matching process and there is free entry on one side, e.g. buyer entry. There are two externalities related to entry: the congestion and thick market externalities. The former is a negative externality that arises because greater buyer entry reduces the matching probability of each buyer. The latter is a positive externality that arises because greater buyer entry increases the matching probability of each seller. Hosios (1990) asked the question: When is entry *constrained efficient*?[1]

The answer provided by Hosios (1990) is remarkably simple: entry is efficient only when buyers' share of the joint match surplus equals the elasticity of the matching function with respect to buyers. This result is now widely known as the "Hosios condition."[2] When this condition holds, markets internalize the search externalities that arise through the frictional matching process. When it fails, markets do not internalize these externalities, leading to inefficiently high or low entry.

The Hosios condition has proven to be widely applicable across a broad range of search-and-matching models. However, it does not guarantee efficiency in settings where the *expected match output* – i.e. the expected output conditional on matching – depends on the market tightness or buyer-seller ratio.[3] In such environments, entry affects not only the number of matches but also the expected match output. An additional externality arises – which we call the *output externality* – that may be positive or negative. The output externality is not internalized by the Hosios condition: entry may be inefficiently high or low when this condition holds.

This paper provides a generalization of the Hosios (1990) condition to a wide class of dynamic search-and-matching models where the expected match output may depend on the market tightness. Our simple, intuitive generalization provides a *unifying lens* for understanding the efficiency of entry across a broad range of models which may appear quite different on the surface.

To see why our generalization is necessary, consider an environment with buyer

[1] By *constrained efficiency*, we mean the social planner is constrained by the same frictions as the decentralized market economy.

[2] This condition is sometimes called the "Hosios-Mortensen" condition. Early versions of it were discussed in Mortensen (1982b), Mortensen (1982a), and Pissarides (1986).

[3] We use the term "output" because we have in mind labor markets, but the term *output* can be interpreted more broadly. Similarly, we use the terms "buyers" and "sellers" but we could instead speak more generally about "organizers" of trade and "visitors" as in Shi and Delacroix (2018).

entry. Entry is efficient only when buyers are paid their marginal contribution to the social surplus. If the expected match output is exogenous, buyers need only be paid for their effect on *match creation*, i.e. on the number of matches, and the standard Hosios condition applies. If the expected match output is endogenous, however, buyers must also be compensated for their effect on *surplus creation*, i.e. on the expected value of the joint match surplus. A generalization of the Hosios condition is thus required.

Our main result is that entry is constrained efficient only when buyers' surplus share equals the *matching elasticity* plus the *surplus elasticity* (i.e. the elasticity of the expected match surplus with respect to buyers). We call this simple condition the "generalized Hosios condition". When this condition holds, both the standard search externalities and the output externality are internalized. Whether or not this condition holds in a particular market depends on how prices are determined.

The importance of the generalized Hosios condition is particularly clear in search-theoretic models of the labor market. First, the Hosios condition is used to determine the efficient level of vacancy entry and thus unemployment. In environments where labor productivity depends on the market tightness, however, the standard Hosios condition does not guarantee efficiency. If this condition is mistakenly used to determine the efficient level of vacancy entry, unemployment may be inefficiently high or low. The generalized Hosios condition is required to ensure that firms are compensated for the effect of job creation on both unemployment and labor productivity.

Second, the Hosios condition is often used to calibrate models in which wages are determined by Nash bargaining. In environments where the standard Hosios condition suffices for efficiency, it is possible to *impose* this condition (and thus efficiency) by using a Cobb-Douglas matching technology and setting firms' bargaining parameter equal to the constant matching elasticity, e.g. Shimer (2005). Importantly, the generalized Hosios condition does not allow this calibration trick. In environments where this condition is necessary for efficiency, we cannot simply restore efficiency by a particular choice of matching technology and bargaining parameter. This is because the surplus elasticity – unlike the matching elasticity – is always endogenous.

In this sense, the inefficiencies that obtain when the generalized Hosios condition fails are harder to eliminate. However, we find that it is possible to decentralize the efficient allocation through directed or competitive search.[4]

[4] See Wright, Kircher, Julien, and Guerrieri (2021) for a survey on directed or competitive search.

Outline. Section 2 presents a simple, motivating example. Section 3 derives our main result, the generalized Hosios condition. Section 4 presents some examples. Section 5 discusses how to apply the condition. Section 6 concludes. The Appendix contains omitted proofs. The Online Appendix includes a generalization of our results to more general matching and output technologies, as well as additional examples.

2 Simple example

To motivate our question and build intuition, consider a static environment. There is a continuum of measure v of vacancies (or "firms") and a continuum of measure u of unemployed workers. All workers are ex ante identical and all firms are ex ante identical. Agents are risk-neutral. The market tightness is $\theta \equiv v/u$.

Each firm meets exactly one worker, but a worker can meet (or receive an "offer" from) many firms. A worker meets $n \in \mathbb{N} = \{0, 1, 2, ...\}$ firms with probability $P_n(\theta)$ where $\sum_{n=0}^{\infty} P_n(\theta) = 1$. The *meeting* probability for a worker (i.e. the probability a worker meets at least one firm) is $1 - P_0(\theta)$.

After meetings occur, each worker draws an i.i.d. match-specific productivity x for each firm he meets and then chooses to work for exactly one of them.[5] Match-specific productivities $x \in X = \{x_L, x_H\} \subseteq \mathbb{R}_+$, where $x_L < x_H$ and $\Pr(x = x_L) = \alpha \in [0, 1]$. Workers who are hired by a firm produce output equal to the match-specific productivity for that firm. Workers who are not hired receive payoff $z \geq 0$ where $z < x_L$, so all matches have positive surplus.

Let $m(\theta)$ denote the *matching* probability for a worker (i.e. the probability a worker is hired). Every worker who meets a firm will be hired, so the matching probability for workers equals the meeting probability, i.e. $m(\theta) = 1 - P_0(\theta)$. The matching probability for a firm (i.e. the probability a firm hires a worker) is $m(\theta)/\theta$.

Suppose workers always choose to work for the firm they meet with the highest productivity.[6] Let $f : X \to [0, 1]$ be the endogenous density of the distribution of output across *employed workers* where $\sum_{x \in X} f(x; \theta) = 1$. The *expected match output* $y(\theta)$ (i.e. expected output conditional on matching) is $y(\theta) \equiv \sum_{x \in X} x f(x; \theta)$. Since

[5] Shi (2002) presents a related model that features two-sided *ex ante* heterogeneity (two types of workers and two types of firms) and shows that directed search delivers constrained efficiency.

[6] This will be true, for example, if wages are determined by either auctions or Nash bargaining between a worker and their chosen firm. We assume that workers randomize when indifferent.

an employed worker produces output x_L if and only if *all* n firms he met drew x_L,

$$f(x_L;\theta) = \frac{\sum_{n=1}^{\infty} P_n(\theta)\alpha^n}{1 - P_0(\theta)}. \tag{1}$$

While the distribution of match-specific productivities is exogenous and given by $\Pr(x = x_L) = \alpha$, the distribution of output across employed workers is endogenous and given by $f(x;\theta)$. For any $\theta > 0$, the expected match output $y(\theta)$ is strictly increasing in θ, i.e. $y'(\theta) > 0$. Intuitively, this is because a higher number of vacancies per unemployed worker allows workers to be more *selective*.

Suppose the social planner can create vacancies at cost $c > 0$. What is the efficient level of vacancy creation? We are interested in *constrained efficiency* in the sense that the planner is constrained by the functions $m(.)$ and $y(.)$. The planner chooses a market tightness θ to maximize the social surplus, i.e. the total market output of employed workers, plus the total payoff for workers who remain unemployed, minus the total costs of vacancy creation. The social surplus per worker, $\Omega(\theta)$, is given by

$$\Omega(\theta) = m(\theta)y(\theta) + (1 - m(\theta))z - c\theta. \tag{2}$$

The *expected match surplus* $s(\theta)$ (i.e. expected surplus conditional on matching) is given by $s(\theta) = y(\theta) - z$ and we can write:

$$\Omega(\theta) = m(\theta)s(\theta) + z - c\theta. \tag{3}$$

Any solution $\theta^P > 0$ must satisfy the following necessary condition:

$$m'(\theta)s(\theta) + m(\theta)s'(\theta) = c. \tag{4}$$

Let $\eta_m(\theta) \equiv m'(\theta)\theta/m(\theta)$, the *matching elasticity*. Let $\eta_s(\theta) \equiv s'(\theta)\theta/s(\theta)$, the *surplus elasticity*. Rearranging (4), an optimal solution $\theta^P > 0$ must satisfy

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{firms' surplus share}}. \tag{5}$$

With free entry of vacancies, the total payoff for firms equals the total entry cost, cv, and the total surplus created is $m(\theta)s(\theta)u$, so the right term of (5) is *firms' surplus* *share*.[7] Intuitively, efficiency requires that firms' surplus share equals the matching elasticity plus the surplus elasticity because this ensures that firms' entry decisions internalize the effects of vacancy creation on both employment and labor productivity.

The original Hosios (1990) condition can be derived as a special case of (5). Suppose that $\alpha = 1$. The expected match output is $y(\theta) = x_L$, the expected match surplus is $s(\theta) = x_L - z$, and (5) simplifies to the Hosios (1990) condition:

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{firms' surplus share}}. \tag{6}$$

In this case, it suffices for efficiency that firms' surplus share equals the matching elasticity because firms' entry decisions need only internalize the effects of vacancy creation on employment (or number of matches).

We refer to condition (5) as the "generalized Hosios condition". Like the original Hosios (1990) condition given by (6), it turns out that this simple condition characterizes efficiency across a wide range of dynamic search-and-matching environments.

3 Generalized Hosios Condition

Consider a general dynamic environment that features buyers and sellers. Time is discrete. In any period $t \in \{0, 1, ...\}$, there is continuum of measure one of sellers and a continuum of potential buyers with sufficiently large measure.[8] All agents are risk-neutral. The measure of buyers who enter is denoted by v. The measure of unmatched sellers is denoted by u. The market tightness is $\theta \equiv v/u$. Matches are destroyed with probability $\delta \in (0, 1]$. The flow payoff for unmatched sellers is $z \geq 0$. For simplicity, we assume all matches have positive surplus.

Buyers and sellers are matched according to a constant-returns-to-scale matching function with matching probabilities for sellers and buyers denoted respectively by $m(\theta)$ and $m(\theta)/\theta$. We call the function $m(.)$ the *matching technology*.

Assumption 1. *The function $m(.)$ satisfies: (i) $m'(\theta) > 0$ and $m''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$, (ii) $\lim_{\theta\to 0} m(\theta) = 0$, (iii) $\lim_{\theta\to 0} m'(\theta) = 1$, (iv) $\lim_{\theta\to\infty} m(\theta) = 1$, (v)*

[7] Alternatively, let $s_f(\theta)$ denote firms' surplus after matching. With free entry of vacancies, we have $\frac{m(\theta)}{\theta} s_f(\theta) = c$, thus the right term of (5) is equivalent to $s_f(\theta)/s(\theta)$, firms' surplus share.

[8] "Sufficiently large" means that seller entry is not constrained by the measure of potential sellers.

$\lim_{\theta\to\infty} m'(\theta) = 0$, *and (vi)* $m(\theta)/\theta$ *is strictly decreasing for all* $\theta \in \mathbb{R}_+$.

Within each period, the timing is as follows. First, match destruction occurs. Next, buyers enter and search takes place. Finally, production occurs.

Match output $x \in X \equiv \{x_1, x_2, ..., x_N\} \subseteq \mathbb{R}_+\backslash\{0\}$ where $x_1 < x_2 < ... < x_N$. Given market tightness θ, the output of a *newly matched* seller is an i.i.d. draw from a discrete probability distribution with density $f : X \to [0,1]$ where $\sum_{x\in X} f(x;\theta) = 1$.[9] The density $f(x;\theta)$ will be determined endogenously by features of the environment. The *expected match output* (i.e. expected output conditional on a match) is

$$y(\theta) \equiv \sum_{x\in X} x f(x;\theta). \tag{7}$$

We call the function $y(.)$ the *output technology*.[10] If the density $f(x;\theta)$ does not depend on the market tightness, i.e. $f(x;\theta) = f(x)$, then $y(\theta) = \bar{y} \in \mathbb{R}_+$.

The market output of *any* seller in the economy (either matched or unmatched) is given by a discrete probability distribution with density $\psi : X \cup \{0\} \to [0,1]$ where $\sum_{X\cup\{0\}} \psi(x) = 1$. Since unmatched sellers produce zero output, we have $\psi(0) = u$.

Let ψ_+ denote the next period's density. This is the density of the distribution of output across all sellers at the production stage, i.e. the beginning of the next period. The density ψ evolves according to the following law of motion:

$$\psi_+(x) = \begin{cases} u(1-m(\theta)) + \delta(1-u) & \text{for } x = 0 \\ um(\theta)f(x;\theta) + (1-\delta)\psi(x) & \text{for } x \in X \end{cases} \tag{8}$$

Since $\psi_+(0) = u_+$, the measure of unmatched sellers in the next period, the first case of (8) is the law of motion for unmatched sellers. The second case of (8) is the law of motion for the measure of matched sellers producing output $x \in X$.

Planner's problem. Suppose the planner can allow buyers to enter at cost $c > 0$ each period. At the start of a period, the planner observes the aggregate state of the economy, given by the density ψ, and chooses a market tightness $\theta = v/u$ where $\theta \in \mathbb{R}_+$. The planner is restricted to take both the matching technology $m(.)$ and the

[9] Our results also hold if the distribution of match output is continuous and bounded.

[10] Online Appendix A generalizes our results to matching functions $M(v,u)$, densities $f(x;(v,u))$, and output technologies $y(v,u)$ that are not necessarily constant-returns-to-scale.

density $f(x;\theta)$ as given, and chooses the market tightness θ to maximize the sum of present and future social surplus, discounted by factor $\beta \in (0,1)$.

The Bellman equation for the planner's value function $W(\psi)$ can be written as:

$$W(\psi) = \max_{\theta \in \mathbb{R}_+} \left\{ \Omega(\theta|\psi) + \beta W(\psi_+) \right\} \tag{9}$$

where ψ_+, the next period's state, is given by the law of motion (8), and

$$\Omega(\theta|\psi) = \sum_{x \in X} x\psi_+(x) + zu_+ - c\theta u. \tag{10}$$

The flow value of the social surplus, $\Omega(\theta|\psi)$, is equal to the total market output of matched sellers, $\sum_{x\in X} x\psi_+(x)$, plus the total flow payoff for unmatched sellers, zu_+, at the production stage, minus the total costs of buyer entry, $cv = c\theta u$.

Our planner's problem generalizes that considered in Menzio and Shi (2011) to environments where the density $f(x;\theta)$ may depend on market tightness.[11]

Lemma 1. *(i) The planner's value function $W(\psi)$ is the unique solution to (9); and (ii) $W(\psi)$ can be written as a linear function:*

$$W(\psi) = W_u u + \sum_{x \in X} W_e(x)\psi(x), \tag{11}$$

where

$$W_u = \max_{\theta \in \mathbb{R}_+} \left\{ -c\theta + (1 - m(\theta))(z + \beta W_u) + m(\theta)(y(\theta) + \beta \tilde{W}_e(\theta)) \right\} \tag{12}$$

and

$$\tilde{W}_e(\theta) \equiv \sum_{x \in X} W_e(x) f(x;\theta) \tag{13}$$

and

$$W_e(x) = \delta(z + \beta W_u) + (1 - \delta)(x + \beta W_e(x)). \tag{14}$$

[11] Theorem 1 in Menzio and Shi (2011) generalizes to our environment. More precisely, the special case of our planner's problem where $f(x;\theta) = f(x)$ is a special case of that considered in Menzio and Shi (2011). We abstract from features such as on-the-job search, aggregate productivity shocks, and signals in order to focus attention on what is novel here.

Before presenting our main result, the dynamic *expected match surplus* $s(\theta)$ is

$$s(\theta) = y(\theta) - z + \beta(\tilde{W}_e(\theta) - W_u). \tag{15}$$

Assumption 2 is sufficient for the existence and uniqueness of the efficient choice θ^P.[12] Assumptions 1 and 2 are maintained throughout the remainder of Section 3.

Assumption 2. *The function* $\Lambda(.)$ *defined by* $\Lambda(\theta) \equiv m(\theta)s(\theta)$ *satisfies: (i)* $\lim_{\theta\to 0}\Lambda(\theta) = 0$; *(ii)* $\lim_{\theta\to 0}\Lambda'(\theta) > c$; *(iii)* $\lim_{\theta\to\infty}\Lambda'(\theta) < c$; *and (iv)* $\Lambda''(\theta) < 0$ *for all* $\theta \in \mathbb{R}_+$.

Proposition 1 says that the planner chooses to set buyers' surplus share equal the *matching elasticity* plus the *surplus elasticity.* When this condition holds, the level of buyer entry is efficient because agents are compensated for their effect on both *match creation* and *surplus creation.* Importantly, this intuitive condition characterizes efficiency along the entire equilibrium path. In the absence of aggregate productivity shocks, the optimal market tightness is constant over time.

Lemma 1 greatly simplifies the proof of Proposition 1.

Proposition 1 (Generalized Hosios Condition). *There exists a unique efficient allocation* $(\theta_t^P)_{t=0}^{\infty}$ *where* $\theta_t^P = \theta^P > 0$ *for all* t *and* θ^P *satisfies the following condition:*

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{buyers' surplus share}}. \tag{16}$$

Proof. Rearranging (12) from Lemma 1, the optimal θ^P is given by

$$\theta^P = \arg\max_{\theta\in\mathbb{R}_+}\left\{-c\theta + z + \beta W_u + m(\theta)(y(\theta) - z + \beta(\tilde{W}_e(\theta) - W_u))\right\}. \tag{17}$$

Using (15), problem (17) above can be rewritten as

$$\theta^P = \arg\max_{\theta\in\mathbb{R}_+}\left\{m(\theta)s(\theta) + z - c\theta + \beta W_u\right\}. \tag{18}$$

Taking the first-order condition for (18), the optimal θ^P satisfies

$$m'(\theta)s(\theta) + m(\theta)s'(\theta) \leq c \tag{19}$$

[12] If Assumption 2 does not hold, (16) is still a necessary condition for efficiency.

and $\theta^P \geq 0$ with complementary slackness. Assumption 2 implies there exists a unique solution $\theta^P > 0$ that satisfies $\Lambda'(\theta) = c$ where $\Lambda(\theta) \equiv m(\theta)s(\theta)$. Therefore, there exists a unique $\theta^P > 0$ that satisfies $m'(\theta)s(\theta) + m(\theta)s'(\theta) = c$. Dividing both sides of this equation by $m(\theta)s(\theta)/\theta$ yields (16) in terms of elasticities. ■

The following provides a useful version of the generalized Hosios condition that is easier to apply in practice. If $y'(\theta) = 0$, we recover the standard Hosios condition.

Proposition 2. *There exists a unique efficient allocation $(\theta_t^P)_{t=0}^{\infty}$ where $\theta_t^P = \theta^P > 0$ for all t and θ^P satisfies the following condition:*

$$\eta_m(\theta) + \frac{y'(\theta)\theta}{(1-\beta(1-\delta))s(\theta)} = \frac{c\theta}{m(\theta)s(\theta)}. \tag{20}$$

Proof. Differentiating (15), we have

$$s'(\theta) = y'(\theta) + \beta \tilde{W}_e'(\theta). \tag{21}$$

Using (14) and definition (13), we obtain

$$\tilde{W}_e(\theta) = \frac{\delta(z + \beta W_u) + (1-\delta)y(\theta)}{1-\beta(1-\delta)}. \tag{22}$$

Differentiating (22) yields

$$\tilde{W}_e'(\theta) = \frac{(1-\delta)y'(\theta)}{1-\beta(1-\delta)}. \tag{23}$$

Substituting (23) into (21) and simplifying yields

$$s'(\theta) = \frac{y'(\theta)}{1-\beta(1-\delta)}. \tag{24}$$

Finally, substituting (24) into (16) delivers condition (20). ■

Output externality. In search-and-matching models with free entry, there are two standard externalities related to the frictional matching process: the congestion and thick market externalities. In environments where the expected match output depends on market tightness, a novel externality – the *output externality* – arises. A higher buyer/seller ratio may either increase or decrease the expected match output.

Consider an environment in which the generalized Hosios condition is necessary for efficiency. When the standard Hosios condition holds, entry decisions fail to internalize the output externality and entry may not be efficient.

Corollary 1 tells us that the direction of the inefficiency depends only on the derivative of the output technology $y(.)$ at the equilibrium θ^*.

Corollary 1. *A steady state equilibrium allocation features* under-entry *(*over-entry*) of buyers under the standard Hosios condition if and only if* $y'(\theta^*) > (<)\ 0$.

When $y'(\theta^*) > 0$, the output externality arising from buyer entry is *positive* and the standard Hosios condition results in under-entry. Alternatively, if $y'(\theta^*) < 0$, the output externality is *negative* and the standard Hosios condition results in over-entry. If $y'(\theta^*) = 0$, there is no output externality and buyer entry is efficient.

Returning to our example in Section 2, the standard Hosios condition would result in under-entry of vacancies since $y'(\theta) > 0$ and the output externality is positive. Intuitively, this is because it does not internalize the fact that higher job creation leads not only to lower unemployment for workers, but also higher labor productivity.[13]

Seller entry. When there is *seller entry* at cost $c > 0$, instead of buyer entry, the direction of the effect of entry is reversed. Since the buyers' surplus share and the sellers' surplus share add to one, an efficient $\theta^P > 0$ must satisfy

$$1 - \underbrace{\eta_m(\theta)}_{\text{matching elasticity}} - \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c}{m(\theta)s(\theta)}}_{\text{sellers' surplus share}} . \tag{25}$$

Corollary 2. *A steady state equilibrium allocation features* over-entry *(*under-entry*) of sellers under the standard Hosios condition if and only if* $y'(\theta^*) > (<)\ 0$.

With seller entry, the direction of Corollary 1 is reversed since $\theta^* < \theta^P$ implies *over-entry* of sellers (because $\theta = v/u$) and $\theta^* > \theta^P$ implies *under-entry*.

[13] In an alternative environment where workers instead apply to firms, we would have $y'(\theta) < 0$. In this case, the output externality is negative and the standard Hosios rule would result in over-entry of vacancies. For examples in the literature, see Gavrel (2012) and Wolthoff (2017).

4 Examples

We present two examples. The first features sequential markets and bilateral meetings. The second features many-on-one meetings and auctions with *ex post* buyer heterogeneity.[14] We choose these examples because they are as different as possible – in terms of the types of heterogeneity, market structure, and meetings.

4.1 Sequential labor market and goods market

One way in which the expected match output may depend on market tightness is through a *sequential channel.* This may arise when there are sequential markets, such as a labor market and a goods market, and the possibility of trade in the second market depends on the matching outcomes in the first.[15] One example is Berentsen, Menzio, and Wright (2011). We present a static, simplified version of that model.[16]

Workers sell their labor to firms in the labor market and then purchase goods from firms in the goods market. All workers search in the goods market, but only *active* firms (i.e. filled vacancies) can trade in the goods market. The labor market tightness affects the goods market tightness by affecting the measure of firms that search in the goods market. In turn, the goods market tightness determines the probability of trade for both workers and firms. This implies that the labor market tightness affects the expected match "output" because this includes both the direct match output in the labor market *and* the expected gains from trade in the goods market.[17]

The labor market is a standard Diamond-Mortensen-Pissarides (DMP) environment with bilateral meetings. There is a continuum of measure u of unemployed workers. The measure of vacancies is v. The labor market tightness is $\theta = v/u$. The matching probabilities for workers and firms respectively are $m(\theta)$ and $m(\theta)/\theta$ where $m(.)$ satisfies Assumption 1. There is free entry of vacancies at cost $c > 0$. Matches produce direct output $\bar{y} > z$, where $z \geq 0$ is the value of non-market activity.

In the goods market, the probabilities of trade for workers and firms are $m^G(\phi)$ and

[14] Online Appendix B presents an example featuring *ex ante* heterogeneity. See Shi (2001) for an alternative approach to implementing constrained efficiency (in the presence of *ex ante* heterogeneity) which incorporates directed search and separation of agents into different submarkets.

[15] Petrosky-Nadeau, Wasmer, and Weil (2018) also study efficiency in sequential markets.

[16] In particular, we eliminate the third market, the Arrow-Debreu market.

[17] Kaplan and Menzio (2016) shares a similar feature because sellers' expected revenue in the product market depends on the unemployment rate and thereby on labor market tightness.

$m^G(\phi)/\phi$ respectively, where ϕ is the seller-buyer ratio and $m^G(.)$ satisfies Assumption 1. Since all workers (including those who remain unemployed) search but only active firms search, we have $\phi = (m(\theta)/\theta)v/u = m(\theta)$ and we therefore write $\phi(\theta)$.[18]

Active firms can produce a single unit of an indivisible good at a production cost $\kappa > 0$. Unemployed workers value the good at $v_u > 0$ and employed workers value the good at $v_e > v_u \geq \kappa$. We assume, for simplicity, that $v_u = \kappa$.

While there is no heterogeneity here, matches in the labor market face different outcomes in terms of the surplus created, depending on whether or not workers trade in the goods market. Match "output" $x \in X = \{\bar{y}, \bar{y} + (v_e - \kappa)\}$. The distribution of output across matches has endogenous density $f : X \to [0, 1]$ where

$$f(x;\theta) = \begin{cases} 1 - m^G(\phi(\theta)) & \text{if } x = \bar{y}, \\ m^G(\phi(\theta)) & \text{if } x = \bar{y} + (v_e - \kappa). \end{cases} \tag{26}$$

In the first case, the worker fails to match and thus trade in the goods market.

The expected match output in the labor market is $y(\theta) = \sum_{x \in X} x f(x;\theta)$ and the expected match surplus is $s(\theta) = y(\theta) - z$. Applying (26), we obtain

$$y(\theta) = \underbrace{\bar{y}}_{\text{direct output}} + \underbrace{m^G(\phi(\theta))(v_e - \kappa)}_{\text{expected gains from trade}}. \tag{27}$$

If Assumption 2 holds, Proposition 1 says there exists a unique efficient choice $\theta^P > 0$ and it satisfies condition (16).[19] Whether or not this condition holds depends on how wages are determined. In this example, the output externality is *positive*, i.e. $y'(\theta^*) > 0$. Since $\phi'(\theta) > 0$ and $\frac{dm^G(\phi)}{d\phi} > 0$, an increase in the labor market tightness θ has a positive effect on the expected gains from trade in the goods market.

4.2 Many-on-one meetings and competing auctions

Another way in which the expected match output may depend on market tightness is through a *selection channel*. Consider an environment that features many-on-one or

[18] More generally, consider a dynamic economy with match destruction rate δ and measure one of workers, measure u of whom are initially unemployed. We have $\phi(\theta) = 1 - u_+$, the measure of employed workers or active firms at the end of the period. Using a standard law of motion for unemployment, $u_+ = u(1-m(\theta)) + \delta(1-u)$, we have $\phi(\theta) = (1-\delta)(1-u) + um(\theta)$. This is equivalent to our static example in the special case where $\delta = 1$ and $u = 1$.

[19] A sufficient condition for Assumption 2 is $\bar{y} - z > c$ and $\frac{-m''(\theta)m(\theta)}{(m'(\theta))^2} > 2$ for all $\theta \in \mathbb{R}_+$.

multilateral meetings (where each seller can meet many buyers) and auctions.[20] Such an environment features a selection channel because the auction mechanism enables sellers to select the buyer with the highest valuation.

Albrecht et al. (2014) examine the efficiency of *seller entry* in a competing auctions environment and find that seller entry is efficient. Although they do not explicitly identify it, the generalized Hosios condition applies in their setting and it is the fact that this condition holds endogenously that ensures efficiency.

Consider a simple version of their model featuring identical sellers with reservation value $z = 0$. Buyers are ex ante identical but heterogeneous ex post. Sellers pay a cost $c > 0$ to enter and attract buyers by posting second-price auctions with reserve prices. The buyer-seller ratio is $\theta \equiv v/u$. Each buyer meets exactly one seller. The probability that a seller meets $n \in \mathbb{N}$ buyers is given by a Poisson distribution, $P_n(\theta) = \frac{\theta^n e^{-\theta}}{n!}$. The matching probabilities for sellers and buyers are $m(\theta)$ and $m(\theta)/\theta$ respectively. In this example, the matching probability for sellers equals their meeting probability, i.e. $m(\theta) = 1 - e^{-\theta}$, which satisfies Assumption 1.

Buyers' valuations x are private information and are drawn *ex post* (i.e. after meetings) independently from a distribution with cdf G, density $g = G' > 0$, and support $X = [0, 1]$. The expected valuation of a *successful* buyer is $y(\theta) = \int_0^1 x f(x;\theta)dx$ where the endogenous density $f : X \to [0, 1]$ is given by

$$f(x;\theta) = \frac{\theta g(x) e^{-\theta(1-G(x))}}{1 - e^{-\theta}}. \tag{28}$$

If $E_G(x) > c$, Assumption 2 holds. Seller entry is constrained efficient if and only if the equilibrium θ^* satisfies (25). In fact, this condition does hold and seller entry is efficient. In this example, the output externality from seller entry is *negative.* Since $y'(\theta) > 0$ and $\theta = v/u$, this is a *negative* externality with regard to seller entry.

5 Applying the condition

We have seen two examples of environments where the generalized Hosios condition is necessary for efficiency. In Section 4.2, where prices are determined by

[20] Such environments are often described as "competing auctions". Following Peters and Severinov (1997), recent papers using competing auctions include Albrecht, Gautier, and Vroman (2012, 2014, 2016); Kim and Kircher (2015); Lester, Visschers, and Wolthoff (2015); and Mangin (2017).

auctions, entry is efficient. In Section 4.1, entry may or may not be efficient depending on how wages are determined. We now discuss the application of the generalized Hosios condition in environments featuring Nash bargaining or competitive search.

5.1 Nash bargaining

Consider the example in Section 2. Suppose that wages are determined by Nash bargaining between a worker and their chosen firm. Workers' bargaining power is $\rho \in (0,1)$. With entry cost $c > 0$ for vacancies, the equilibrium $\theta^* > 0$ satisfies:

$$\frac{m(\theta)}{\theta}(1-\rho)s(\theta) = c, \tag{29}$$

or, equivalently, the equilibrium $\theta^* > 0$ satisfies

$$\underbrace{1-\rho}_{\text{firms' bargaining power}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{firms' surplus share}}. \tag{30}$$

Applying the generalized Hosios condition, and using (30), entry is efficient only if

$$\underbrace{\eta_m(\theta^*)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta^*)}_{\text{surplus elasticity}} = \underbrace{1-\rho}_{\text{firms' bargaining power}}. \tag{31}$$

In general, there is no reason why condition (31) should hold since ρ is exogenous. Thus, entry is generically inefficient if wages are determined by Nash bargaining.

Diagnosis of inefficiency. Given a specific bargaining parameter ρ, how can we determine whether or not vacancy entry is efficient? In the example in Section 2, expected match output $y(\theta)$ can be interpreted as *labor productivity.* Defining $\eta_y(\theta) \equiv y'(\theta)\theta/y(\theta)$, the elasticity of labor productivity with respect to market tightness, the version of the generalized Hosios condition in Proposition 2 can be rearranged as

$$\eta_m(\theta) + \left(\frac{1}{1-\beta(1-\delta)}\right)\left(\frac{y(\theta)}{s(\theta)}\right)\eta_y(\theta) = 1-\rho. \tag{32}$$

We can use (32) not just for diagnosing inefficiency, but also for determining the quantitative significance of the generalized Hosios condition. In particular, if the ratio

$y(\theta)/s(\theta)$ is higher (lower), or the elasticity $\eta_y(\theta)$ is higher (lower), then the middle term of (32) is quantitatively more (less) important and imposing the standard Hosios condition would result in a greater (lesser) deviation from the efficient allocation.

Implementation of efficiency. In environments where the standard Hosios condition holds, we have efficiency of entry if and only if $\eta_m(\theta^*) = 1 - \rho$. In such environments, the choice of parameters of the matching technology can be used to restore efficiency. Suppose the matching technology is Cobb-Douglas and $m(\theta)$ has constant elasticity, i.e. $\eta_m(\theta) = \eta$ for all $\theta \in \mathbb{R}_+$. We can *impose* the standard Hosios condition by setting $\eta = 1 - \rho$. Importantly, this ensures efficient entry regardless of the equilibrium market tightness θ^*. While there is no reason why these two unrelated parameters would be equal, a large literature has used this approach to calibrate search-theoretic models of the labor market following Shimer (2005).

Given that the surplus elasticity $\eta_s(\theta^*)$ is endogenous, it is not usually possible to use this approach to impose the generalized Hosios condition in environments featuring Nash bargaining. This highlights the importance of competitive search.

5.2 Competitive search

It is well-known that directed or competitive search typically decentralizes the efficient allocation in environments where the standard Hosios condition is required for efficiency. It turns out that competitive search also decentralizes the efficient allocation in environments where the generalized Hosios condition is required. Intuitively, this is because competitive search allows agents to trade off prices against both the probability of trade *and* the expected match surplus if trade occurs, thus internalizing both the search externalities and the output externality.

Online Appendix B extends the competitive search (price posting) approach of Moen (1997) to an environment where the expected match output depends on the market tightness. Since the generalized Hosios condition holds *endogenously*, competitive search (price posting) provides a way of decentralizing the constrained efficient allocation in environments where meetings are bilateral, such as Example 4.1. In environments where meetings are many-on-one or multilateral, competitive search (auctions) decentralizes the constrained efficient allocation, as seen in Example 4.2.

6 Conclusion

This paper generalizes the well-known Hosios (1990) condition that characterizes efficient entry in search-and-matching models. We extend this simple rule to dynamic environments where the expected match output depends on the market tightness. Such environments give rise to a novel externality – the *output externality* – that is not captured by the standard Hosios condition. To ensure efficiency, markets must internalize the effect of entry on both the number of matches created and the average value created by each match. We show that this occurs precisely when buyers' surplus share equals the *matching elasticity* plus the *surplus elasticity*. We call this intuitive condition the "generalized Hosios condition". When it holds, agents are fully compensated for the effect of entry on both *match creation* and *surplus creation*.

Appendix: Omitted proofs

Proof of Lemma 1. *Part (i).* Let Ψ denote the standard simplex in $\mathbb{R}^{1+|X|}$. Let $C(\Psi)$ be the set of bounded, continuous functions $R : \Psi \to \mathbb{R}$ with the sup norm $\|\mathbb{R}\| = \sup_{\psi\in\Psi} R(\psi)$. We can define an operator T with domain $C(\Psi)$ by

$$(TR)(\psi) = \max_{\theta\in\mathbb{R}_+} \Omega(\theta|\psi) + \beta R(\psi_+) \tag{33}$$

where ψ_+ is given by (8) and Ω is defined by (10). First, TR is bounded. Consider any function $R \in C(\Psi)$. Since R is bounded, there exist R_0 and $\bar{R}$ such that $R_0 \leq R(\psi_+) \leq \bar{R}$ for all $\hat{\psi} \in \Psi$. Therefore, using (33) and (10), $(TR)(\psi)$ is bounded below by $\min\{z, x_1\} + \beta R_0$ and above by $\max\{z, x_N\} + \beta\bar{R}$. Next, TR is continuous in ψ. To see this, observe that since X is bounded we can replace the constraint $\theta \in \mathbb{R}_+$ with $\theta \in [0, \bar{\theta}]$ where $\bar{\theta} \equiv c^{-1}u^{-1}\{[\max\{z, x_N\} - \min\{z, x_1\}] + \beta[\bar{R} - R_0]\}$. For the modified problem, the maximand is continuous in (ψ, θ) and the set of feasible choices for θ is compact, so the Theorem of the Maximum implies TR is continuous in ψ (Theorem 3.6 in Stokey, Lucas, and Prescott, 1989). Thus, $T : C(\Psi) \to C(\Psi)$. It is straightforward to verify that T satisfies Blackwell's sufficient conditions for a contraction (Theorem 3.3 in Stokey et al., 1989). Therefore, T is a contraction mapping and it has exactly one fixed point $R^* \in C(\Psi)$. Since $\lim_{t\to\infty} \beta^t R^*(\psi) = 0$ for all $\psi \in \Psi$, R^* is equal to the planner's function W (Theorem 4.3 in Stokey et al., 1989).

Part (ii). Define a set $C'(\Psi) \subseteq C(\Psi)$ as follows. We have $R \in C'(\Psi)$ if and only if $R \in C(\Psi)$ and there exist R_u and $R_e : X \to \mathbb{R}$ such that $R(\psi) = R_u u + \sum_{x \in X} R_e(x)\psi(x)$. Consider any $R \in C'(\Psi)$. Substituting (8) into (10), and then substituting into the maximand in (33) and simplifying, using the fact that $1 - u = \sum_{x \in X} \psi(x)$, we obtain

$$(TR)(\psi) = \hat{R}_u u + \sum_{x \in X} \hat{R}_e(x)\psi(x) \tag{34}$$

where $\hat{R}_u$ is given by

$$\hat{R}_u = \max_{\theta \in \mathbb{R}_+} \left\{ -c\theta + (1 - m(\theta))(z + \beta R_u) + m(\theta) \left(y(\theta) + \beta \sum_{x \in X} R_e(x) f(x;\theta) \right) \right\}, \tag{35}$$

and $y(\theta)$ is given by (7), and $\hat{R}_e(x)$ is given by

$$\hat{R}_e(x) = \delta(z + \beta R_u) + (1 - \delta)(x + \beta R_e(x)). \tag{36}$$

Therefore, we have $T : C'(\Psi) \to C'(\Psi)$ and, since $C'(\Psi)$ is a closed subset of $C(\Psi)$, we have $W \in C'(\Psi)$ by Corollary 1 to Theorem 3.2 in Stokey et al. (1989). ■

Proof of Corollary 1. Suppose that $\eta_m(\theta^*) = c\theta^*/m(\theta^*)s(\theta^*)$. First, (24) implies that $s'(\theta^*) > 0$ if and only if $y'(\theta^*) > 0$. Next, we show there is under-entry (over-entry) of buyers if and only if $s'(\theta^*) > (<)0$. Letting $\Lambda(\theta) = m(\theta)s(\theta)$, Proposition 1 says there exists a unique efficient $\theta^P > 0$ that satisfies $\Lambda'(\theta^P) = c$. By assumption, $m'(\theta^*)s(\theta^*) = c$ and therefore $\Lambda'(\theta^P) = m'(\theta^*)s(\theta^*)$. Now, $\Lambda'(\theta^*) = m'(\theta^*)s(\theta^*) + m(\theta^*)s'(\theta^*)$, and thus $\Lambda'(\theta^P) = \Lambda'(\theta^*) - m(\theta^*)s'(\theta^*)$, so if $s'(\theta^*) > 0$ then $\Lambda'(\theta^P) < \Lambda'(\theta^*)$. Assumption 2 implies that $\Lambda''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$ and therefore $\Lambda'(\theta^P) < \Lambda'(\theta^*)$ implies that $\theta^* < \theta^P$. If $s'(\theta^*) < 0$, there is *over-entry* of buyers, $\theta^* > \theta^P$. ■

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Online Appendix

A. Generalization of Propositions 1 & 2

Suppose that buyers and sellers are matched according to a general matching function that gives the total number of matches, $M(v,u)$, as a function of the measure of unmatched buyers v and the measure of unmatched sellers u. The matching probability for a seller is $M(v,u)/u$ and the matching probability for a buyer is $M(v,u)/v$.

Assumption 1′. *The function $M(.,.)$ satisfies: (i) $\frac{\partial M}{\partial v} > 0$ and $\frac{\partial^2 M}{\partial v^2} < 0$ for all $v, u \in \mathbb{R}_+$, (ii) $\lim_{v\to 0} M(v,u) = 0$, (iii) $\lim_{v\to 0} \frac{\partial M}{\partial v} = 1$, (iv) $\lim_{v\to\infty} M(v,u) = 1$, (v) $\lim_{v\to\infty} \frac{\partial M}{\partial v} = 0$, and (vi) $M(v,u)/v$ is strictly decreasing in v.*

Match output $x \in X \equiv \{x_1, x_2, ..., x_N\} \subseteq \mathbb{R}_+ \backslash \{0\}$ where $x_1 < x_2 < ... < x_N$. Given measure v of unmatched buyers and measure u of unmatched sellers, the output of a *newly matched* seller is an i.i.d. draw from a discrete probability distribution with density $f : X \to [0,1]$ where $\sum_{x\in X} f(x;(v,u)) = 1$.

The *expected match output* (i.e. expected output conditional on a match) is

$$y(v,u) \equiv \sum_{x\in X} x f(x;(v,u)). \tag{37}$$

The market output of *any* seller in the economy is given by a discrete probability distribution with density $\psi : X \cup \{0\} \to [0,1]$ where $\sum_{X\cup\{0\}} \psi(x) = 1$. Since unmatched sellers produce zero output, we have $\psi(0) = u$.

Let ψ_+ denote the next period's density. The density ψ evolves according to:

$$\psi_+(x) = \begin{cases} u - M(v,u) + \delta(1-u) & \text{for } x = 0 \\ M(v,u) f(x;(v,u)) + (1-\delta)\psi(x) & \text{for } x \in X. \end{cases} \tag{38}$$

Suppose the planner can allow buyers to enter at cost $c > 0$ each period. At the start of a period, the planner observes the aggregate state of the economy, given by the density ψ, and chooses a measure of entering buyers $v \in \mathbb{R}_+$. The planner is restricted to take both the matching technology $M(.,.)$ and the output technology $y(.,.)$ as given, and chooses the measure of entering buyers v to maximize the sum of present and future social surplus, discounted by factor $\beta \in (0,1)$.

The Bellman equation for the planner's value function $W(\psi)$ can be written as:

$$W(\psi) = \max_{v \in \mathbb{R}_+} \left\{ \Omega(v|\psi) + \beta W(\psi_+) \right\} \tag{39}$$

where ψ_+, the next period's state, is given by the law of motion (38), and

$$\Omega(v|\psi) = \sum_{x \in X} x\psi_+(x) + zu_+ - cv. \tag{40}$$

Lemma 1$'$. *(i) The planner's value function $W(\psi)$ is the unique solution to (39); and (ii) $W(\psi)$ can be written as a linear function:*

$$W(\psi) = W_u u + \sum_{x \in X} W_e(x)\psi(x), \tag{41}$$

where

$$W_u u = \max_{v \in \mathbb{R}_+} \left\{ -cv + (u - M(v,u))(z + \beta W_u) + M(v,u)(y(v,u) + \beta \tilde{W}_e(v,u)) \right\} \tag{42}$$

and

$$\tilde{W}_e(v,u) \equiv \sum_{x \in X} W_e(x) f(x; (v,u)) \tag{43}$$

and

$$W_e(x) = \delta(z + \beta W_u) + (1-\delta)(x + \beta W_e(x)). \tag{44}$$

Proof. *Part (i).* The proof is essentially identical to the proof of Lemma 1 except that we define an operator T with domain $C(\Psi)$ by

$$(TR)(\psi) = \max_{v \in \mathbb{R}_+} \ \Omega(v|\psi) + \beta R(\psi_+) \tag{45}$$

where ψ_+ is given by (38) and Ω is defined by (40), and we can replace the constraint $v \in \mathbb{R}_+$ with $v \in [0, \bar{v}]$, where $\bar{v} \equiv c^{-1}\{[\max\{z, x_N\} - \min\{z, x_1\}] + \beta[\bar{R} - R_0]\}$.

Part (ii). The proof is almost identical to Lemma 1 except it uses (45), (38), and (40) to obtain the linear expression for $W(\psi)$. ■

The dynamic *expected match surplus* $s(v,u)$ is given by

$$s(v,u) = y(v,u) - z + \beta(\tilde{W}_e(v,u) - W_u). \tag{46}$$

Assumption 2′. *The function* $\Lambda(.,.)$, *defined by* $\Lambda(v,u) \equiv M(v,u)s(v,u)$ *for* $v \in \mathbb{R}_+$ and $u \in \mathbb{R}_+ \backslash \{0\}$, *satisfies the following: (i)* $\lim_{v\to 0} \Lambda(v,u) = 0$; *(ii)* $\lim_{v\to 0} \frac{\partial \Lambda}{\partial v} > c$; *(iii)* $\lim_{v\to\infty} \frac{\partial \Lambda}{\partial v} < c$; *and (iv)* $\frac{\partial^2 \Lambda}{\partial v^2} < 0$ *for all* $v \in \mathbb{R}_+$ and $u \in \mathbb{R}_+ \backslash \{0\}$.

Assumptions 1′ and 2′ are assumed to hold throughout Online Appendix A.

Proposition 1′. *There exists a unique efficient allocation* $(v_t^P)_{t=0}^{\infty}$ *where* $v_t^P = v^P > 0$ *for all* t *and* v^P *satisfies the following condition:*

$$\underbrace{\frac{\partial M}{\partial v}\frac{v}{M(v,u)}}_{\text{matching elasticity}} + \underbrace{\frac{\partial s}{\partial v}\frac{v}{s(v,u)}}_{\text{surplus elasticity}} = \underbrace{\frac{cv}{M(v,u)s(v,u)}}_{\text{buyers' surplus share}}. \tag{47}$$

Proof. Rearranging (42) from Lemma 1′, the optimal v^P is given by

$$v^P = \arg\max_{v\in\mathbb{R}_+} \left\{ -cv + zu + \beta W_u u + M(v,u)(y(v,u) - z + \beta(\tilde{W}_e(v,u) - W_u)) \right\}, \tag{48}$$

which can be rewritten using (46) as

$$v^P = \arg\max_{v\in\mathbb{R}_+} \{M(v,u)s(v,u) + zu - cv + \beta W_u u\}. \tag{49}$$

Taking the first-order condition, the optimal v^P satisfies

$$\frac{\partial M}{\partial v} s(v,u) + M(v,u)\frac{\partial s}{\partial v} \leq c \tag{50}$$

and $v^P \geq 0$ with complementary slackness. Assumption 2′ implies that, for any given $u \in \mathbb{R}_+ \backslash \{0\}$, there exists a unique solution $v^P > 0$ that satisfies $\frac{\partial \Lambda}{\partial v} = c$ where $\Lambda(v,u) \equiv M(v,u)s(v,u)$. Therefore, there exists a unique $v^P > 0$ that satisfies

$$\frac{\partial M}{\partial v} s(v,u) + M(v,u)\frac{\partial s}{\partial v} = c.$$

Dividing both sides of the above equation by $M(v,u)s(v,u)/v$ yields (47). ■

Proposition 2′. *There exists a unique efficient allocation* $(v_t^P)_{t=0}^{\infty}$ *where* $v_t^P = v^P > 0$ *for all* t *and* v^P *satisfies the following condition:*

$$\frac{\partial M}{\partial v}\frac{v}{M(v,u)} + \frac{\partial y}{\partial v}\frac{v}{(1-\beta(1-\delta))s(v,u)} = \frac{cv}{M(v,u)s(v,u)}. \tag{51}$$

Proof. Differentiating (46) with respect to v,

$$\frac{\partial s}{\partial v} = \frac{\partial y}{\partial v} + \beta\frac{\partial \tilde{W}_e}{\partial v}. \tag{52}$$

Using (43), we obtain

$$\tilde{W}_e(v,u) = \frac{\delta(z+\beta W_u) + (1-\delta)y(v,u)}{1-\beta(1-\delta)}. \tag{53}$$

Differentiating the above yields

$$\frac{\partial \tilde{W}_e}{\partial v} = \frac{(1-\delta)}{1-\beta(1-\delta)}\frac{\partial y}{\partial v}. \tag{54}$$

Substituting (54) into (52) and simplifying yields

$$\frac{\partial s}{\partial v} = \frac{1}{1-\beta(1-\delta)}\frac{\partial y}{\partial v}. \tag{55}$$

Finally, substituting (55) into (47) delivers condition (51). ■

B. Constrained planner

In the following example featuring *ex ante* heterogeneity, we suppose that the planner is constrained not only by the matching frictions but also by the entry decisions agents would choose in the decentralized equilibrium. This is because the function $y(.)$ arises as a consequence of these entry decisions. Since the planner is restricted to take both the matching technology $m(.)$ *and* the output technology $y(.)$ as given, the planner is constrained by these.[21]

[21] In this example, the constrained efficiency is "doubly constrained" since the planner's problem is solved subject to an additional constraint which is one of the equilibrium conditions.

Ex ante heterogeneity and market composition

When there is *ex ante* heterogeneity among buyers or sellers, dependence of the expected match output on market tightness can arise naturally through market composition. If the market tightness influences the individual entry decisions of buyers or sellers that are ex ante heterogeneous with respect to characteristics that affect match output, then average output per match will depend on market tightness.[22] We call this the *composition channel*.

Suppose there is a measure u of unemployed workers and a fixed measure M of firms that may choose to search. Firms' productivities x are distributed according to a twice differentiable distribution with cdf G and density g where $G(0) = 0$ and $g(x) > 0$ for all $x \in X = [0, 1]$. Firms learn their own productivity before deciding whether to pay the entry cost $c > 0$ and search. Expected wages paid by a firm with productivity x is $w(x, \theta) \leq x$.

Let v be the measure of *searching* firms and define $\theta \equiv v/u$. Meetings are bilateral and the probabilities of matching for workers and firms are $m(\theta)$ and $m(\theta)/\theta$ respectively, where we assume $m(.)$ satisfies Assumption 1.

A firm with productivity x chooses to search for a worker if and only if

$$\frac{m(\theta)}{\theta}(x - w(x, \theta)) > c. \tag{56}$$

If $\partial w(x, \theta)/\partial x < 1$, there is a unique cut-off productivity $x^*(\theta)$ such that firms enter if and only if $x \geq x^*(\theta)$.[23] The distribution of output across matches has density

$$f(x; \theta) = \frac{g(x)}{1 - G(x^*(\theta))} \tag{57}$$

and the expected match output, or labor productivity, is given by

$$y(\theta) = \int_{x^*(\theta)}^{1} \frac{xg(x)}{1 - G(x^*(\theta))} dx. \tag{58}$$

[22] For example, Albrecht, Navarro, and Vroman (2010) consider an environment where workers are ex ante heterogeneous with respect to their market productivity. There is both firm entry *and* a labor force participation decision. See also Albrecht, Navarro, and Vroman (2009), Gavrel (2011), and Masters (2015). In a follow-up paper to the present one, Julien and Mangin (2017) applies and extends the generalized Hosios condition to the environment in Albrecht et al. (2010).

[23] This is true, for example, if wages are determined by Nash bargaining with $\rho < 1$.

It can be verified that x^* is strictly increasing in θ provided that $\partial w(x,\theta)/\partial x < 1$. This is intuitive: as the market tightness increases, the probability of finding a worker is lower so only high productivity firms choose to pay the cost c and search. At the same time, the average match output $y(\theta)$ is increasing in the cut-off productivity x^*. Therefore, $y'(\theta) > 0$ for all $\theta \in \mathbb{R}_+$ and the output externality is positive.

Suppose the planner chooses a market tightness θ to maximize the total social surplus minus the total entry costs. We assume the planner uses the same cut-off productivity rule $x^*(\theta)$ as in the decentralized economy. If Assumption 2 is satisfied, there exists a unique social optimum θ^P and we have constrained efficiency if and only if θ^* satisfies the generalized Hosios condition in Proposition 1.[24] Since $y'(\theta^*) > 0$, Corollary 1 implies there is under-entry of firms under the standard Hosios condition.

C. Competitive search (posting)

It is well-known that competitive search equilibrium is typically (but not always) constrained efficient in the sense that it decentralizes the planner's allocation (Shimer, 1996; Moen, 1997). In competitive search models where the expected match output is constant, agents simply trade off prices against the probability of trade. The fact that competitive search allows agents to do so is what delivers efficiency. In environments where the expected match output depends on the market tightness, agents trade off prices against both the probability of trade *and* the expected match surplus if trade occurs. Again, the fact that agents can do so is what delivers efficiency.

Consider a simple competitive search model in the spirit of Moen (1997). There is a continuum of submarkets indexed by $i \in [0,1]$ and free entry of vacancies at cost $c > 0$. Workers in submarket i post the same wage w_i and face the same market tightness θ_i, the ratio of vacancies to workers in that submarket. Firms' search is *directed* by observing the posted wages and deciding which submarkets to enter. Within each submarket, workers and firms are matched according to a frictional meeting technology. Matching probabilities for workers and firms are $m(\theta_i)$ and $m(\theta_i)/\theta_i$ respectively, where $m(.)$ satisfies Assumption 1.

In any submarket, match output $x \in X = [x_{\min}, x_{\max}] \subseteq \mathbb{R}_+$ where $x_{\max} \in \mathbb{R}_+ \cup \{\infty\}$. In submarket i, match output is an i.i.d. draw from a probability distribution with density $f(x;\theta_i)$ and a finite mean. Let $y(\theta_i) \equiv \int_{x_{\min}}^{x_{\max}} x f(x;\theta_i)dx$, the expected

[24] For example, if G is uniform on $[0,1]$ and wages are determined by Nash bargaining, Assumption 2 holds provided that $c < 1/2$ and $\rho < 1/2$.

match output. The flow payoff for unmatched sellers is $z \geq 0$ and we assume that $x_{\min} > z$. The expected match surplus in submarket i is $s(\theta_i) = y(\theta_i) - z$.

The expected payoff for firms in submarket i with wage w_i and tightness θ_i is

$$\Pi(\theta_i, w_i) = \frac{m(\theta_i)}{\theta_i}(y(\theta_i) - w_i), \tag{59}$$

and the expected payoff for workers in submarket i with market tightness θ_i is

$$V(\theta_i, w_i) = m(\theta_i)w_i + (1 - m(\theta_i))z. \tag{60}$$

Workers in submarket i choose a wage w_i^* and market tightness θ_i^* that solve

$$\max_{w_i, \theta_i \in \mathbb{R}_+} (m(\theta_i)w_i + (1 - m(\theta_i))z) \tag{61}$$

subject to $\Pi(\theta_i, w_i) \leq c$ and $\theta_i \geq 0$ with complementary slackness. To induce participation by firms in submarket i, i.e. $\theta_i > 0$, the constraint $\Pi(\theta_i, w_i) \leq c$ is binding:

$$\frac{m(\theta_i)}{\theta_i}(y(\theta_i) - w_i) = c. \tag{62}$$

Solving for w_i as a function of θ_i using (62), we obtain

$$w(\theta_i) = y(\theta_i) - \frac{c\theta_i}{m(\theta_i)}. \tag{63}$$

Choosing a wage w_i^* is thus equivalent to choosing a market tightness θ_i^* where

$$\theta_i^* = \arg\max_{\theta_i \in \mathbb{R}_+} (m(\theta_i)w(\theta_i) + (1 - m(\theta_i))z) \tag{64}$$

and using (63), this is equivalent to

$$\theta_i^* = \arg\max_{\theta_i \in \mathbb{R}_+} (m(\theta_i)y(\theta_i) + (1 - m(\theta_i))z - c\theta_i). \tag{65}$$

The equilibrium θ_i^* satisfies the first-order condition

$$m'(\theta_i)s(\theta_i) + m(\theta_i)s'(\theta_i) = c, \tag{66}$$

or, equivalently, the equilibrium θ_i^* solves

$$\underbrace{\eta_m(\theta_i)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta_i)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta_i}{m(\theta_i)s(\theta_i)}}_{\text{firms' surplus share}} . \tag{67}$$

The generalized Hosios condition holds endogenously *within each active submarket* i. If we consider a symmetric equilibrium in which firms are indifferent across submarkets and all workers post the same wage, then $\theta_i^* = \theta^*$ for all submarkets i. If Assumption 2 holds, then Proposition 1 tells us that the equilibrium level of vacancy entry is constrained efficient. While we consider only a static model here, the same result holds in dynamic environments where Proposition 1 applies.