(For online publication)

Appendix to: Unemployment and the Labor Share

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Contents

A Proofs and derivations

A.0 Useful facts

Here we reproduce some useful facts presented in Lemma 3 and Fact 1 in Mangin (2017) that will be used in the proofs in this Appendix.

For any $s \in \mathbb{R}^+$and $x \in \mathbb{R}^+$, the Lower Incomplete Gamma function is defined by

$$\gamma(s,x) \equiv \int_0^x t^{s-1}e^{-t}\, dt. \tag{1}$$

Observe that $\lim_{x\to\infty} \gamma(s,x) = \Gamma(s)$, the Gamma function, and $\gamma(1,x) = 1 - e^{-x}$.

Fact 1. *Recurrence relation:* $\gamma(s,x) = (s-1)\gamma(s-1,x) - x^{s-1}e^{-x}$

Fact 2. $\frac{\partial}{\partial x}\gamma(s,x) = x^{s-1}e^{-x}$

Fact 3. $\frac{\partial}{\partial s}\gamma(s,x) = \int_0^x t^{s-1}e^{-t}(\ln t)dt$

Fact 4. *For $x > 0$, the elasticity of $\gamma(s,x)$ with respect to x is*

$$\varepsilon(s,x) = \frac{x^s e^{-x}}{\gamma(s,x)}. \tag{2}$$

Fact 5. *For $x > 0$, the derivative of $\varepsilon(s,x)$ with respect to s is*

$$\frac{\partial}{\partial s}\varepsilon(s,x) = x^s e^{-x}\left(\frac{\int_0^x(\ln x - \ln t)t^{s-1}e^{-t}dt}{\gamma(s,x)^2}\right) > 0.$$

Fact 6. *For $x > 0$, the derivative of $\varepsilon(s,x)$ with respect to x is*

$$\frac{\partial}{\partial x}\varepsilon(s,x) = \frac{x^{s-1}e^{-x}}{\gamma(s,x)}\left(s - x - \frac{x^s e^{-x}}{\gamma(s,x)}\right) < 0.$$

Fact 7. $\lim_{x\to 0}\, \varepsilon(s,x) = s$ *and* $\lim_{x\to\infty}\varepsilon(s,x) = 0$

A.1 Proof there exists a unique cut-off productivity

First, we assume that both workers and firms accept when indifferent.

In a bilateral meeting, workers always accept the wage offered by definition of the reservation wage b_τ. A match is therefore acceptable to both workers and firms in a bilateral meeting if and only if the firm accepts, which is true if and only if $J^1_{t,\tau}(x) \geq 0$ or $x \geq x^{1c}_t$ where x^{1c}_t is the unique solution to $J^1_{t,\tau}(x) = 0$.

In a multilateral meeting, firms always accept since $A_t x_1 > A_t x_2$ for all t, where x_1 and x_2 are respectively the highest and second-highest productivities among the firms competing to hire the worker. So match acceptability depends on workers alone. A firm with productivity x can offer the worker (or "bid" in an auction) a wage contract that is at best equal to $V^{\max}_{\tau,\tau}(x)$ where

$$V^{\max}_{t,\tau}(x) = A_t x + \beta((1-\delta)\mathbb{E}_t V^{\max}_{t+1,\tau}(x) + \delta \mathbb{E}_t V^u_{t+1}). \tag{3}$$

At the time of hiring τ, workers will only consider "bids" from firms with productivities x that satisfy $V^{\max}_{\tau,\tau}(x) \geq V^1_{\tau,\tau}(b_\tau)$ or $x \geq x^{2c}_t$ where x^{2c}_t is the unique solution to $V^{\max}_{\tau,\tau}(x) - V^1_{\tau,\tau}(b_\tau) = 0$, since $V^1_{\tau,\tau}(b_\tau) = z + \beta V^u_{\tau+1}$, the value of remaining unemployed. Now, using (3) above, and (14) from Section 2.3 of the main text, we have

$$V^{\max}_{\tau,\tau}(x) - V^1_{\tau,\tau}(b_\tau) = A_\tau x - b_\tau + \beta(1-\delta)\mathbb{E}_\tau(V^{\max}_{\tau+1,\tau}(x) - V^1_{\tau+1,\tau}(b_\tau)),$$

and since $J^1_{\tau,\tau}(x)$ is given by

$$J^1_{\tau,\tau}(x) = A_\tau x - b_\tau + \beta(1-\delta)\mathbb{E}_t J^1_{\tau+1,\tau}(x),$$

we have $V^{\max}_{\tau,\tau}(x) - V^1_{\tau,\tau}(b_\tau) = J^1_{\tau,\tau}(x)$.

Therefore $x^{1c}_t = x^{2c}_t$ and we call this unique cut-off productivity x^c_t. Only firms with productivity $x \geq x^c_t$ will decide to *compete*, and all matches with $x \geq x^c_t$ will be acceptable by both workers and firms, at least at the time of hiring.[1]

A.2 Derivation of value of a filled vacancy

The expected value of an filled vacancy $J_{\tau,\tau}$ at the time of match creation τ is

$$q(\theta_\tau)J_{\tau,\tau} = e^{-\theta_\tau} J^1_{\tau,\tau} + (1-e^{-\theta_\tau})J^2_{\tau,\tau}, \tag{4}$$

and $J^1_{t,\tau}$ and $J^2_{t,\tau}$ are the respective expected payoffs in period t from a bilateral meeting (which occurs with probability $e^{-\theta_\tau}$),[2] or from a multilateral meeting (which occurs

[1] Note that whether a match is either bilateral or multilateral (i.e. there are either one or two or more *competing* firms) itself depends on the cut-off productivities x^{1c}_t and x^{2c}_t. Since we show that the same cut-off x^c_t applies for both types of matches, this is not a problem.

[2] Note that these are the Poisson probabilities of n firms approaching from the perspective of the *firm*, which are the probabilities from the workers' perspective divided by θ.

with probability $1 - e^{-\theta_\tau}$) during the matching period τ.[3]

The expected value at time t of a vacancy that is filled in a bilateral meeting in period τ is

$$J^1_{t,\tau} = \int_{x_{0t}}^{\infty} J^1_{t,\tau}(x) dG^c_\tau(x) \tag{5}$$

and the expected value of a vacancy, with productivity drawn from $G^c_\tau(x)$, that is competing to hire a worker in a multilateral meeting is

$$J^2_{t,\tau} = \int_{x_{0t}}^{\infty} \eta_\tau(x) J^2_{t,\tau}(x) dG^c_\tau(x), \tag{6}$$

where $\eta_\tau(x)$ is the probability that a firm with productivity x successfully hires, conditional on two or more firms competing.

Using (4), (5), and (6) above, and equations (16) and (19) from the main text,

$$\begin{aligned} q(\theta_\tau)J_{t,\tau} &= e^{-\theta_\tau} \int_{x_{0\tau}}^{\infty} (A_t x - b_\tau) dG^c_\tau(x) + (1 - e^{-\theta_\tau}) \int_{x_{0\tau}}^{\infty} \eta_\tau(x)(A_t x - w^2_{t,\tau}(A_t x)) dG^c_\tau(x) \\ &\quad + \beta(1-\delta) q(\theta_\tau) \mathbb{E}_t J_{t+1,\tau} \end{aligned}$$

Defining a new variable $y = A_t x$ and letting $G_{t,\tau}(y) \equiv G^c_\tau(y/A_t)$ and $\eta_{t,\tau}(y) \equiv \eta_\tau(y/A_t)$,

$$\begin{aligned} q(\theta_\tau)J_{t,\tau} &= e^{-\theta_\tau} \int_{A_t x_{0\tau}}^{\infty} (y - b_\tau) dG_{t,\tau}(y) + (1 - e^{-\theta_\tau}) \int_{A_t x_{0\tau}}^{\infty} \eta_{t,\tau}(y)(y - w^2_{t,\tau}(y)) dG_{t,\tau}(y) \\ &\quad + \beta(1-\delta) q(\theta_\tau) \mathbb{E}_t J_{t+1,\tau} \end{aligned}$$

Applying a result found in the Appendix of Mangin (2017) we have

$$\int_{A_t x_{0\tau}}^{\infty} \eta_{t,\tau}(y)(y - w^2_{t,\tau}(y)) dG_{t,\tau}(y) = \int_{A_t x_{0\tau}}^{\infty} \eta_{t,\tau}(y)(1 - G_{t,\tau}(y)) dy$$

and therefore we have:

$$q(\theta_\tau)J_{t,\tau} = e^{-\theta_\tau} \int_{A_t x_{0\tau}}^{\infty} (y - b_\tau) dG_{t,\tau}(y) + (1 - e^{-\theta_\tau}) \int_{A_t x_{0\tau}}^{\infty} \eta_{t,\tau}(y)(1 - G_{t,\tau}(y)) dy + \beta(1-\delta) q(\theta_\tau) \mathbb{E}_t J_{t+1,\tau}$$

Next, it can be shown that

$$\eta_{t,\tau}(y) = \frac{e^{-\theta_\tau(1 - G_{t,\tau}(y))} - e^{-\theta_\tau}}{1 - e^{-\theta_\tau}}$$

[3]Observe that $J^1_{t,\tau}$ is the value of a filled vacancy since the probability of hiring is one in a bilateral match, but $J^2_{t,\tau}$ is not the value of a filled vacancy since it incorporates the probability the firm will be successful in hiring (i.e. it will have the highest productivity).

and, simplifying further, using a result from the Appendix of Mangin (2017) we obtain:

$$J_{t,\tau}=\frac{1}{q(\theta_\tau)}\left(\int_{A_t x_{0\tau}}^{\infty} e^{-\theta_\tau(1-G_{t,\tau}(y))}(1-G_{t,\tau}(y))dy + e^{-\theta_\tau}(A_t x_{0\tau}-b_\tau)\right)+\beta(1-\delta)E_t J_{t+1,\tau}$$

Rearranging, and using the fact that $\mu(\theta_\tau)=e^{-\theta_\tau}/q(\theta_\tau)$, we have

$$\begin{aligned}
& \frac{1}{q(\theta_\tau)}\int_{A_t x_{0\tau}}^{\infty} e^{-\theta_\tau(1-G_{t,\tau}(y))}(1-G_{t,\tau}(y))dy \\
= & \frac{1}{m(\theta_\tau)}\int_{A_t x_{0\tau}}^{\infty}\left(\frac{1-G_{t,\tau}(y)}{y g_{t,\tau}(y)}\right)\theta_\tau e^{-\theta_\tau(1-G_{t,\tau}(y))} g_{t,\tau}(y) y dy \\
= & \int \frac{1-G_{t,\tau}(y)}{y g_{t,\tau}(y)} y dH^e_{t,\tau}(y;\theta_\tau)
\end{aligned}$$

where $H^e_{t,\tau}(y;\theta_\tau)\equiv H^e_\tau(y/A_t;\theta_\tau)$, the distribution at time t of output $y=A_t x$ across matches formed at time τ. Substituting into the above, we obtain:

$$J_{t,\tau}=\underbrace{\pi_{t,\tau}(\theta_\tau)}_{\text{productivity rents}}+\underbrace{\mu(\theta_\tau)(A_t x_{0\tau}-b_\tau)}_{\text{matching rents}}+\beta(1-\delta)\mathbb{E}_t J_{t+1,\tau} \tag{7}$$

where $\mu(\theta_\tau)$ is the share of bilateral meetings at time τ, and $\pi_{t,\tau}(\theta_\tau)$ is the expected value of productivity rents at time t for a match formed in period τ, defined by

$$\pi_{t,\tau}(\theta_\tau)\equiv\int_{A_t x_{0\tau}}^{\infty}\left(\frac{1-G_{t,\tau}(y)}{y g_{t,\tau}(y)}\right) y dH^e_{t,\tau}(y;\theta_\tau). \tag{8}$$

A.3 Generalized Pareto distribution

The Generalized Pareto distribution (GPD) also yields tractable expressions for steady state expected wages, factor income shares, and the expected output per match.

Let $G(x)$ with support $[x_{\min},\infty)$ be defined as follows:

$$G(x)=\begin{cases} 1-\left(1+\frac{\lambda(x-x_{\min})}{\sigma}\right)^{-1/\lambda} & \text{if } \lambda>0 \\ 1-e^{-\left(\frac{x-x_{\min}}{\sigma}\right)} & \text{if } \lambda=0\end{cases} \tag{9}$$

where $\lambda\in[0,1)$ and $\sigma\in(0,\infty)$. We assume that $\sigma\geq\lambda$ and $x_{\min}=1$. This distribution nests the Pareto as the special case where $\sigma=\lambda>0$, and it nests the exponential as the special case where $\lambda=0$.

We again define $G_b(y)\equiv G^c(y/A)$. For the generalized Pareto distribution, we have

$$\frac{1-G_b(y)}{y g_b(y)}=\lambda+\frac{A(\sigma-\lambda)}{y}. \tag{10}$$

Substituting into (21) and (22) from Section 2.3 of the main text yields the following

$$\widetilde{w} = (1-\lambda)Ap(\theta) - A(\sigma-\lambda) - (Ax_0 - b)\mu(\theta). \tag{11}$$

Steady state labor's share s_L is

$$s_L = 1 - \lambda - \left(\frac{\sigma-\lambda}{p(\theta)}\right) - \frac{\mu(\theta)\,(Ax_0 - b)}{Ap(\theta)}. \tag{12}$$

In the special case where $\sigma = \lambda > 0$, we recover the corresponding expressions for the Pareto distribution. In the special case where $\lambda = 0$, we obtain expressions for wages and labor's share for the exponential distribution.

In general, whenever $b \geq A$ in the steady state, we have $x_0 = b/A$ and the last term on the right of (12) disappears. However, observe that steady state factor shares are *not* constant whenever $\sigma \neq \lambda$ since we have

$$s_L = 1 - \lambda - \left(\frac{\sigma-\lambda}{p(\theta)}\right) \text{ if } b \geq A. \tag{13}$$

Regardless of the level of the endogenous reservation wage b, labor share is increasing in the degree of firm competition θ since $p'(\theta) > 0$. Lemma 1 extends to the GPD.

If $b < A$, we can also obtain a tractable expression for $p(\theta)$. For the generalized Pareto distribution, if $\lambda > 0$ the average match productivity is

$$p(\theta) = \left(\frac{\sigma}{\lambda}\right)\frac{\gamma(1-\lambda,\theta)\theta^{\lambda}}{1-e^{-\theta}} - \left(\frac{\sigma-\lambda}{\lambda}\right). \tag{14}$$

In the special case where $\sigma = \lambda$, we recover the corresponding expression for the Pareto distribution when $x_0 = 1$.

A.4 Existence and uniqueness of function ϕ_r

Existence. Let $F(\theta) = (1-G(x^c))\left(\int_{Ax_0}^{\infty} e^{-\theta(1-G_b(y))}(1-G_b(y))dy + e^{-\theta}(Ax_0 - b)\right)$, $b \in \mathbb{R}^+$. The zero profit condition holds if and only if $F(\theta) = C(1-\beta(1-\delta))$, where $C(1-\beta(1-\delta) > 0$. Now $F(\theta)$ is continuous in θ on $[0,\infty)$ and $F(\theta) \to 0$ as $\theta \to \infty$. If we can ensure that $F(0) > C(1-\beta(1-\delta))$, the intermediate value theorem implies there exists $\theta > 0$ such that $F(\theta) = C(1-\beta(1-\delta))$. Now, $F(0) = (1-G(x^c))\left(\int_{Ax_0}^{\infty}(1-G_b(y))dy + (1-b)\right) = (1-G(x^c))(E_{G_b}(y) - b)$. If $G(x)$ is Pareto, we have $F(0) > C(1-\beta(1-\delta))$ provided the following condition holds:

$$C < \frac{x_0^{-1/\lambda}}{1-\beta(1-\delta)}\left(\frac{Ax_0}{1-\lambda} - b\right). \tag{15}$$

If condition (15) holds, there exists $\theta > 0$ and hence there exists $\phi > 0$ such that the zero profit condition holds, where $\phi = \theta/(1 - G(x^c))$. If (15) fails, then no firms enter and $\theta = \phi = 0$.

If Assumption 1 holds, there is a unique critical value $\bar{b}(\lambda, \beta, \delta, C) > z$ such that condition (15) holds whenever $b < \bar{b}$. To see this, let $f(b) = \frac{x_0^{-1/\lambda}}{1-\beta(1-\delta)}\left(\frac{Ax_0}{1-\lambda} - b\right) - C$. Condition (15) holds if and only if $f(b) > 0$. First we prove that $f'(b) < 0$. If $b \leq A$, we have $x_0 = 1$ so $f(b) = \frac{1}{1-\beta(1-\delta)}\left(\frac{A}{1-\lambda} - b\right) - C$ and $f'(b) = \frac{-1}{1-\beta(1-\delta)} < 0$. If $b > A$, we have $f(b) = \frac{\lambda b^{1-1/\lambda}A^{-1/\lambda}}{(1-\lambda)(1-\beta(1-\delta))} - C$, so $f'(b) = \frac{-(b/A)^{-1/\lambda}}{1-\beta(1-\delta)} < 0$. So for all $b \in \mathbb{R}^+$, we have $f'(b) < 0$. Now if $f(z) > 0$, then since $f'(b) < 0$ and $f(b) \to -C$ as $b \to \infty$, there exists a unique $\bar{b} > z$ such that $f(\bar{b}) = 0$. Since $z < A$, we have $f(z) > 0$ provided that

$$C < \frac{1}{1-\beta(1-\delta)}\left(\frac{A}{1-\lambda} - z\right). \tag{16}$$

Condition (16) ensures that there exists a unique critical value $\bar{b} > z$ such that $f(\bar{b}) = 0$. We know that for any $b < \bar{b}$ condition (15) also holds and therefore there exists $\theta > 0$ and $\phi > 0$ that satisfy the zero profit condition, where $\phi = \theta/(1-G(b))$. If $b \geq \bar{b}$, then (15) fails and hence $\theta = \phi = 0$.

Uniqueness. To prove the uniqueness of θ which satisfies $F(\theta) = C(1-\beta(1-\delta))$, and hence the uniqueness of $\phi = \theta/(1 - G(b))$, it suffices to show that $F'(\theta) < 0$. Applying Leibniz' rule, $F'(\theta) = -(1-G(x^c))\left(\int_{Ax_0}^{\infty}(1 - G_b(y))^2 e^{-\theta(1-G_b(y))}dy + (1-b)e^{-\theta}\right) < 0$. So for any given $b \in \mathbb{R}^+$, there exists a unique θ and hence a unique ϕ that satisfies the zero profit condition. In other words, we have a function $\phi_r(b) : \mathbb{R}^+ \to \mathbb{R}^+$.

A.5 Proof that $\phi_r'(b) \leq 0$

If $b \geq \bar{b}$, we have $\phi_r(b) = 0$ and so $\phi_r'(b) = 0$ for $b > \bar{b}$. Assume instead that $b < \bar{b}$. Let $F_1(\theta, b) = x_0^{-1/\lambda}(Ax_0\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}(Ax_0 - b)) - C(1-\beta(1-\delta)) = 0$ where $x_0 = \max\{1, b/A\}$. When $b < A$, $F_1(\theta, b) = A\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}(A-b) - C(1-\beta(1-\delta))$ and $\theta = \phi$, so $\partial F_1/\partial b = -e^{-\theta}$ and $\partial F_1/\partial\theta = -(A\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) + e^{-\theta}(A-b))$. By the implicit function theorem, $\theta_r'(b) = \phi_r'(b) = -\frac{\partial F_1/\partial b}{\partial F_1/\partial\theta}$, which gives the following:

$$\phi_r'(b) = \theta_r'(b) = \frac{-e^{-\theta}}{A\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) + e^{-\theta}(A-b)} < 0,\ b < A. \tag{17}$$

When $b \geq A$, we have $F_1(\theta(\phi, b), b) = A^{-1/\lambda}b^{1-1/\lambda}\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) - C(1-\beta(1-\delta))$ where $\theta(\phi, b) = \phi(1 - G(x^c)) = \phi(b/A)^{-1/\lambda}$ and hence $\frac{\partial\theta}{\partial b} = -\frac{1}{\lambda}\theta b^{-1}$. Now $\phi_r'(b) = -\frac{dF_1/db}{dF_1/d\phi}$, where $\partial F_1/\partial\theta = -A^{-1/\lambda}b^{1-1/\lambda}\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)$ is obtained by differentiating and then applying Fact 1. Again using Fact 1 to simplify the following, we have $\frac{dF_1}{db} = \frac{\partial F_1}{\partial\theta}\frac{\partial\theta}{\partial b} + \frac{\partial F_1}{\partial b} = -A^{-1/\lambda}b^{-1/\lambda}e^{-\theta}$. Also, $\frac{dF_1}{d\phi}$ is given by $\frac{dF_1}{d\phi} = \frac{\partial F_1}{\partial\theta}\frac{\partial\theta}{\partial\phi} = -b^{1-1/\lambda}\lambda\theta^{\lambda-2}\gamma(2-$

$\lambda, \theta)b^{-1/\lambda}$, so we have the following expression for $\phi_r'(b)$, which is again negative:

$$\phi_r'(b) = -\frac{dF_1/db}{dF_1/d\phi} = \frac{-e^{-\theta}}{b^{1-1/\lambda}\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)} < 0, \;\; b \geq A.$$

A.6 Steady state reservation wage

Let $V^1(w)$ be the expected value of being employed at wage w and let V^u be the expected value of being unemployed. We have

$$V^1(w) = w + \beta((1-\delta)V^1(w) + \delta V^u). \tag{18}$$

Workers decide whether to accept or reject wage offers w, taking ϕ as given. If $V^1(w) \geq z + \beta V^u$, workers accept the wage offer w , while if $V^1(w) < z + \beta V^u$ they reject it. We show that for any given ϕ there exists a unique reservation wage b such that workers will accept a wage offer w if and only if $w \geq b$. The reservation wage b satisfies $V^1(b) = z + \beta V^u$. Rearranging (18), we have

$$V^1(w) = \frac{w + \beta\delta V^u}{1-\beta(1-\delta)}.$$

Since $dV^1(w)/dw > 0$, $\lim_{w\to\infty} V^1(w) = +\infty$, and we can verify that $V^1(0) < z + \beta V^u$, there exists a unique reservation wage b for any given ϕ such that $V^1(b) = z + \beta V^u$. This gives us a function $b_r(\phi) : \mathbb{R}^+ \to \mathbb{R}^+$. Given the existence of the unique reservation wage b, we can then derive the expression for V^u. Letting $\theta = \phi(1 - G(x^c))$, the expected value of being unemployed at the start of a period is

$$V^u = m(\theta)\int_{x_0}^{\infty} \max(V^1(w),\; z + \beta V^u)d\tilde{F}(w,\theta) + (1 - m(\theta)(z + \beta V^u),$$

where $\tilde{F}(w,\theta)$ is the distribution of wage offers w given θ. Substituting in $V^1(b)$, we have $\max(V^1(w), z + \beta V^u) = \max(V^1(w), V^1(b)) = V^1(w)$ since $b \leq w$ for all w. Since $\int_{x_0}^{\infty} V^1(w)d\tilde{F}(w,\theta) = V^e$ where V^e is the expected value of employment, we obtain

$$V^u = m(\theta)V^e + (1 - m(\theta))(z + \beta V^u).$$

A.7 Proof that $b_r'(\phi) > 0$

Start with expression (35) in Section 3.2 of the main text for the reservation wage:

$$b = \frac{z(1-\beta(1-\delta)) + \beta(1-\delta)w(\theta, b)}{1-\beta(1-\delta)e^{-\theta}}.$$

Let $F_2(\theta, b) = b(1-\beta(1-\delta)e^{-\theta}) - z(1-\beta(1-\delta)) - \beta(1-\delta)w(\theta, b) = 0$ and let $w(\theta, b) = Ax_0(1-\lambda)\theta^{\lambda}\gamma(1-\lambda, \theta) - \theta e^{-\theta}(Ax_0 - b)$ where $x_0 = \max\{1, b/A\}$. We have

$$\frac{\partial F_2}{\partial \theta} = \beta(1-\delta)\left(be^{-\theta} - \frac{\partial w}{\partial \theta}\right), \tag{19}$$

$$\frac{\partial F_2}{\partial b} = 1 - \beta(1-\delta)e^{-\theta} - \beta(1-\delta)\frac{\partial w}{\partial b}. \tag{20}$$

If $b < A$, then $\theta = \phi$ and $w(\theta, b) = (1-\lambda)A\theta^{\lambda}\gamma(1-\lambda, \theta) - \theta e^{-\theta}(A-b)$. Differentiating,

$$\frac{\partial w}{\partial \theta} = (1-\lambda)A(\lambda\theta^{\lambda-1}\gamma(1-\lambda, \theta) + e^{-\theta}) - (A-b)(e^{-\theta}(1-\theta)) \text{ and } \frac{\partial w}{\partial b} = \theta e^{-\theta}.$$

Substituting these into (19) and (20) and simplifying, we have

$$\frac{\partial F_2}{\partial \theta} = -\beta(1-\delta)(\lambda A\theta^{\lambda-1}\gamma(2-\lambda, \theta) + (A-b)\theta e^{-\theta}),$$

$$\frac{\partial F_2}{\partial b} = 1 - \beta(1-\delta)e^{-\theta} - \beta(1-\delta)\theta e^{-\theta}.$$

and hence using $b_r'(\phi) = b_r'(\theta) = -\frac{\partial F_2/\partial \theta}{\partial F_2/\partial b}$ we have

$$b_r'(\phi) = b_r'(\theta) = \frac{\beta(1-\delta)(\lambda A\theta^{\lambda-1}\gamma(2-\lambda, \theta) + (A-b)\theta e^{-\theta})}{1 - \beta(1-\delta)e^{-\theta} - \beta(1-\delta)\theta e^{-\theta}} > 0,\ b < A. \tag{21}$$

The numerator is positive when $b < A$ and the denominator is positive since $1 - e^{-\theta} - \theta e^{-\theta} > 0$ and $\beta(1-\delta) < 1$, so $b_r'(\phi) > 0$ when $b < A$.

If $b \geq A$, then we have $F_2(\theta(\phi, b), b) = b(1-\beta(1-\delta)e^{-\theta}) - z(1-\beta(1-\delta)) - \beta(1-\delta)w(\theta, b)$ where $w(\theta, b) = b(1-\lambda)\theta^{\lambda}\gamma(1-\lambda, \theta)$ and $\theta(\phi, b) = \phi(1-G(x^c)) = \phi(b/A)^{-1/\lambda}$, and hence $\frac{\partial \theta}{\partial b} = -\frac{1}{\lambda}\theta b^{-1}$. Differentiating, we have

$$\frac{\partial w}{\partial \theta} = b(1-\lambda)(\lambda\theta^{\lambda-1}\gamma(1-\lambda, \theta) + e^{-\theta}) \text{ and } \frac{\partial w}{\partial b} = (1-\lambda)\theta^{\lambda}\gamma(1-\lambda, \theta).$$

Substituting into (19) and (20) and simplifying using Fact 1,

$$\frac{\partial F_2}{\partial \theta} = -\beta(1-\delta)b\lambda\theta^{\lambda-1}\gamma(2-\lambda, \theta),$$

$$\frac{\partial F_2}{\partial b} = 1 - \beta(1-\delta)e^{-\theta} - \beta(1-\delta)(1-\lambda)\theta^{\lambda}\gamma(1-\lambda, \theta).$$

Now $b_r'(\phi) = -\frac{dF_2/\partial \phi}{dF_2/\partial b}$, where $\frac{dF_2}{d\phi} = \frac{\partial F_2}{\partial \theta}\frac{\partial \theta}{\partial \phi} = -b^{1-1/\lambda}A^{1/\lambda}\beta(1-\delta)\lambda\theta^{\lambda-1}\gamma(2-\lambda, \theta)$ and

$\frac{dF_2}{db} = \frac{\partial F_2}{\partial \theta}\frac{\partial \theta}{\partial b} + \frac{\partial F_2}{\partial b} = 1 - \beta(1-\delta)e^{-\theta} - \beta(1-\delta)\theta e^{-\theta}$, by applying Fact 3. So we have

$$b_r'(\phi) = -\frac{dF_2/\partial\phi}{dF_2/\partial b} = \frac{b^{1-1/\lambda}A^{1/\lambda}\beta(1-\delta)\lambda\theta^{\lambda-1}\gamma(2-\lambda,\theta)}{1-\beta(1-\delta)e^{-\theta}-\beta(1-\delta)\theta e^{-\theta}} > 0,\ b \geq A.$$

A.8 Proof of Proposition 2

Here we establish some comparative statics results for the equilibrium (ϕ^*, b^*) with respect to the parameters $p_i \in \mathbf{p} = (\lambda, z, C)$. We restrict our attention to the case where $b < A$ and $\phi = \theta$.

Consider the functions $\theta_r(b; \mathbf{p})$ and $b_r(\theta; \mathbf{p})$. The function θ_r is differentiable for any $b < \bar{b}$ and the function b_r is differentiable for any $\theta \geq 0$. Let $x = (\theta, b)$ and define the following function:

$$G(\mathbf{x}; \mathbf{p}) = \begin{bmatrix} b_r(\theta; \mathbf{p}) - b \\ \theta_r(b; \mathbf{p}) - \theta \end{bmatrix}.$$

By definition, $x^* = (\theta^*, b^*)$ is an equilibrium if and only if $G(x, p) = 0$. By the implicit function theorem, for any $p_i \in \mathbf{p}$ we have

$$\begin{aligned} D\mathbf{x}^*(p_i) &= -(D_{\mathbf{x}}G(\mathbf{x}^*(p_i); p_i))^{-1} D_{p_i}G(\mathbf{x}^*(p_i); p_i) \\ &= -\begin{bmatrix} \frac{\partial b_r}{\partial \theta} & -1 \\ -1 & \frac{\partial \theta_r}{\partial b} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial b_r}{\partial p_i} \\ \frac{\partial \theta_r}{\partial p_i} \end{bmatrix}. \end{aligned} \tag{22}$$

For notational simplicity, denote the matrix $D_{\mathbf{x}}G(x^*(p_i); p_i)$ by J_G. Using the derivatives $\partial b_r/\partial\theta$ and $\partial\theta_r/\partial b$ given by (21) and (17) respectively, we obtain

$$\det J_G = \frac{-(1-\beta(1-\delta)e^{-\theta})}{1-\beta(1-\delta)e^{-\theta}-\beta(1-\delta)\theta e^{-\theta}}, \tag{23}$$

where $\det J_G < -1$ and hence J_G is invertible. Multiplying out (22), for any $p_i \in \mathbf{p}$,

$$\frac{\partial \theta^*}{\partial p_i} = -(\det J_G)^{-1}\left(\frac{\partial \theta_r}{\partial b}\frac{\partial b_r}{\partial p_i} + \frac{\partial \theta_r}{\partial p_i}\right),$$

$$\frac{\partial b^*}{\partial p_i} = -(\det J_G)^{-1}\left(\frac{\partial b_r}{\partial p_i} + \frac{\partial b_r}{\partial \theta}\frac{\partial \theta_r}{\partial p_i}\right).$$

The functions $\theta_r(b; \mathbf{p})$ and $b_r(\theta; \mathbf{p})$ are implicitly defined by (24) and (25):

$$F_1(\theta, b; \mathbf{p}) = \lambda A\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}(A-b) - C(1-\beta(1-\delta)) = 0, \tag{24}$$

$$F_2(\theta, b; \mathbf{p}) = b(1-\beta(1-\delta)e^{-\theta}) - z(1-\beta(1-\delta)) - \beta(1-\delta)w(\theta, b; \mathbf{p}) = 0, \tag{25}$$

where $w(\theta, b; \mathbf{p}) = (1-\lambda)A\theta^{\lambda}\gamma(1-\lambda,\theta) - \theta e^{-\theta}(A-b)$.

We can now use the implicit function theorem in relation to $F_1(\theta, b; \mathbf{p})$ and $F_2(\theta, b; \mathbf{p})$ and apply some earlier results to obtain the following comparative statics.

Comparative statics for z.

$$\frac{\partial \theta^*}{\partial z} = -(\det J_G)^{-1}\left(\frac{\partial \theta_r}{\partial b}\frac{\partial b_r}{\partial z} + \frac{\partial \theta_r}{\partial z}\right)$$

$$= \frac{-(1-\beta(1-\delta))}{1-\beta(1-\delta)e^{-\theta}}\left(\frac{e^{-\theta}}{\lambda A\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(A-b)}\right) < 0 \tag{26}$$

$$\frac{\partial b^*}{\partial z} = -(\det J_G)^{-1}\left(\frac{\partial b_r}{\partial z} + \frac{\partial b_r}{\partial \theta}\frac{\partial \theta_r}{\partial z}\right)$$

$$= \frac{1-\beta(1-\delta)}{1-\beta(1-\delta)e^{-\theta}} > 0 \tag{27}$$

Comparative statics for C.

$$\frac{\partial \theta^*}{\partial C} = -(\det J_G)^{-1}\left(\frac{\partial \theta_r}{\partial b}\frac{\partial b_r}{\partial C} + \frac{\partial \theta_r}{\partial C}\right)$$

$$= \frac{-(1-\beta(1-\delta))}{1-\beta(1-\delta)e^{-\theta}}\left(\frac{1-\beta(1-\delta)e^{-\theta}-\beta(1-\delta)\theta e^{-\theta}}{\lambda A\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(A-b)}\right) < 0 \tag{28}$$

$$\frac{\partial b^*}{\partial C} = -(\det J_G)^{-1}\left(\frac{\partial b_r}{\partial C} + \frac{\partial b_r}{\partial \theta}\frac{\partial \theta_r}{\partial C}\right)$$

$$= \frac{-(1-\beta(1-\delta))\beta(1-\delta)\theta}{1-\beta(1-\delta)e^{-\theta}} < 0 \tag{29}$$

Comparative statics for λ. For $b < A$, using (17) we have

$$\frac{\partial \theta^*}{\partial \lambda} = -(\det J_G)^{-1}\left(\frac{\partial \theta_r}{\partial b}\frac{\partial b_r}{\partial \lambda} + \frac{\partial \theta_r}{\partial \lambda}\right) \tag{30}$$

$$= \frac{A\theta^{\lambda-1}\left(\gamma(1-\lambda,\theta)+(\lambda-\mu)B\right)}{\lambda A\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(A-b)}, \text{ where } \mu = \frac{\beta(1-\delta)\theta e^{-\theta}}{1-\beta(1-\delta)e^{-\theta}}$$

and B is given by the following:

$$B = \int_0^\theta t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt. \tag{31}$$

We have $\frac{\partial \theta^*}{\partial \lambda} > 0$ if and only if $\gamma(1-\lambda,\theta)+(\lambda-\mu)B > 0$. If $\lambda \geq \mu$, this is clearly true for $\theta > 0$. Suppose instead that $\mu > \lambda$. Rearranging and multiplying both sides by $1-\lambda$, we have $\frac{\partial\theta^*}{\partial\lambda} > 0$ if and only if $\frac{B(1-\lambda)}{\gamma(1-\lambda,\theta)} < \frac{1-\lambda}{\mu-\lambda}$. Now $(1-\lambda)/(\mu-\lambda) > 1/\mu$ provided $\mu < 1$, which is true since $1-\beta(1-\delta)e^{-\theta} - \beta(1-\delta)\theta e^{-\theta} > 0$. So it suffices to show that

$$\frac{B(1-\lambda)}{\gamma(1-\lambda,\theta)}\frac{\beta(1-\delta)\theta e^{-\theta}}{1-\beta(1-\delta)e^{-\theta}} < 1. \tag{32}$$

It follows from Lemma 4 in the Appendix of Mangin (2017) that

$$\frac{B(1-\lambda)}{\gamma(1-\lambda,\theta)} < \frac{(2-\lambda)\gamma(2-\lambda,\theta)}{\theta^{2-\lambda}e^{-\theta}}.$$

Hence to establish (32), it is sufficient to show that $m(\theta) \leq 1/(2-\lambda)$ where $m(\theta) = \frac{\beta(1-\delta)\theta^{\lambda-1}\gamma(2-\lambda,\theta)}{1-\beta(1-\delta)e^{-\theta}}$. It can be shown that $\max m(\theta) = \frac{1-\zeta}{2-\lambda-\zeta}$ where ζ is the unique solution to $1-\beta(1-\delta)e^{-\zeta} = \zeta$. To ensure that $\frac{\partial\theta^*}{\partial\lambda} > 0$, it suffices to show that $\frac{1-\zeta}{2-\lambda-\zeta} \leq \frac{1}{2-\lambda}$, which is always true since $\lambda < 1$ and $\zeta > 0$. So we have $\frac{\partial\theta^*}{\partial\lambda} > 0$ for $b < 1$.

We also have $\frac{\partial b^*}{\partial\lambda} > 0$ for $b < A$ and $\theta > 0$.

$$\begin{aligned}\frac{\partial b^*}{\partial\lambda} &= -(\det J_G)^{-1}\left(\frac{\partial b_r}{\partial\lambda} + \frac{\partial b_r}{\partial\theta}\frac{\partial\theta_r}{\partial\lambda}\right)\\ &= \frac{AB\beta(1-\delta)\theta^{\lambda}}{1-\beta(1-\delta)e^{-\theta}} > 0.\end{aligned} \tag{33}$$

Comparative statics for A.

$$\begin{aligned}\frac{\partial\theta^*}{\partial A} &= -(\det J_G)^{-1}\left(\frac{\partial\theta_r}{\partial b}\frac{\partial b_r}{\partial A} + \frac{\partial\theta_r}{\partial A}\right)\\ &= \frac{(\lambda-\mu)\theta^{\lambda-1}\gamma(1-\lambda,\theta)+e^{-\theta}}{\lambda A\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(A-b)},\ \text{where } \mu = \frac{\beta(1-\delta)\theta e^{-\theta}}{1-\beta(1-\delta)e^{-\theta}}.\end{aligned} \tag{34}$$

Since $b/A < 1$, we have $\frac{\partial\theta^*}{\partial A} > 0$ if and only if

$$(\lambda-\mu)\theta^{\lambda-1}\gamma(1-\lambda,\theta)+e^{-\theta} > 0, \tag{35}$$

which is true if and only if

$$\lambda + \frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)} > \frac{\beta(1-\delta)\theta e^{-\theta}}{1-\beta(1-\delta)e^{-\theta}}. \tag{36}$$

Since $\beta(1-\delta) < 1$, we have $\frac{\beta(1-\delta)\theta e^{-\theta}}{1-\beta(1-\delta)e^{-\theta}} < \frac{\theta e^{-\theta}}{1-e^{-\theta}}$ and it suffices to show that

$$\lambda + \frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)} \geq \frac{\theta e^{-\theta}}{1-e^{-\theta}}, \tag{37}$$

which is true provided that $\frac{\partial h}{\partial \lambda} > 0$ where $h(\lambda, \theta) = \lambda + \frac{\theta^{1-\lambda} e^{-\theta}}{\gamma(1-\lambda,\theta)}$, since $h(0, \theta) = \frac{\theta e^{-\theta}}{1-e^{-\theta}}$. Differentiating $h(\lambda, \theta)$ with respect to λ using Fact 5, we have

$$\frac{\partial h}{\partial \lambda} = 1 - \frac{\theta^{1-\lambda} e^{-\theta} B}{\gamma(1-\lambda, \theta)^2} \tag{38}$$

So $\frac{\partial h}{\partial \lambda} > 0$ if and only if $\theta^{1-\lambda} e^{-\theta} B < \gamma(1-\lambda, \theta)^2$. Substituting in the expression for B derived in the Appendix of Mangin (2017), we require that

$$\left(\frac{\theta^{1-\lambda}}{1-\lambda} \right)^2 e^{-\theta} F_{2,2}(1-\lambda, 1-\lambda; 2-\lambda, 2-\lambda; -\theta) < \gamma(1-\lambda, \theta)^2 .$$

Using the identity $\gamma(x, z) = z^x x^{-1} F_{1,1}(x; x+1; -z)$ from Andrews, Askey, and Roy (2000) this is equivalent to

$$e^{-\theta} F_{2,2}(1-\lambda, 1-\lambda; 2-\lambda, 2-\lambda; -\theta) < F_{1,1}(1-\lambda; 2-\lambda; -\theta)^2. \tag{39}$$

Now Lemma 4 in Mangin (2014) implies that the left-hand side of (39) is less than or equal to $F_{1,1}(1-\lambda; 2-\lambda; -\theta) F_{1,1}(2-\lambda; 3-\lambda; -\theta)$, so it suffices to show that $F_{1,1}(2-\lambda; 3-\lambda; -\theta) < F_{1,1}(1-\lambda; 2-\lambda; -\theta)$. Applying Kummer's first transformation, $F_{1,1}(y; z; -x) = e^{-x} F_{1,1}(z-y; z; x)$ from Andrews, Askey, and Roy (2000), we require that $F_{1,1}(1; 3-\lambda; \theta) < F_{1,1}(1; 2-\lambda; \theta)$. This is true since the function $F_{1,1}(a_1; b_1; x)$ is decreasing in its second argument. Hence $\frac{\partial h}{\partial \lambda} > 0$, so the original inequality (37) holds and therefore $\partial \theta^* / \partial A > 0$.

We also have $\frac{\partial b^*}{\partial A} > 0$ for $b < A$ and $\theta > 0$ since

$$\begin{aligned} \frac{\partial b^*}{\partial A} &= -(\det J_G)^{-1} \left(\frac{\partial b_r}{\partial A} + \frac{\partial b_r}{\partial \theta} \frac{\partial \theta_r}{\partial A} \right) \\ &= \frac{\beta(1-\delta)\theta^\lambda \gamma(1-\lambda, \theta)}{1-\beta(1-\delta)e^{-\theta}} > 0. \end{aligned} \tag{40}$$

Unemployment. Consider $u^* = u(\theta^*)$. The steady state unemployment rate is $u(\theta) = \delta/(\delta + m(\theta))$, which is clearly decreasing in θ since $m'(\theta) > 0$ so $u'(\theta) < 0$ for $\theta > 0$. Since $\frac{\partial \theta^*}{\partial z} < 0$, $\frac{\partial \theta^*}{\partial C} < 0$, $\frac{\partial \theta^*}{\partial \lambda} > 0$, and $\frac{\partial \theta^*}{\partial A} > 0$ for $b < A$, if $u'(\theta) < 0$ then $\frac{\partial u^*}{\partial z} = \frac{du}{d\theta} \frac{\partial \theta^*}{\partial z} > 0$, $\frac{\partial u^*}{\partial C} = \frac{du}{d\theta} \frac{\partial \theta^*}{\partial C} > 0$, $\frac{\partial u^*}{\partial \lambda} = \frac{du}{d\theta} \frac{\partial \theta^*}{\partial \lambda} < 0$ and $\frac{\partial u^*}{\partial A} = \frac{du}{d\theta} \frac{\partial \theta^*}{\partial A} < 0$ for $b < A$.

Output per capita. Consider $y^* = Y/L = y(\theta^*)$. First, we can write output per capita as follows: $y(\theta) = Ap(\theta)(1-u(\theta))$. Since $p'(\theta) > 0$ and $u'(\theta) < 0$, we have $y'(\theta) > 0$. Now, since $\frac{\partial \theta^*}{\partial C} \leq 0$, we have $\frac{\partial y^*}{\partial C} = \frac{\partial y}{\partial \theta} \frac{\partial \theta^*}{\partial C} < 0$ since $\frac{dy}{d\theta} > 0$. Next, $\frac{\partial y^*}{\partial z} = \frac{dy}{d\theta} \frac{\partial \theta^*}{\partial z} < 0$ if $b < A$, since $\frac{\partial \theta^*}{\partial z} < 0$. Finally, $\frac{\partial y^*}{\partial A} = \frac{dy}{d\theta} \frac{\partial \theta^*}{\partial A} + \frac{\partial y}{\partial A}$ where $\frac{\partial \theta^*}{\partial A} > 0$, $\frac{dy}{d\theta} > 0$, and $\frac{\partial y}{\partial A} > 0$, and therefore $\frac{\partial y^*}{\partial A} > 0$. It remains only to show that $\frac{\partial y^*}{\partial \lambda} > 0$. We can write $y(\theta) = \frac{A\gamma(1-\lambda,\theta)\theta^\lambda}{\delta+m(\theta)}$ for $b < A$. So $\frac{\partial y^*}{\partial \lambda} = \frac{dy}{d\theta} \frac{\partial \theta^*}{\partial \lambda} + \frac{\partial y}{\partial \lambda}$, where $\frac{\partial \theta^*}{\partial \lambda} > 0$ and $\frac{dy}{d\theta} > 0$, and it therefore suffices to show that $\frac{\partial y}{\partial \lambda} > 0$. Differentiating $y(\theta)$ with respect to λ yields

$$\frac{\partial y}{\partial \lambda} = \frac{A\theta^{\lambda}\left(\int_0^{\theta} t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt\right)}{\delta + m(\theta)} > 0.$$

Labor productivity. Consider $p^* = Ap(\theta^*)$. If $b < A$, we have $p(\theta) = \frac{A\gamma(1-\lambda,\theta)\theta^{\lambda}}{1-e^{-\theta}}$. Lemma 2 of Mangin (2017) establishes that if G is well-behaved then $p'(\theta) > 0$. This result applies when G is Pareto. We have $\frac{\partial p^*}{\partial \lambda} = \frac{dp}{d\theta}\frac{\partial \theta^*}{\partial \lambda} + \frac{\partial p}{\partial \lambda}$ where $\frac{\partial \theta^*}{\partial \lambda} > 0$, so $\frac{\partial p^*}{\partial \lambda} > 0$ provided that $\frac{\partial p}{\partial \lambda} > 0$, which is true since $\frac{\partial y}{\partial \lambda} > 0$ from above. Next, we have $\frac{\partial p^*}{\partial z} = \frac{dp}{d\theta}\frac{\partial \theta^*}{\partial z} < 0$ and $\frac{\partial p^*}{\partial C} = \frac{dp}{d\theta}\frac{\partial \theta^*}{\partial C} < 0$ since $\frac{\partial \theta^*}{\partial z} < 0$ and $\frac{\partial \theta^*}{\partial C} < 0$. Finally, $\frac{\partial p^*}{\partial A} = \frac{dp}{d\theta}\frac{\partial \theta^*}{\partial A} + \frac{\partial p}{\partial A} > 0$ since $\frac{\partial \theta^*}{\partial A} > 0$ and $\frac{\partial p}{\partial A} > 0$.

A.9 Proof of Proposition 3

For the Pareto distribution, we have the following:

$$\frac{x_0\mu(\theta)}{p(\theta)} = \varepsilon(1-\lambda,\theta) \tag{41}$$

and therefore when $x_0 = 1$ (i.e. $b < A$) steady state labor share can be expressed as

$$s_L = 1 - \lambda - \left(1 - \frac{b}{A}\right)\varepsilon(1-\lambda,\theta). \tag{42}$$

Differentiating $s_K^* = 1 - s_L^*$ with respect to z, we have

$$\frac{ds_K^*}{dz} = -\frac{1}{A}\frac{\partial b^*}{\partial z}\varepsilon(1-\lambda,\theta) + \left(1 - \frac{b^*}{A}\right)\frac{\partial \theta^*}{\partial z}\frac{\partial}{\partial \theta}\varepsilon(1-\lambda,\theta).$$

Substituting in $\frac{\partial}{\partial x}\varepsilon(s,x)$ from Fact 6, where $s = 1-\lambda$ and $x = \theta$, we have

$$\frac{ds_K^*}{dz} = -\frac{1}{A}\frac{\partial b^*}{\partial z}\varepsilon(1-\lambda,\theta) + \left(1 - \frac{b}{A}^*\right)\frac{\partial \theta^*}{\partial z}\frac{\theta^{-\lambda}e^{-\theta}\left(1-\lambda-\theta-\varepsilon(1-\lambda,\theta)\right)}{\gamma(1-\lambda,\theta)}.$$

Substituting in $\frac{\partial b^*}{\partial z}$ from (27) and $\frac{\partial \theta^*}{\partial z}$ from (26), and using Fact 4, we have

$$\frac{ds_K^*}{dz} = \frac{-\theta^{-\lambda}e^{-\theta}(1-\beta(1-\delta))}{\gamma(1-\lambda,\theta)(1-\beta(1-\delta)e^{-\theta})}\left(\theta + \frac{e^{-\theta}(A-b)\left(1-\lambda-\theta-\varepsilon(1-\lambda,\theta)\right)}{\lambda A\theta^{\lambda-2}\gamma(2-\lambda,\theta) + e^{-\theta}(A-b)}\right)$$

Rearranging and simplifying, we have $\frac{ds_K^*}{dz} < 0$ if $\lambda A\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}(A-b) > 0$, which is true. Hence $\frac{ds_K^*}{dz} < 0$ or equivalently $\frac{ds_L^*}{dz} > 0$.

A.10 Proof of Proposition 4

Using (42) and differentiating $s_K^* = 1 - s_L^*$ with respect to C, we have

$$\frac{ds_K^*}{dC} = -\frac{1}{A}\frac{\partial b^*}{\partial C}\varepsilon(1-\lambda,\theta) + \left(1 - \frac{b}{A}^*\right)\frac{\partial \theta^*}{\partial C}\frac{\partial}{\partial \theta}\varepsilon(1-\lambda,\theta).$$

Since $\frac{\partial b^*}{\partial C} < 0$ from (29), $\frac{\partial}{\partial \theta}\varepsilon(1-\lambda,\theta) < 0$ by Fact 6, and $\frac{\partial \theta^*}{\partial C} < 0$ from (28), we have $\frac{ds_K^*}{dC} > 0$ or equivalently $\frac{ds_L^*}{dC} < 0$.

A.11 Proof of Proposition 5

Again using (42) and differentiating with respect to A, we have

$$\begin{aligned}\frac{ds_L^*}{dA} &= \left(\frac{1}{A}\frac{db^*}{dA} - \frac{b^*}{A^2}\right)\varepsilon(1-\lambda,\theta^*) - \left(1 - \frac{b^*}{A}\right)\frac{\partial \theta^*}{\partial A}\frac{\partial}{\partial \theta}\varepsilon(1-\lambda,\theta)\\ &= \frac{\varepsilon(1-\lambda,\theta)}{A}\left(\frac{b^*}{A}\left(\varepsilon_{b^*}(A) - 1\right) - \left(1 - \frac{b^*}{A}\right)\varepsilon_{\theta^*}(A)\eta_\varepsilon(\theta^*)\right)\end{aligned}$$

and therefore $\frac{ds_L^*}{dA} > 0$ if and only if

$$\varepsilon_{b^*}(A) > \frac{A}{b^*}\left(1 - \frac{b^*}{A}\right)\varepsilon_{\theta^*}(A)\eta_\varepsilon(\theta^*) + 1$$

where $\varepsilon_{b^*}(A)$, the elasticity of b^* with respect to A, is

$$\varepsilon_{b^*}(A) = \frac{\beta(1-\delta)\theta^\lambda\gamma(1-\lambda,\theta)}{1-\beta(1-\delta)e^{-\theta}}\frac{A}{b^*}$$

and $\varepsilon_{\theta^*}(A)$, the elasticity of θ^* with respect to A, is

$$\varepsilon_{\theta^*}(A) = \frac{(\lambda-\mu)\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}}{\lambda A\theta^{\lambda-2}\gamma(2-\lambda,\theta) + e^{-\theta}(A-b)}\frac{A}{\theta^*}, \text{ where } \mu = \frac{\beta(1-\delta)\theta e^{-\theta}}{1-\beta(1-\delta)e^{-\theta}}$$

and $\eta_\varepsilon(\theta)$, the elasticity of $\varepsilon(1-\lambda,\theta)$ with respect to θ, is

$$\eta_\varepsilon(1-\lambda,\theta) = 1 - \lambda - \theta - \varepsilon(1-\lambda,\theta).$$

B Quantitative analysis

B.1 Estimation details

This Appendix provides additional details on the estimation presented in Section 4.4. The goal of that exercise is to use the calibrated model to estimate the underlying aggregate TFP and investment-specific shocks. To do so, the linearized model is set into state-space form and the time-paths of the two aggregate shocks are estimated using the Kalman filter where the observable variables are real GDP and the unemployment rate. Note that the model is calibrated to a monthly frequency. Therefore, in the model, real GDP is measured as a 3-month average of aggregate output, observed only every three months. The Kalman filter can conveniently deal with the within-quarter months as missing observations. Unemployment, on the other hand, is observed monthly and poses no additional complication for the estimation. Finally, both variables are expressed in log-deviations from their respective HP-filter trends (with a smoothing coefficient of 10^5 for quarterly GDP as in Shimer (2005) and a smoothing coefficient of $10^5 * 3^4$ for monthly unemployment, using the adjustment factor of Ravn and Uhlig (2002)).

An important by-product of estimating the model is that we obtain model-predicted time series for all the variables in the model (using the Kalman smoother). For our purposes, this means that we also obtain a model-predicted time-series for the aggregate labor share. The latter is depicted, together with the observed data on the labor share, taken from Rios-Rull and Santaeulalia-Llopis (2010), in Figure 5 in the main text.

Figure 1 shows the time-paths of other model variables predicted by the model and compares them to those observed in the data. The fact that the model tracks closely the business cycle fluctuations of several variables, not directly used in the estimation, is reassuring. Only labor market tightness is somewhat less volatile than in the data owing to the fact that the model underpredicts the volatility of vacancies.

B.2 Wage elasticities of new and existing workers

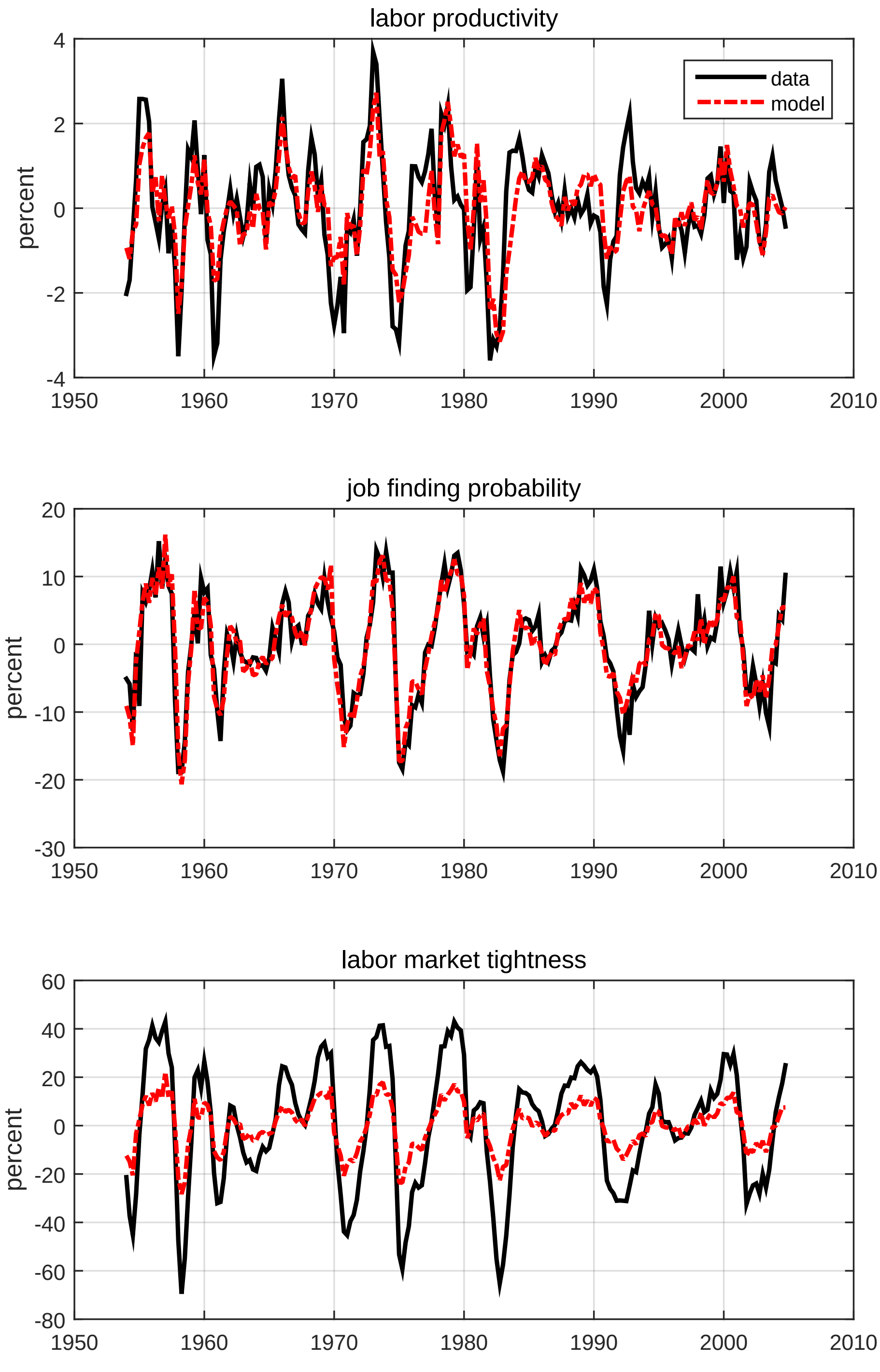
The model mechanism rests on the fact that the wages of newly hired workers are more flexible (i.e. more responsive to TFP shocks) than the wages of existing workers. This Appendix provides details on the estimation of the wage elasticities of new and existing workers in the model.

Existing studies document that wages of newly hired workers are indeed more flexible than those of existing employment relationships (see e.g. Bils, 1985; Haefke, Sonntag, and van Rens, 2013). The typical regression in these studies, estimating the wage elasticity of new and existing workers, is given by

$$\Delta \log w_{j,t} = \alpha_j + \eta_j \Delta \log(Y_t/N_t) + \epsilon_{j,t}, \qquad (43)$$

where Δ indicates first differences, $w_{j,t}$ are average wages of subgroup j of workers

Figure 1: Labor market variables: data and model



Notes: time-paths of variables in the "data" and in the model estimated using data on real GDP and unemployment. All variables in log-deviations from their respective HP-filter trends.

($j =$ all, new), α_j is a group-specific intercept, Y_t/N_t is (aggregate) labor productivity and $\epsilon_{j,t}$ are group-specific residuals. The main coefficient of interest, the group-specific wage elasticity, is η_j.

We follow the above methodology in our model. We simulate our model $1,000$ for $1,600$ periods, dropping the first $1,000$ periods in each simulation. This leaves us with 600 periods, corresponding to 50 years of data, roughly the sample period used in the above studies. Using this simulated data, we estimate the regression defined in (43) in each of the $1,000$ simulations. As in the above studies, we define newly hired workers as those with less than three months' tenure. The values reported below are the average values of η_j over the $1,000$ simulations.

The model predicts that while the wage elasticity for all workers is about one half, $\eta_{all} = 0.51$, that of new hires is around 1.5 times higher, $\eta_{new} = 0.86$. These values are consistent with the range of estimates found in existing studies.[4] Therefore, the micro-founded distinction between new and existing matches – which is key to the model's dynamics – is not only qualitatively but also quantitatively plausible.

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[4]Depending on how wages and productivity are measured, estimates of η_{all} range between 0.19 and 0.43, while estimates of η_{new} vary between 0.54 and 1.07 (see Haefke, Sonntag, and van Rens, 2013, for detailed empirical robustness exercises).