Private and Social Learning in Frictional Product Markets*

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Abstract

We introduce dynamic learning in the static search-theoretic framework of imperfect competition of Butters (1977), Varian (1980) and Burdett and Judd (1983). We consider two forms of learning: memory and word of mouth. In the model with memory, long-lived buyers not only learn about some new sellers in every period, but they also remember some of the sellers about which they learned in the past. In the model with word of mouth, short-lived buyers not only learn about sellers through search, but they also learn about sellers by talking to previous buyers, which refer them to the best seller of which they are aware. Both models are tractable. We establish the existence and uniqueness of equilibrium and characterize the dynamics of the product market. Memory increases the quantity of information available to buyers and, for this reason, leads to higher concentration and competition. Word of mouth increases the quality of information available to buyers and, for this reason, leads to higher concentration, but does not increase competition as much as memory.

JEL Codes: D43, D83, J31 .

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1 Introduction

The paper contributes to the development of the search-theoretic model of imperfect competition in product markets of Butters (1977), Varian (1980), and Burdett and Judd (1983). The simple idea behind the search-theoretic model of imperfect competition is that, due to information or physical frictions, buyers cannot purchase from any seller in the market but only from a discrete subset of sellers. Since buyers have a limited choice set, the equilibrium of the market is imperfectly competitive—where the extent of actual competition depends on the distribution of the size of buyers' choice sets. The model has been fruitfully applied to study price dispersion (e..g., Sorensen 2000, Galenianos et al. 2010, Kaplan and Menzio 2015, Kaplan et al. 2019), price stickiness (Head et al. 2012, Burdett and Menzio 2018), the variation of markups across groups of buyers (Pytka 2017, Nord 2023) and over time (Kaplan and Menzio 2016), specialization patterns (Menzio 2023), and store locations (Cai et al. 2023). These applications, even those that are explicitly dynamic, assume that the quantity and the quality of the information possessed by buyers about sellers does not improve over time. The assumption keeps the model static, and the analysis of equilibrium simple. The assumption, however, may not be a good description of product markets in which buyers are long-lived, or markets in which buyers can gather information from others who were in the market before them.

In this paper, we contribute to the development of the search-theoretic framework of imperfect competition by introducing learning. We consider two forms of learning: memory and word of mouth. In the model with memory, we consider a version of the framework in which long-lived buyers not only learn about some new sellers in every period, but they also remember some of the sellers that they contacted in the past. As a result of this process of private learning, the quantity of information that buyers have about sellers increases over time. In the model with word of mouth, we consider a version of the framework in which short-lived buyers not only learn about sellers through search, but they also learn about sellers from talking to previous buyers, which refer them to the best seller of which they are aware. As a result of this process of social learning, the quality of the information that buyers have about sellers increases over time. Both types of learning lead to non-trivial, albeit different, market dynamics.

In Section 2, we consider a version of the search-theoretic model of imperfect competition in which buyers have memory. Specifically, we consider a product market that is populated by a continuum of long-lived sellers and by a double continuum of long-lived buyers. Sellers are heterogeneous with respect to the quality of their variety of the product. In every period, sellers post prices. Buyers are ex-ante homogeneous and demand one unit of the product per period. In every period, buyers search the market and contact m new sellers, where m is distributed as a Poisson with some coefficient μ. Buyers also have a probability $1-\delta$ of remembering a seller that they had contacted in the past.

We show that equilibrium exists and is unique. The dynamics of equilibrium are driven by the evolution of the number nt of sellers in the choice set of buyers, which is distributed as a Poisson with a coefficient λ_t that is equal to $(1-\delta)\lambda_{t-1}+\mu$. Since λ_t is strictly increasing over time, the choice set of buyers expands and the market becomes progressively more competitive. The increase in competition induces sellers to lower prices. The increase in competition allows buyers to purchase the good from sellers with higher quality, since sellers with higher quality offer higher surplus to buyers. While competition increases over time, we show that the market does not become perfectly competitive as long as buyer's memory is imperfect. Intuitively, as long as memory is imperfect, each seller has a strictly positive probability of meeting a captive buyers and, hence, can always secure itself strictly positive profits. We also show that the price distribution in any period is strictly decreasing, in the sense of first-order stochastic dominance, with respect to the buyers' memory δ.

We also consider a version of the memory model in which buyers enter and exit the market. The properties of equilibrium described above carry over. In addition, this version of the model allows us to study the dynamics of individual buyers independently from the dynamics of the market. We show that the surplus captured by a buyer is higher, in expectation, the longer the buyer has been in the market. We show that a buyer purchases from higher quality sellers, in expectation, the longer the buyer has been in the market. In realization, however, a buyer may experience both declines and increases in its terms of trade.

In Section 3, we consider a version of the search-theoretic model of imperfect competition with word of mouth. Specifically, we consider a product market that is populated by a continuum of long-lived sellers and by a double continuum of short-lived buyers. Sellers are heterogeneous with respect to the quality of their variety of the product. In every period, sellers post prices. Buyers are homogeneous and demand one unit of the good. In every period, buyers search the market and contact m sellers, where m is distributed as a Poisson with coefficient μ. Buyers also meet r old buyers, where r is distributed as a Poisson with coefficient ρ. When new and an old buyers meet, the old buyers tell the new buyers about the best seller they have met when they were in the market.

We show that equilibrium exists and is unique. The dynamics of equilibrium are driven by the evolution of the distribution of sellers from which buyers sample through word of mouth. Buyers active in period t sample m sellers through search from the distribution of sellers, and they sample r sellers through word of mouth from the distribution of the highest quality seller known to buyers in period $t-1$. Buyers active in period t purchase the good from the highest quality seller among those contacted through search and those contacted through word of mouth. Along the equilibrium, the distribution of the highest quality seller known to buyers in period t is better than the distribution of the highest quality seller known to buyers in period $t-1$. Therefore, over time, the size of the buyers' choice sets does not increase, but its quality does. The increase in the quality of the buyers' choice sets leads to higher sales concentration—just as memory does. The increase in the quality of the buyers' choice sets does lead to low prices to the same degree as memory. Specifically, we show that, for the same level of sales concentration, prices in the model with word of mouth are always higher than prices in the model with memory.

2 Memory

In this section, we construct a version of the search-theoretic model of Butters (1977), Varian (1980) and Burdett and Judd (1983) in which buyers have memory about sellers about which they learned in the past. We construct the model so that, even though buyers and sellers are long-lived, the buyer's problem of choosing where to purchase the good is static, and the seller's problem of choosing what price to post is static. We establish that the equilibrium exists and is unique and we characterize the evolution of equilibrium outcomes over time.

2.1 Environment

We consider the market for some consumer good. On one side of the market, there is a measure 1 of infinitely-lived sellers. Sellers are heterogeneous with respect to the quality y of their variety of the good.[1] The distribution of sellers with respect to y is given by a twice differentiable cumulative distribution function $\Phi(y)$, with support $[y_\ell, y_h]$, $0 < y_\ell < y_h$. A seller produces its variety of the good at a constant marginal cost, which we set to 0 for the sake of simplicity.[2] In every period $t = 1, 2, ...$, the seller posts a price p for its variety of the good.[3] If the seller trades q units of the good at the price p, its periodical profit is qp.

On the other side of the market, there is a measure b of infinitely-lived buyers per

[1] We assume that sellers are heterogeneous with respect to the quality of their variety of the product. In a version of the model with homogeneous sellers, the price posted by an individual seller is indeterminate, since the equilibrium involves sellers mixing over prices according to an equilibrium distribution. In a version of the model with heterogeneous sellers, the price posted by an individual seller is uniquely pinned down. Heterogeneity purifies mixed strategies. The reader is free to interpret the heterogeneity in the model as a purification device, in which case the reader can take the extent of heterogeneity to be arbitrarily small, or as economically meaningful heterogeneity, in the sense that some sellers carry a better variety of the product than others. Supporting the second interpretation, Albrecht, Menzio and Vroman (2023) show that, if sellers can invest in the quality of their product, the unique equilibrium of the model features heterogeneity.

[2] The assumption that the marginal cost of production is 0 is made for the sake of algebraic simplicity, and all the results presented in the paper extend trivially to the case in which the marginal cost of production is strictly positive. The assumption that the marginal cost of production is constant, rather than strictly increasing, is substantive. Menzio (2023) shows that the structure of equilibrium is qualitatively different when sellers operate a production function with decreasing rather than constant returns to scale.

[3] We assume that a seller can only post a price. A seller cannot post more a schedule of prices in an attempt to discriminate buyers. Specifically, a seller is not allowed to post a price schedule $p(h_t)$, where the price paid by the buyer depends on its purchasing history h_t. The assumption seems natural in the context of retail markets.

seller.[4] Buyers are ex-ante homogeneous. In every period t, a buyer demands one unit of the good. If the buyer purchases a variety of the good of quality y at the price p, its periodical utility is $y - p$. If the buyer does not purchase the good, its periodical utility is 0.

The product market is frictional, in the sense that a buyer cannot purchase the good from any seller in the market, but only from those sellers with which it is in contact. At the beginning of each period t, a buyer is in contact with n_{t-1} sellers. During period t, the buyer permanently loses contact with one of the n_{t-1} sellers with probability $\delta \in [0, 1]$. During period t, The buyer contacts m_t additional sellers, where $m_t = 0, 1, 2, \ldots$ is drawn from a Poisson distribution with coefficient $\mu > 0$. At the end of period t, the buyer chooses whether and where to purchase the good among the sellers with which it is in contact. In period 1, buyers enter the market without any contacts.

The environment described above is quite natural. In every period, buyers search the market and discover m sellers. Buyers are also aware of the sellers that they have discovered in the past. The ability of buyers to remember sellers that they have discovered in the past is limited by the fact that a buyer "forgets" about a previously discovered seller with probability δ. If $\delta = 1$, buyers have no memory, and the environment is the same as in Butters (1977), Varian (1980) and Burdett and Judd (1983). If $\delta = 0$, buyers have perfect memory. If $\delta \in (0, 1)$, buyers have some memory, but their memory is imperfect. The parameter δ may be interpret literally as the probability that a buyer forgets about a seller. The parameter δ may also capture the probability of a negative shock to a time-varying, match-specific component of the gains from trade between the buyer and the seller that drives the gains from trade to zero. Notice that buyers have the same probability of "forgetting" a seller whether or not they purchased the good from it. This assumption is the polar opposite of the assumption made in the search model of imperfect competition in the labor market of Burdett and Mortensen (1998). Indeed, in this model, workers are assumed to have no memory of any of the previously contacted firms, except for the firm at which they are currently employed. This assumption greatly simplifies the analysis of equilibrium, as it makes the buyer's choice of where to purchase the good static.

[4] We assume that the market is populated by a continuum of sellers and a double continuum of buyers. If we were to assume that the market is populated by a continuum of sellers and a continuum of buyers, the actual number of buyers that meet an individual seller and, in turn, the actual number of trades made by an individual seller would be a random variable. The realization of the random variable would be informative about the choice set of the buyers that purchased the good from the seller and, in turn, the seller would use this information to set its price in the next period. For instance, if the seller made more trades than expected, the seller would infer that it is in contact with a larger than expected number of buyers for which the seller is the most attractive choice. Since the identity of the sellers in the buyers' choice sets is persistent, the seller could use this information to set its price in the next period. In contrast, under the assumption of a double continuum of buyers, the measure of buyers that meet an individual seller is deterministic and so are the measure of trades made by the seller. Therefore, the seller cannot infer anything about buyers from observing the quantity of the good that it has traded.

2.2 Equilibrium

In order to characterize equilibrium, we start by analyzing the properties of the buyer's choice set. At the beginning of period 1, a buyer enters the market without any contact. During period 1, the buyer searches the market and comes into contact with m_1 sellers, where m_1 is a Poisson random variable with coefficient μ. Hence, at the end of period 1, the buyer can purchase the good from $n_1 = m_1$ sellers. At the beginning of period 2, the buyer is in contact with n_1 sellers. During period 2 and for each one of the n_1 sellers, the buyer has a probability δ of losing contact with the seller, and a probability $1-\delta$ of staying in contact with the seller. Let $\hat{n}_2$ denote the number of sellers with which the buyer remains in contact. During period 2, the buyer searches the market and comes into contact with m_2 additional sellers, where m_2 is a Poisson random variable with coefficient μ. Therefore, at the end of period 2, the buyer can purchase the good from $n_2 = \hat{n}_2 + m_2$ sellers. The buyer keeps adding and losing contacts in the same way in period $t = 3, 4, \ldots$

By assumption, the number of buyer's contacts n_1 is distributed like a Poisson with coefficient μ. The next lemma establishes that the number of buyer's contacts n_t is distributed like a Poisson also in any period $t = 2, 3, 4, \ldots$. There is a simple logic behind this result. If the buyer enters period t with a number of contacts n_{t-1} that is Poisson with coefficient λ_{t-1} and each contact is maintained with probability $1-\delta$, the number of contacts $\hat{n}_t$ maintained by the buyer is distributed like a Poisson with coefficient $(1-\delta)\lambda_{t-1}$. In turn, if the number of contacts $\hat{n}_t$ maintained by the buyer is distributed like a Poisson with coefficient $(1-\delta)\lambda_{t-1}$ and the buyer contacts a number m_t of additional sellers that is distributed like a Poisson with coefficient μ, the buyer's number n_t of contacts at the end of period t is Poisson with coefficient λ_t, where λ_t is given by $\mu + (1-\delta)\lambda_{t-1}$.

Lemma 1: *The number n_t of buyer's contacts in period $t = 1, 2, \ldots$ is distributed as a Poisson with coefficient λ_t, where λ_t is given by*

$$\lambda_t = \mu \sum\nolimits_{i=1}^{t} (1-\delta)^{t-i}. \tag{2.1}$$

Proof: Suppose that the buyer enters the period with n contacts, where n is Poisson with some arbitrary coefficient $\gamma > 0$. Moreover, suppose that the buyer maintains a contact with probability $1-\delta$ and loses a contact with probability δ, with $\delta \in [0, 1]$. Let $\hat{n}$ denote the number of contacts retained by the buyer. The probability that $\hat{n}$ equals

$k = 0, 1, 2, ...n$ is given by

$$\begin{aligned} \Pr(\hat{n} = k) &= \sum_{n=k}^{\infty} \frac{e^{-\gamma}\gamma^n}{n!} \frac{n!}{k!(n-k)!}(1-\delta)^k \delta^{n-k} \\ &= \frac{e^{-\gamma}e^{\gamma\delta}\gamma^k(1-\delta)^k}{k!} \left[\sum_{n=k}^{\infty} \frac{\gamma^{n-k}}{1} \frac{e^{-\gamma\delta}}{(n-k)!} \delta^{n-k} \right] \\ &= \frac{e^{-\gamma(1-\delta)}(\gamma(1-\delta))^k}{k!} \end{aligned} \tag{2.2}$$

The expression in the first line of (2.2) is easy to understand. The probability that $\hat{n} = k$ is equal to the sum of the probability that the buyer enters the market with n contacts and retains k of them, for $n = k, k+1, k+2,$.... The first term on the right-hand side of (2.2) is the probability that the buyer enters the market with n contacts, given that n is distributed as a Poisson with coefficient γ. The second term is the probability that the buyer maintains exactly k of its n contacts, given that the buyer retains a contact with probability δ. The second line of (2.2) is obtained by collecting terms. The third line of (2.2) follows from the fact that the summation in the second line is equal to 1. The third line reveals that $\hat{n}$ is distributed like a Poisson with coefficient $\gamma(1-\delta)$.

Next, suppose that the buyer retains a number $\hat{n}$ of old contacts, where $\hat{n}$ is Poisson with some arbitrary coefficient $\hat{\gamma} > 0$. Moreover, suppose that the buyer makes m new contacts, where m is Poisson with coefficient $\mu > 0$. Let n_+ denote the total number of contacts of the buyer. The probability that n_+ equals k is given by

$$\begin{aligned} \Pr(n_+ = k) &= \sum_{\hat{n}=0}^{k} \frac{e^{-\hat{\gamma}}\hat{\gamma}^{\hat{n}}}{\hat{n}!} \frac{e^{-\mu}\mu^{k-\hat{n}}}{(k-\hat{n})!} \\ &= \frac{e^{-(\hat{\gamma}+\mu)}(\hat{\gamma}+\mu)^k}{k!} \left[\sum_{\hat{n}=0}^{k} \frac{k!}{\hat{n}!(k-\hat{n})!} \left(\frac{\hat{\gamma}}{\hat{\gamma}+\mu} \right)^{\hat{n}} \left(\frac{\mu}{\hat{\gamma}+\mu} \right)^{k-\hat{n}} \right] \\ &= \frac{e^{-(\hat{\gamma}+\mu)}\left(\hat{\gamma}+\mu\right)^k}{k!} \end{aligned} \tag{2.3}$$

The expression in the first line of (2.3) is easy to understand. The probability that the buyer has k total contacts is equal to the sum of the probability that the buyer retains $\hat{n}$ contacts and adds $k - \hat{n}$ new contacts, for $\hat{n} = 0, 1, 2....k$. The first term on the right-hand side of (2.3) is the probability that the buyer retains $\hat{n}$ contacts, given that $\hat{n}$ is distributed as a Poisson with coefficient $\hat{\gamma}$. The second term is the probability that the buyer adds $m = k - \hat{n}$ contacts, given that m is Poisson with coefficient μ. The second line of (2.3) is obtained by collecting terms. The third line of (2.3) follows from the fact that the summation in the second line is equal to 1. The third line reveals that n_+ is distributed like a Poisson with coefficient $\hat{\gamma} + \mu$.

By assumption, n_1 is Poisson with coefficient $\lambda_1 = \mu$. From the above observations, it follows that $\hat{n}_2$ is Poisson with coefficient $\lambda_1(1-\delta)$ and, in turn, n_2 is Poisson with coefficient $\lambda_2 = \lambda_1(1-\delta) + \mu$. Similarly, it follows that $\hat{n}_3$ is Poisson with coefficient $\lambda_2(1-\delta)$ and, in turn, n_3 is Poisson with coefficient $\lambda_3 = \lambda_2(1-\delta) + \mu$. In general, $\hat{n}_t$ is Poisson with coefficient $\lambda_{t-1}(1-\delta)$, and n_t is Poisson with coefficient $\lambda_t = \lambda_{t-1}(1-\delta) + \mu$. ■

Having characterized the properties of the buyer's choice set, we can now examine the problem of a buyer. In period t, the buyer is in contact with n_t sellers. For each of these sellers, the buyer observes the quality y of their variety of the product and the price p that they charge. The buyer's decision affects its payoff in the current period. In particular, if the buyer purchases a variety of the good of quality y at the price p, the buyer enjoys a payoff of $y - p$ in the current period. If the buyer does not purchase the good, the buyer enjoys a payoff of 0 in the current period. The buyer's decision does not affect the buyers' future payoffs. The buyer's decision does not affect the buyer's contacts in the future (since the buyer is equally likely to remember a seller from which it has purchased the good as a seller from which it has not purchased), nor does it affect the buyer's prices in the future (since sellers are not allowed to discriminate based on the purchasing history of a buyer). The buyer's problem is static, and it simply involves comparing the surplus $s = y - p$ offered by the n_t sellers and the payoff from not purchasing the good. Obviously, the solution to the buyer's problem is such that the buyer purchases from the seller that offers the highest surplus s if such surplus is positive, and the buyer does not purchase the good if s is strictly negative. If the highest positive surplus s is offered by multiple sellers, the buyer purchases the good from any of them with equal probability.

We now turn to examine the problem of a seller with a variety of quality y in period t. The seller has to decide the price p for its variety of the good or, equivalently, the surplus $s = y - p$ offered to buyers. The seller's decision has no effect on the seller's demand in the future, since the measure and the distribution of buyers that come into contact with the seller in the future is independent of the seller's current price. The seller's decision has no effect on the seller's ability to set prices in the future, since the current price does not restrict the seller's future prices. Therefore, the seller's problem is static, and it amounts to choosing s to maximize profits in period t.

For any $s \geq 0$, the seller's profit in period t is given by

$$V_t(y, s) = \left[\sum_{k=0}^{\infty} b_{k,t}\pi_{k,t}(s)\right](y - s), \tag{2.4}$$

where

$$b_{k,t} = b\frac{e^{-\lambda_t}\lambda_t^{k+1}}{(k+1)!}(k+1), \tag{2.5}$$

and

$$\pi_{k,t}(s) = F_t(s-)^k + \sum_{j=1}^{k} \frac{j!(k-j)!}{k!} \frac{\zeta_t(s)^j F_t(s-)^{k-j}}{(j+1)}. \tag{2.6}$$

Let us explain the expressions above. The seller meets a measure $b_{k,t}$ of buyers that are in contact with k other sellers, with $k = 0, 1, 2\ldots$ The measure of buyers $b_{k,t}$ who have k other contacts is given by (2.5), which is the measure of buyers that have $k+1$ contacts, given that the number of contacts of each buyer is Poisson with coefficient λ_t, multiplied by $k+1$, the number of contacts for each one of these buyers. A buyer with k other contacts purchases from the seller with probability $\pi_{k,t}(s)$. The probability $\pi_{k,t}(s)$ is given by (2.6), where $F_t(s)$ denotes the cumulative distribution function of surplus offered by sellers, $F_t(s-)$ denotes the left limit of F_t and is the fraction of sellers offering strictly less than s, and $\zeta_t(s)$ denotes the fraction of sellers that offer s. The probability $\pi_{k,t}(s)$ is the sum of the probability of two events. The first event is that the buyer's k other contacts offer a surplus strictly smaller than s. The second event is that j of the buyer's k other contacts offer a surplus equal to s and $k-j$ of them offer a surplus strictly smaller than s and the buyer breaks the indifference in favor of the seller. The seller's periodical profit in (2.4) is given by the number of buyers that purchase from the seller multiplied by the seller's profit per unit sold.

The next three lemmas are standard fare in the analysis of models in the style of Butters (1977), Varian (1980), and Burdett and Judd (1983). The first lemma establishes that the profit of the seller is strictly positive. The second lemma establishes that the surplus distribution $F_t(s)$ cannot have any mass points. The third lemma establishes that the support of the surplus distribution $F_t(s)$ is an interval $[s_{\ell,t}, s_{h,t}]$, with $s_{\ell,t} = 0$.

Lemma 2: *For any $y \in [y_\ell, y_h]$, the seller's maximized profit is strictly positive.*

Proof: If a seller with quality $y \in [y_\ell, y_h]$ offers a surplus s equal to 0, its profit is

$$\begin{aligned} V_t(y,0) &= \left[\sum\nolimits_{k=0}^{\infty} b_{k,t}\pi_{k,t}(0)\right] y \\ &\geq b_{0,t}\pi_{0,t}(0)y \end{aligned} \tag{2.7}$$

It is clear from (2.5) that $b_{0,k} > 0$. That is, the seller meets a strictly positive measure of buyers that are captive. It is clear from (2.6) that $\pi_{0,t}(0) = 1$. That is, a captive buyer purchases the good from the seller with probability 1. Lastly, since $y \in [y_\ell, y_h]$ and $y_\ell > 0$, the seller makes a strictly positive profit per sale by offering a surplus of 0. These observations imply that the second line in (2.7) is strictly positive and so is $V_t(y, 0)$. Since offering a surplus of 0 is feasible but not necessarily optimal, it follows that the seller's maximized profit is strictly positive. ■

Lemma 3: *The equilibrium surplus distribution $F_t(s)$ does not have mass points.*

Proof: On the way to a contradiction, suppose that the distribution $F_t(s)$ has a mass point at some surplus s_0. Let y_0 denote the quality of one of the sellers offering s_0. From Lemma 2, it follows that $y_0 > s_0$.

By offering the surplus s_0, the seller enjoys a profit of

$$V_t(y_0, s_0) = \left[\sum\nolimits_{k=0}^{\infty} b_{k,t}\pi_{k,t}(s_0)\right](y_0 - s_0). \tag{2.8}$$

By offering a surplus of $s_0 + \epsilon$, with $\epsilon > 0$, the seller enjoys a profit of

$$\begin{aligned}
& V_t(y_0, s_0+\epsilon) \\
= \; & \left[\sum\nolimits_{k=0}^{\infty} b_{k,t}\pi_{k,t}(s_0+\epsilon)\right](y_0 - s_0 - \epsilon) \\
\geq \; & \left[\sum\nolimits_{k=0}^{\infty} b_{k,t}\pi_{k,t}(s_0)\right](y_0 - s_0 - \epsilon) \\
+ \; & \left[\sum\nolimits_{k=0}^{\infty} b_{k,t}\left(\sum\nolimits_{j=1}^{k} \frac{j!(k-j)!}{k!}\zeta_t(s)^j F_t(s-)^{k-j}\left(1-\frac{1}{j+1}\right)\right)\right](y_0 - s_0 - \epsilon),
\end{aligned} \tag{2.9}$$

where the inequality in the third line follows from the fact that buyers that have j other contacts offering s_0 and $k-j$ other contacts offering strictly less than s_0 purchase the good from the seller with probability less than 1 if the seller offers s_0 and with probability 1 if the seller offers $s_0+\epsilon$. Since $\zeta_t(s) > 0$, the difference in trading probability in the fourth line of (2.9) is strictly positive. Also, the difference in the trading probability in the fourth line is independent of ϵ. Since $s_0 < y_0$, there is some ϵ small enough such that $V_t(y_0, s_0+\epsilon)$ is strictly greater than $V_t(y_0, s_0)$. Since a seller must maximize its profit and $V_t(y_0, s_0+\epsilon) > V_t(y_0, s_0)$, a seller with quality y_0 cannot offer the surplus s_0. Therefore, $F_t(s)$ cannot be an equilibrium distribution. ■

Since the surplus distribution does not have any mass points, $\zeta_t(s) = 0$ and $F_t(s-) = F_t(s)$ for all s. Using these observations, we can simplify (2.4) as

$$\begin{aligned}
V_t(y,s) &= \left[\sum\nolimits_{k=0}^{\infty} b\frac{e^{-\lambda_t}\lambda_t^{k+1}}{(k+1)!}(k+1)F_t(s)^k\right](y-s) \\
&= b\lambda_t e^{-\lambda_t}e^{\lambda_t F_t(s)}\left[\sum\nolimits_{k=0}^{\infty} \frac{e^{-\lambda_t F_t(s)}\lambda_t^k F_t(s)^k}{k!}\right](y-s) \\
&= b\lambda_t e^{-\lambda_t(1-F_t(s))}(y-s).
\end{aligned} \tag{2.10}$$

The first line is obtained by substituting $b_{k,t}$ and $\pi_{k,t}(s)$ from (2.5) and (2.6) into (2.4), and by using the fact that $\zeta_t(s) = 0$ and $F_t(s-) = F_t(s)$. The second line is obtained by collecting terms. The third line is obtained by recognizing that the summation in the second line is equal to 1.

Lemma 4: *The support of the equilibrium surplus distribution F_t is an interval $[s_{\ell,t}, s_{h,t}]$, with $s_{\ell,t} = 0$.*

Proof: We first establish that the support of F_t is an interval $[s_{\ell,t}, s_{h,t}]$. On the way to a contradiction, suppose that the support of F_t has a gap between s_0 and s_1, with $s_0 < s_1$, and s_0 and s_1 both on the support of F_t. Let y_1 denote the quality of a seller that offers the surplus s_1. The profit of this seller is given by

$$V_t(y_1, s_1) = b\lambda_t e^{-\lambda_t(1-F_t(s_1))}(y_1 - s_1). \tag{2.11}$$

If the seller offers the surplus s_0, the seller's profit is given by

$$
\begin{aligned}
V_t(y_1, s_0) &= b\lambda_t e^{-\lambda_t(1-F_t(s_0))}(y_1 - s_0) \\
&= b\lambda_t e^{-\lambda_t(1-F_t(s_1))}(y_1 - s_0) \\
&> b\lambda_t e^{-\lambda_t(1-F_t(s_1))}(y_1 - s_1) = V_t(y_1, s_1),
\end{aligned}
\tag{2.12}
$$

where the second line follows from the fact that $F(s_1) = F(s_0)$, and the third line follows from the fact that $s_0 < s_1$. Since a seller must maximize its profit and $V_t(y_1, s_0) > V_t(y_1, s_1)$, a seller with quality y_1 cannot offer the surplus s_1. Therefore, $F_t(s)$ cannot be an equilibrium distribution.

Next, we establish that $s_{\ell,t} = 0$. On the way to a contradiction, suppose $s_{\ell,t} < 0$. Let y_0 denote the quality of a seller that offers the surplus $s_{\ell,t}$. The seller attains a profit $V_t(y_0, s_{\ell,t})$ equal to 0, since no buyer purchases the good for a strictly negative surplus. If, however, the seller were to offer a surplus of 0, it would attain a profit $V_t(y_0, 0)$, which is strictly positive by Lemma 1. Therefore, $s_{\ell,t}$ cannot be strictly negative. Next, suppose $s_{\ell,t} > 0$. Let y_0 denote the quality of a seller that offers the surplus $s_{\ell,t}$. The seller attains a profit $V_t(y_0, s_{\ell,t})$ equal to $b\lambda_t \exp(-\lambda_t)(y_0 - s_{\ell,t})$. If, however, the seller were to offer a surplus of 0, it would attain a profit $V_t(y_0, 0)$ equal to $b\lambda_t \exp(-\lambda_t) y_0$, which is strictly greater than $V_t(y_0, s_{\ell,t})$. Therefore, $s_{\ell,t}$ cannot be strictly positive. Combining the above observations yields $s_{\ell,t} = 0$. ■

The next lemma establishes that the surplus offered by a seller is a strictly increasing function of the seller's quality. The finding is intuitive. The benefit of increasing the surplus s offered to buyers—which is to the increase the quantity of the good sold—is strictly greater for a seller with a variety of the good of higher rather than lower quality, since the profit per sale enjoyed by a seller is strictly increasing in y.

Lemma 5: *In equilibrium, the surplus offered by a seller is a strictly increasing function $s_t(y)$ of its quality.*

Proof: The proof has two parts. The first part establishes that surplus offered by a seller is strictly increasing in the seller's quality. The second part establishes that the mapping between the seller's quality and the surplus is a function, in the sense that every seller with the same quality offers the same surplus.

(i) Consider a seller with quality y_0 and a seller with quality y_1, with $y_0 < y_1$. Denote as s_0 the surplus offered by the seller with quality y_0, and denote as s_1 the surplus offered by the seller with quality y_1. Since the seller with quality y_0 enjoys a higher profit offering s_0 rather than s_1, we have

$$
b\lambda_t e^{-\lambda(1-F_t(s_0))}(y_0 - s_0) \geq b\lambda_t e^{-\lambda(1-F_t(s_1))}(y_0 - s_1). \tag{2.13}
$$

Since the seller with quality y_1 enjoys a higher profit offering s_1 rather than s_0, we have

$$
b\lambda_t e^{-\lambda_t(1-F_t(s_1))}(y_1 - s_1) \geq b\lambda_t e^{-\lambda_t(1-F_t(s_0))}(y_1 - s_0). \tag{2.14}
$$

Combining (2.13) and (2.14) yields

$$\left[e^{-\lambda_t(1-F_t(s_1))} - e^{-\lambda_t(1-F_t(s_0))}\right](y_1 - y_0) \geq 0.$$

Since $y_1 > y_0$ and $F_t(s)$ is strictly increasing in s, (2.16) implies that s_1 is greater or equal to s_0. Now, suppose that $s_0 = s_1 = s$. Then, every seller with quality $y \in (y_0, y_1)$ must offer the surplus s and, hence, there is a mass point in the surplus distribution F_t, which contradicts Lemma 3. Hence, it must be the case that s_1 is strictly greater than s_0.

(ii) On the way to a contradiction, suppose that there is a seller with quality y_0 that offers the surplus s_0 and another seller with quality y_0 that offers the surplus s_1, with $s_0 < s_1$. Every seller with quality $y > y_0$ must offer a surplus strictly greater than s_1. Every seller with quality $y < y_0$ must offer a surplus strictly smaller than s_0. Therefore, $F_t(s_0)$ must be equal to $F_t(s_1)$ or, in other words, there must be a gap on the support of F_t between s_0 and s_1, which contradicts Lemma 4. ■

The necessary condition for the optimality of the surplus $s_t(y)$ offered by a seller of quality y is

$$b\lambda_t e^{-\lambda_t(1-F_t(s_t(y)))}\lambda_t F_t'(s_t(y))(y - s_t(y)) - b\lambda_t e^{-\lambda_t(1-F_t(s_t(y)))} = 0. \tag{2.16}$$

The first term on the left-hand side of (2.16) is the marginal benefit of offering buyers an additional unit of surplus, which is given by the increase in the volume of sales multiplied by the profit per sale. The second term on the left-hand side of (2.16) is the negative of the marginal cost of offering buyers an additional unit of surplus, which is given by the reduction in profit per sale multiplied by the volume of sales. The optimality condition (2.16) states that the marginal benefit of offering an additional unit of surplus must be equal to the marginal cost.

The surplus distribution F_t is such that

$$F_t(s_t(y)) = \Phi(y). \tag{2.17}$$

The left-hand side of (2.17) is the fraction of sellers that offer surplus smaller than $s_t(y)$. The right-hand side of (2.17) is the fraction of sellers with a quality smaller than y. Since $s_t(y)$ is strictly increasing function, the left and the right-hand sides of (2.17) must be equal for every y.

Differentiating (2.17) with respect to y implies that $F_t'(s_t(y))s_t'(y)$ is equal to $\Phi'(y)$. Combining this observation with (2.17) allows us to rewrite (2.16) as

$$s_t'(y) = \Phi'(y)\lambda_t(y - s_t(y)). \tag{2.18}$$

The expression above is a differential equation for the surplus function $s_t(y)$. The relevant solution of the differential equation satisfies the boundary condition $s_t(y_\ell) = 0$, since Lemma 4 guarantees that the lowest surplus on the distribution F_t is 0, and Lemma 5 guarantees that the seller than offers the lowest surplus on the distribution F_t is the seller with the lowest quality y_ℓ.

The analysis above identifies a unique candidate equilibrium, which is given by the solution to the differential equation (2.18) together with the boundary condition $s_t(y_\ell) = 0$. To verify that the candidate equilibrium is an equilibrium, consider a seller with an arbitrary quality $y_0 \in [y_\ell, y_h]$. In the candidate equilibrium, the seller offers the surplus $s_t(y_0)$, where $s_t(y_0)$ satisfies the optimality condition (2.16). In the candidate equilibrium, the seller makes a strictly positive profit. The seller does not want to deviate and offer a surplus $\hat{s}$ that is strictly negative, since by doing so its profit would be zero. The seller does not want to deviate and offer a surplus $\hat{s} \in [0, s_t(y_0))$. In fact, $\hat{s} = s_t(\hat{y})$ for some $\hat{y} < y_0$ and, hence, the derivative of the seller's objective function (2.16) is equal to zero for $\hat{y}$, and strictly positive for y_0. Similarly, the seller does not want to deviate and offer a surplus $\hat{s} \in (s_t(y_0), s_t(y_h)]$. Finally, the seller does not want to deviate and offer a surplus $\hat{s} > s_t(y_h)$, since in doing so it would attain a profit that is strictly smaller than by offering the surplus $s_t(y_h)$.

We have thus established the existence and uniqueness of equilibrium.

Theorem 1: (Existence and uniqueness of equilibrium with memory) *An equilibrium exists and is unique. For $t = 1, 2, ...,$ equilibrium is given by a surplus function $s_t(y)$ that satisfies the differential equation (2.18) together with the boundary condition $s_t(y_\ell) = 0$.*

2.3 Market dynamics

In this subsection, we characterize the dynamics of equilibrium. In particular, we are interested in the evolution of the distribution of surplus offered by sellers, the distribution of prices posted by sellers, the distribution of transactions across sellers and the evolution of sales concentration.

The dynamics of equilibrium are driven by the accumulation of the amount of information that buyers have about sellers in the market. In period t, a buyer is in contact with n_t sellers, where n_t is distributed like a Poisson with coefficient λ_t. In period $t+1$, a buyer is in contact with n_{t+1} sellers, where n_{t+1} is distributed like a Poisson with coefficient λ_{t+1}. From Lemma 1, we know that λ_t and λ_{t+1} are respectively given by

$$\lambda_t = \mu \sum\nolimits_{i=1}^{t} (1-\delta)^{t-i} \tag{2.19}$$

and

$$\lambda_{t+1} = \mu \sum\nolimits_{i=1}^{t+1} (1-\delta)^{t+1-i}. \tag{2.20}$$

Subtracting (2.19) from (2.20) yields

$$\lambda_{t+1} - \lambda_t = \mu(1-\delta)^t. \tag{2.21}$$

As long as buyers have some memory of past contacts, in the sense that $\delta < 1$, the expression in (2.21) shows that λ_t is strictly increasing over time. Since the coefficient λ_t is equal to the buyer's average number of contacts in period t, the fact that λ_t is strictly increasing over time means that, on average, buyers are in contact with an increasing number of sellers.

As buyers accumulate contacts, the market becomes more competitive and, for this reason, the surplus offered by sellers increases and, for the same reason, the prices posted by sellers decline. In order to formalize this observation, note that the distribution of surplus offered by sellers, $F_t(s)$, is given by

$$F_t(s) = \Phi(y_t(s)), \tag{2.22}$$

where $y_t(s)$ is a strictly increasing function that denotes the inverse of the surplus function $s_t(y)$. Similarly, the distribution of prices posted by sellers, $G_t(p)$, is given by

$$G_t(p) = \int_{y:p_t(y)\leq y} \Phi'(y)dy, \tag{2.23}$$

where $p_t(y)$ is the price function defined as $p_t(y) = y - s_t(y)$. Note that, even though the surplus function $s_t(y)$ is strictly increasing in y, the pricing function need not be strictly increasing or strictly decreasing in y.[5] Therefore, the price distribution $G_t(p)$ has to be written as an integral of the quality density $\Phi'(y)$ over the qualities y such that $p_t(y)$ is smaller than p.

Proposition 1: (Surplus and price dynamics) *For any $\delta \in [0, 1)$, the surplus distribution $F_t(s)$ is strictly decreasing in t, and the price distribution $G_t(p)$ is strictly increasing in t.*

Proof: The surplus function $s_t(x)$ and the surplus function $s_{t+1}(x)$ are respectively given by

$$s'_t(y) = \Phi'(y)\lambda_t(y - s_t(y)), \tag{2.24}$$

and

$$s'_{t+1}(y) = \Phi'(y)\lambda_{t+1}(y - s_{t+1}(y)), \tag{2.25}$$

together with the boundary conditions $s_t(y_\ell) = 0$ and $s_{t+1}(y_\ell) = 0$.

Notice that the surplus function $s_{t+1}(y)$ is strictly greater than $s_t(y)$ for every $y \in (y_\ell, y_h]$. To see why this is the case, the following two observations are sufficient. First, notice that $s_{t+1}(y) = s_t(y)$ implies $s'_{t+1}(y) > s'_t(y)$, since $\lambda_{t+1} > \lambda_t$. In turn, this implies that $s_{t+1}(y)$ can equal $s_t(y)$ at most once, say for $y = y_c$, and that $s_{t+1}(y) > s_t(y)$ for all $y > y_c$. Second, notice that $s_{t+1}(y_\ell) = s_t(y_\ell)$. Taken together, these two observations imply that $s_{t+1}(y) > s_t(y)$ for all $y \in (y_\ell, y_h]$.

Since the surplus function $s_{t+1}(y)$ is strictly greater than $s_t(y)$, it follows from (2.22) that $F_{t+1}(s) < F_t(s)$. Since $s_{t+1}(y)$ is strictly greater than $s_t(y)$, the price function

[5] If the extent of heterogeneity in the sellers' quality is small enough, the pricing function $p_t(y)$ is strictly decreasing in y.

$p_{t+1}(y) = y - s_{t+1}(y)$ is strictly smaller than $p_t(y)$. It then follows from (2.23) that $G_{t+1}(p) > G_t(p)$. ■

As buyers accumulate contacts, they are not only offered more surplus by each seller, but they also get to purchase the good from sellers that offer more surplus, which are those that carry a variety of the good of higher quality. Both the increase in the surplus offered by each seller of a given quality y and the shift of transactions from low to high-surplus sellers (or, equivalently, from low to high-quality sellers) contribute to increase the buyer's surplus. In order to formalize the observation that transactions move towards better sellers, let $H_t(y)$ denote the fraction of transactions at sellers with quality smaller than y. Note that $H_t(y)$ is given by

$$\begin{aligned} H_t(y) &= \frac{\int_{y_\ell}^{y} b\lambda_t e^{-\lambda_t(1-\Phi(\hat{y}))}\Phi'(\hat{y})d\hat{y}}{\int_{y_\ell}^{y_h} b\lambda_t e^{-\lambda_t(1-\Phi(\hat{y}))}\Phi'(\hat{y})d\hat{y}} \\ &= \frac{e^{-\lambda_t(1-\Phi(y))} - e^{-\lambda_t}}{1 - e^{-\lambda_t}}, \end{aligned} \tag{2.26}$$

where the first line uses the fact that a seller with quality y trades $b\lambda_t \exp(-\lambda_t(1-\Phi(y)))$ units of the good, and the second line is obtained by solving the integrals in the first line.

Proposition 2: (Transaction dynamics) *For any $\delta \in [0, 1)$, the distribution of transactions across sellers of different quality, $H_t(y)$, is strictly decreasing in t.*

Proof: For $\gamma > 0$, let $h(\gamma)$ be defined as

$$h(\gamma) = \frac{e^{-\gamma(1-x)} - e^{-\gamma}}{1 - e^{-\gamma}}. \tag{2.27}$$

The derivative of $h(\gamma)$ with respect to γ is given by

$$h'(\gamma) = \frac{e^{-\gamma}\left[1 - xe^{-\gamma(1-x)} - (1-x)e^{\gamma x}\right]}{(1-e^{-\gamma})^2}. \tag{2.28}$$

The term in square brackets at the numerator of (2.28) determines the sign of $h'(\gamma)$. The term takes the value 0 for $x = 0$, it takes the value 0 for $x = 1$, and it is strictly convex with respect to x for all $x \in (0, 1)$. Therefore, the term in square brackets is strictly negative for all $x \in (0, 1)$ and, in turn, $h'(\gamma)$ is strictly negative for all $x \in (0, 1)$. Since $H_t(y) = h(\lambda_t)$ and $H_{t+1}(y) = h(\lambda_{t+1})$ for $x = \Phi(y)$ and $\lambda_t < \lambda_{t+1}$, it follows that $H_{t+1}(y) < H_t(y)$ for all $y \in (y_\ell, y_h)$. ■

As shown in Proposition 2, the distribution of transactions moves towards higher quality sellers as buyers accumulate contacts. Since high quality sellers trade more than low quality sellers, trade becomes more concentrated as buyers accumulate contacts. This finding may appear surprising, since the market becomes progressively more competitive. Indeed, it is true that, as buyers accumulate contacts and their choice sets expand, the increase in competition drives prices down. At the same time, though, as buyers accumulate contacts and their choice sets expand, buyers can purchase the good from better sellers. Since all buyers have the same ranking over sellers, trade concentration increases. Indeed, in a perfectly competitive version of the market, only the highest-quality seller would trade.

To formalize the above observation, let $Q_t(x)$ denote the fraction of transactions made by the fraction x of largest sellers. Since larger sellers are sellers with higher quality, $Q_t(x)$ is given by

$$Q_t(x) = 1 - H_t(y(1-x)), \tag{2.29}$$

where $y(x)$ denotes the quality of a seller at the x-th quantile of the distribution $\Phi(y)$. In Proposition 2, we established that $H_{t+1}(y) < H_t(y)$ for all $y \in (y_\ell, y_h)$. Therefore, $Q_{t+1}(x) > Q_t(x)$ for all $x \in (0,1)$.

Proposition 3: (Concentration dynamics) *For any $\delta \in [0,1)$, the fraction $Q_t(x)$ of sales made by the fraction x of the largest sellers is strictly incraesing over time.*

We now want to examine the role of memory in shaping the equilibrium of the product market. Let us first compare the properties of equilibrium if buyers have some memory of the sellers that they have contacted in the past, in the sense that $\delta \in [0,1)$, with the properties of equilibrium if buyers have no memory, in the sense that $\delta = 1$. In period 1, the buyers' number of contacts n_1 is distributed as a Poisson with coefficient $\lambda_1 = \mu$ whether buyers have memory or not. For this reason, in period 1, the surplus distribution, the price distribution and the transaction distribution are the same whether buyers have memory or not. Formally, $F_1(s|\delta)$, $G_1(p|\delta)$ and $H_1(y|\delta)$ are independent of δ. In periods $t \geq 2$, memory matters. If $\delta \in [0,1)$, $\lambda_2 > \lambda_1$, $\lambda_3 > \lambda_2$, etc... If $\delta = 1$, $\lambda_2 = \lambda_1$, $\lambda_3 = \lambda_2$, etc... For this reason, if $\delta \in [0,1)$, the market becomes more competitive, the surplus offered by sellers increases, the prices posted by sellers decline, and buyers shift their transactions towards from sellers with a lower quality to sellers with a higher quality. Formally, $F_t(s|\delta)$ is strictly decreasing in t, $G_t(p|\delta)$ is strictly increasing in t, and $H_t(y|\delta)$ is strictly decreasing in t. If $\delta = 1$, the surplus distribution $F_t(s|\delta)$ remains equal to $F_1(s|\delta)$, the price distribution $G_t(p|\delta)$ remains equal to $G_1(p|\delta)$, and the transaction distribution $H_t(y|\delta)$ remains equal to $H_1(y|\delta)$.

Next, let us compare the properties of equilibrium if buyers have better or worse memory. That is, let us compare the equilibrium if buyers have memory δ_1 or δ_2, with $\delta_1 < \delta_2$. In period 1, $F_1(s|\delta_1) = F_1(s|\delta_2)$, $G_1(p|\delta_1) = G_1(p|\delta_2)$ and $H_1(y|\delta_1) = H_1(y|\delta_2)$. In periods $t \geq 2$, λ_t is strictly greater if buyers have better memory (δ_1) than it is if buyers have worse memory (δ_2). Therefore, in any subsequent period, the surplus distribution is higher, the price distribution is lower, and the transaction distribution is better if buyers have better memory (δ_1) than if they have worse memory (δ_2). Formally, for $t \geq 2$, $F_t(s|\delta_1) < F_t(s|\delta_2)$, $G_t(p|\delta_1) > G_t(p|\delta_2)$ and $H_t(y|\delta_1) < H_t(y|\delta_2)$.

The above observations are summarized in the following proposition.

Proposition 4: (Memory). *Take any δ_1 and δ_2, with $0 \leq \delta_1 < \delta_2 \leq 1$.*

(i) *In period 1, equilibrium surplus, prices and transactions are the same for δ_1 and δ_2,* *in the sense that* $F_1(s|\delta_1) = F_1(s|\delta_2)$, $G_1(p|\delta_1) = G_1(p|\delta_2)$ *and* $H_1(y|\delta_1) = H_1(y|\delta_2)$. *In period* $t \geq 2$, *equilibrium surplus is higher, prices are lower, and transactions are better for* δ_1 *than* δ_2, *in the sense that* $F_t(s|\delta_1) < F_t(s|\delta_2)$, $G_t(p|\delta_1) > G_t(p|\delta_2)$ *and* $H_t(y|\delta_1) < H_t(y|\delta_2)$.

(ii) *Let* $\delta_2 = 1$. *Equilibrium surplus, prices and transactions are constant over time, in the sense that* $F_t(s|\delta_2) = F_1(s|\delta_2)$, $G_t(p|\delta_2) = G_1(p|\delta_2)$, *and* $H_t(y|\delta_2) = H_1(y|\delta_2)$.

Lastly, we want to understand the limiting behavior of the product market. In particular, we want to understand whether the market eventually becomes perfectly competitive, in the sense that every buyer purchases the good from the seller with the highest quality at a price equal to the seller's marginal cost. To understand the limiting behavior of the product market, it is sufficient to examine the limiting behavior of λ_t. If buyers have imperfect memory of sellers met in the past, in the sense that $\delta \in (0,1)$, λ_t is strictly increasing and converges to $\lambda^* = \mu/\delta$. Since λ_t is strictly increasing, the surplus offered by sellers increases over time, the prices posted by sellers decline over time, and the distribution of transactions moves from lower to higher quality sellers. Since λ^* is finite, however, the equilibrium does not become perfectly competitive. Indeed, since λ^* is finite, a seller with any quality $y \in [y_\ell, y_h]$ meets a strictly positive measure $b\lambda^* \exp(-\lambda^*)$ of captive buyers and, hence, can guarantee itself a strictly positive profit. For this reason, the surplus function $s^*(y)$ is such that $s^*(y) < y$ for all $y \in [y_\ell, y_h]$, the price function $p^*(y)$ is such that $p^*(y) > 0$ for all $y \in [y_\ell, y_h]$, and the volume of sales, $b\lambda^* \exp(-\lambda^*(1 - \Phi(y)))$, is strictly positive for all $y \in [y_\ell, y_h]$.

2.4 Entry and exit of buyers

A natural extension of the model with memory is one in which buyers enter and exit the market. In particular, we consider a version of the model described in Section 2.1 in which new buyers enter the market in every period, and old buyers exit the market in every period. Specifically, we assume that there is a double continuum of buyers with measure $b > 0$ per seller that enter the market in period $t = 1, 2, \ldots$. We assume that a fraction $\sigma \in [0,1]$ of buyers that were in the market in period $t-1$ permanently exit the market in period $t = 1, 2, \ldots$ Since the market opens in period 1, initially all buyers are new entrants.

Let us start by examining the population of buyers in the version of the model with entry and exit. In period 1, the market is populated by a measure $b_1 = b$ of buyers that have searched the market for one period. These buyers are in contact with n_1 sellers, where n_1 is distributed as a Poisson random variable with coefficient $\lambda_1 = \mu$. In period 2, the market is populated by a measure $b_1 = b$ of buyers that have just entered the market, and by a measure $b_2 = b(1-\sigma)$ of buyers that entered the market in period 1 and did not exit the market in period 2. The b_1 buyers with 1 period of experience have n_1 contacts, where n_1 is Poisson with coefficient λ_1. As shown in Lemma 1, the b_2

buyers with 2 periods of experience have n_2 contacts, where n_2 is Poisson with coefficient $\lambda_2 = \mu + \mu(1-\delta)$. In a generic period t, the market is populated by $b_i = b(1-\sigma)^{i-1}$ buyers that have been in the market for i periods, with $i = 1, 2, \dots t$, and, hence, that are in contact with n_i sellers, where, as shown in Lemma 1, n_i is distributed as a Poisson with coefficient $\lambda_i = \mu \sum_{j=1}^{i} (1-\delta)^{i-j}$.

As in the baseline model, the maximized profit for a seller with quality $y \in [y_\ell, y_h]$ is strictly positive (Lemma 2). As in the baseline model, the distribution $F_t(s)$ of surplus offered by sellers does not have any mass points (Lemma 3). Since the surplus distribution $F_t(s)$ does not have any mass points and the b_i buyers with experience i are in contact with a number of sellers distributed as a Poisson with coefficient λ_i, the profit for a seller with quality y that offers the surplus $s \geq 0$ can be written as

$$V_t(y, s) = \left[\sum_{i=1}^{t} b_i \lambda_i e^{-\lambda_i (1 - F_t(s))} \right] (y - s). \tag{2.30}$$

As in the baseline model, the support of the surplus distribution $F_t(s)$ is an interval $[s_{\ell,t}, s_{h,t}]$, with $s_{\ell,t} = 0$ (Lemma 4). Moreover, the surplus offered by a seller is a strictly increasing function $s_t(y)$ of the quality y of the variety carried by the seller (Lemma 5). Therefore, as in the baseline model, the fraction $F_t(s_t(y))$ of sellers offering surplus smaller than $s_t(y)$ is equal to the fraction $\Phi(y)$ of sellers with quality smaller than y. That is,

$$F_t(s_t(y)) = \Phi(y). \tag{2.31}$$

The necessary condition for the optimality of the surplus $s_t(y)$ offered by a seller of quality y in period t is

$$\begin{aligned} & \left[\sum_{i-1}^{t} b_i \lambda_i^2 F_t'(s_t(y)) e^{-\lambda_i (1 - F_t(s_t(y)))} \right] (y - s_t(y)) \\ = \ & \sum_{i-1}^{t} b_i \lambda_i e^{-\lambda_i (1 - F_t(s_t(y)))}. \end{aligned} \tag{2.32}$$

The left-hand side of (2.32) is the seller's marginal benefit from offering an additional unit of surplus to its buyers, which is given by the increase in the seller's volume of trade multiplied by the seller's profit per trade. The right-hand side of (2.32) is the seller's marginal cost from offering an additional unit of surplus to its buyers, which is given by the seller's volume of trade.

Using (2.31) to substitute $F_t(s_t(y))$ with $\Phi(y)$ and $F_t'(s_t(y))$ with $\Phi'(y)/s_t'(y)$, we can rewrite the optimality condition (2.32) as

$$s_t'(y) = \Phi'(y) \Lambda_t(y) (y - s_t(y)) \tag{2.33}$$

where $\Lambda_t(y)$ is given by

$$\Lambda_t(y) = \frac{\sum_{i=1}^{t} b_i \lambda_i^2 F_t'(s_t(y)) e^{-\lambda_i (1 - F_t(s_t(y)))}}{\sum_{i=1}^{t} b_i \lambda_i e^{-\lambda_i (1 - F_t(s_t(y)))}} \tag{2.34}$$

The surplus function $s_t(y)$ must satisfy the differential equation (2.33), together with the boundary condition $s_t(y_\ell) = 0$. The differential equation has the same structure as in the baseline model, except that $\Lambda_t(y)$ takes the place of λ_t. It is easy to see that $\Lambda_t(y)$ is simply a weighted average of the Poisson coefficient λ_i for buyers with experience $i = 1, 2, \ldots t$.

The unique candidate equilibrium of the product market is given by the solution $s_t(y)$ of the differential equation (2.33) together with the boundary condition $s_t(y) = 0$. Using the same argument used for the baseline model, it is easy to verify that the candidate equilibrium is indeed an equilibrium. Associated with the equilibrium surplus function $s_t(y)$, there are a surplus distribution $F_t(s)$, a price distribution $G_t(p)$, and a transaction distribution $H_t(y)$. The surplus distribution $F_t(s)$ is given by

$$F_t(s) = \Phi(y_t(s)), \tag{2.35}$$

where $y_t(s)$ is the inverse of the surplus function $s_t(y)$. The price distribution $G_t(p)$ is given by

$$G_t(p) = \int_{y: p_t(y) \leq p} \Phi'(y) dy, \tag{2.36}$$

where $p_t(y)$ is the pricing function $y - s_t(y)$. The transaction distribution $H_t(y)$ is given by

$$H_t(y) = \frac{\sum_{i=1}^{t} b_i \left(e^{-\lambda_i (1 - \Phi(y))} - e^{-\lambda_i} \right)}{\sum_{j=1}^{t} b_j \left(1 - e^{-\lambda_j} \right)}, \tag{2.37}$$

where the numerator in (2.37) is the quantity of the good traded by sellers with quality less than y, and the denominator in (2.37) is the quantity of the good traded by all sellers.

As in the baseline model, the market becomes more competitive over time because the fraction of more experienced buyers increases over time. To formalize this intuition, it is sufficient to examine the difference between $\Lambda_{t+1}(y)$ and $\Lambda_t(y)$, which can be written as

$$\Lambda_{t+1}(y) - \Lambda_t(y) = \left[\sum_{i=1}^{t} \left(\omega_{i,t+1} - \omega_{i,t} \right) \lambda_i \right] + \omega_{t+1,t+1} \lambda_{t+1}, \tag{2.38}$$

where

$$\omega_{i,t} = \frac{b_i \lambda_i e^{-\lambda_i (1 - \Phi(y))}}{\sum_{j=1}^{t} b_j \lambda_j e^{-\lambda_j (1 - \Phi(y))}} \tag{2.39}$$

and

$$\omega_{i,t+1} = \frac{b_i \lambda_i e^{-\lambda_i (1 - \Phi(y))}}{\sum_{j=1}^{t+1} b_j \lambda_j e^{-\lambda_j (1 - \Phi(y))}} \tag{2.40}$$

If buyers have some memory, in the sense that $\delta \in [0, 1)$, the average number λ_i of contacts for a buyer is strictly increasing in the buyer's experience i. If buyers have a positive probability of staying in the market for multiple periods, in the sense that $\sigma \in [0, 1)$, the measure of buyers $b_i = b(1 - \sigma)^{i-1}$ with experience $i = 1, 2, \ldots t$ is strictly positive, and so are $\omega_{i,t}$ and $\omega_{i,t+1}$. Using these observations yields

$$\begin{aligned}\Lambda_{t+1}(y)-\Lambda_t(y) &= \left[\sum_{i=1}^{t}(\omega_{i,t+1}-\omega_{i,t})\lambda_i\right]+\omega_{t+1,t+1}\lambda_{t+1}\\ &> \left[\sum_{i=1}^{t}(\omega_{i,t+1}-\omega_{i,t})\lambda_{t+1}\right]+\omega_{t+1,t+1}\lambda_{t+1}\\ &= \left[\sum_{i=1}^{t}(\omega_{i,t+1}-\omega_{i,t})+\omega_{t+1,t+1}\right]\lambda_{t+1}=0.\end{aligned} \tag{2.41}$$

Specifically, the second line in (2.41) makes use of the fact that $b_i > 0$ for $i = 1, 2, ...t + 1$ and, hence, (2.39) and (2.40) imply that $\omega_{i,t+1} - \omega_{i,t} < 0$ for $i = 1, 2, \ldots t$. Moreover, the second line makes use of the fact that $\lambda_{t+1} > \lambda_i$ for $i = 1, 2, \ldots t$. The third line makes use of the fact that $\omega_{i,t}$ and $\omega_{i,t+1}$ are weights, in the sense that $\sum_{i=1}^{t}\omega_{i,t} = 1$ and $\sum_{i=1}^{t+1}\omega_{i,t+1} = 1$ and, hence, the term in square brackets multiplying λ_{t+1} is equal to 0.

Following the same arguments as in Proposition 1, it is easy to show that, since $\Lambda_t(y)$ is strictly increasing in t, the surplus function $s_t(y)$ is strictly increasing over time. Since $s_t(y)$ is strictly increasing in t, it follows from (2.35) that $F_t(s)$ is strictly decreasing in t. Since $s_t(y)$ is strictly increasing in t, the price function $p_t(y) = y - s_t(y)$ is strictly decreasing over time. Therefore, it follows from (2.36) that $G_t(p)$ is strictly increasing in t. Following the same arguments as in Propositions 2 and 3, it is also easy to see that $H_t(y)$ is strictly decreasing in t and, hence, trade concentration is increasing over time.

The above observations are summarized in the following proposition.

Proposition 5: (Market dynamics with entry and exit) *Let $\sigma \in [0, 1)$ and $\delta \in [0, 1)$. The surplus distribution $F_t(s)$ is strictly decreasing in t. The price distribution $G_t(p)$ is strictly increasing in t. The transaction distribution $H_t(y)$ is strictly decreasing in t.*

If the buyers' exit probability σ is equal to 1, the distribution of buyers across experience levels is degenerate, in the sense that $b_1 = b$ and $b_i = 0$ for $i = 2, 3, \ldots t + 1$. It then follows from (2.38) that $\Lambda_{t+1}(y) - \Lambda_t(y) = 0$. It also follows from (2.34) that $\Lambda_t(y) = \mu$ for $t = 1, 2, \ldots$. Hence, if $\sigma = 1$, the surplus function $s_t(y)$, the price function $p_t(y)$, the surplus distribution $F_t(s)$, the price distribution $G_t(p)$, and the transaction distribution $H_t(y)$ remain constant over time. Intuitively, if $\sigma = 1$, buyers cannot accumulate any experience and, hence, the extent of competitiveness of the product market remains constant over time and so do all the equilibrium outcomes. The same conclusions apply to the case in which buyers have no memory ($\delta = 1$), since in this case buyers do accumulate experience but experience is worthless.

We now want to compare the properties of equilibrium if the buyers' exit probability is lower (σ_1) or higher (σ_2), given that buyers have some memory. In period 1, $\Lambda_1(y) = \mu$ irrespective of σ and, hence, the equilibrium is the same whether the buyers' exit probability is lower (σ_1) or higher (σ_2). Formally, in period 1, $F_1(s|\sigma_1) = F_1(s|\sigma_2)$, $G_1(p|\sigma_1) = G_1(p|\sigma_2)$ and $H_1(y|\sigma_1) = H_1(y|\sigma_2)$. In any period $t \geq 2$, $\Lambda_t(y)$ is strictly higher if the buyer's exit probability is lower (σ_1) than higher (σ_2), since the distribution of buyers is tilted towards buyers with more contacts when the buyers' exit probability is lower. For this reason, in any period $t \geq 2$, the market is more competitive if the buyer's exit probability is lower. In turn, this implies that, in any period $t \geq 2$, the surplus offered by sellers is higher, the prices posted by sellers are lower, and the transactions are shifted toward higher-quality sellers if the buyers' exit probability is σ_1 rather than σ_2. Formally, for $t \geq 2$, $F_t(s|\sigma_1) < F_t(s|\sigma_2)$, $G_t(p|\sigma_1) > G_t(p|\sigma_2)$ and $H_t(y|\sigma_1) < H_t(y|\sigma_2)$.

The above observations are summarized in the following proposition.

Proposition 6: (Exit). *Take any $\delta \in [0, 1)$ and any σ_1 and σ_2, with $0 \leq \sigma_1 < \sigma_2 \leq 1$.*

(i) *In period 1, equilibrium surplus, prices and transactions are the same for σ_1 and σ_2, in the sense that $F_1(s|\sigma_1) = F_1(s|\sigma_2)$, $G_1(p|\sigma_1) = G_1(p|\sigma_2)$ and $H_1(y|\sigma_1) = H_1(y|\sigma_2)$. In any period $t \geq 2$, equilibrium surplus is higher, prices are lower, and transactions are better for σ_1 than for σ_2, in the sense that $F_t(s|\sigma_1) < F_t(s|\sigma_2)$, $G_t(p|\sigma_1) > G_t(p|\sigma_2)$ and $H_t(y|\sigma_1) < H_t(y|\sigma_2)$.*

(ii) *Let $\sigma_2 = 1$. Equilibrium surplus, prices and transactions are constant over time, in the sense that $F_t(s|\sigma_2) = F_1(s|\sigma_2)$, $G_t(p|\sigma_2) = G_1(p|\sigma_2)$, and $H_t(y|\sigma_2) = H_1(y|\sigma_2)$.*

The limiting behavior of the product market is determined by the limiting behavior of $\Lambda_t(y)$. In the limit for $t \to \infty$, $\Lambda_t(y)$ is given by

$$\lim_{t\to\infty} \Lambda_t(y) = \lim_{t\to\infty} \frac{\sum_{i=1}^{t} b(1-\sigma)^{i-1}\lambda_i^2 F_t'(s_t(y))e^{-\lambda_i(1-F_t(s_t(y)))}}{\sum_{i=1}^{t} b(1-\sigma)^{i-1}\lambda_i e^{-\lambda_i(1-F_t(s_t(y)))}}. \tag{2.42}$$

Since $\Lambda_t(y)$ is increasing in t, $\Lambda_t(y)$ converges to some finite $\Lambda^*(y)$ if and only if $\Lambda_t(y)$ is bounded from above. For any $\sigma \in (0, 1]$, the numerator is the sum of products between a term that declines exponentially, $(1-\sigma)^{i-1}$, and a term λ_i^2 that is bounded above by $(\mu i)^2$. Therefore, the numerator in (2.42) is bounded from above. Since the denominator in (2.42) is strictly increasing in t and the numerator in (2.42) is bounded from above, $\Lambda_t(y)$ is bounded from above and, hence, $\Lambda_t(y)$ converges to some finite $\Lambda^*(y)$. For $\sigma = 1$ and any $\delta \in (0, 1]$, $\Lambda_t(y)$ is a weighted average of λ_i for $i = 1, \ldots t$. Since λ_i is bounded above by μ/δ, it follows that (2.42) is bounded from above, and, hence, $\Lambda_t(y)$ converges to some finite $\Lambda^*(y)$.

From the above observations, it follows that the limiting behavior of the product market remains imperfectly competitive as long as either buyers have imperfect memory, in the sense that $\delta \in (0, 1]$, or buyers have a strictly positive probability of exiting, in the sense that $\sigma \in (0, 1]$. In fact, if $\delta \in (0, 1]$ or $\sigma \in (0, 1]$, $\Lambda_t(y)$ converges to some finite $\Lambda^*(y)$ and the equilibrium associated with a finite $\Lambda^*(y)$ is such that sellers enjoy a strictly positive profit, prices are strictly greater than marginal cost, and all sellers, including those with lower quality, trade the good. These findings are intuitive. If buyers have imperfect memory, there is always a positive fraction of buyers that are captive. Hence, sellers' profits are always bounded away from zero and the market remains imperfectly competitive. Similarly, if buyers have a strictly positive probability of exit, there is always a positive fraction of buyers who have just entered the market and have limited contacts with sellers. Hence, sellers' profits are always bounded away from zero and the market remains imperfectly competitive.

Up to this point, we focused on the dynamics of the product market dynamics. First, we showed that, as long as buyers have some memory and as long as buyers have a probability of exiting that is less than 1, the market becomes progressively more competitive and, for this reason, the surplus offered by sellers increases over time, the prices posted by sellers decrease over time, and the distribution of transaction moves from lower to higher quality sellers. Second, we showed that, as long as buyers do not have perfect memory or as long as buyers have a strictly positive probability of exiting, the market does not become perfectly competitive.

The version of the model with entry and exit of buyers also allows us to study the effect of experience on buyers' outcomes independently from the dynamics of the product market. To analyze the effect of experience, suppose that $\delta \in (0,1)$ and $\sigma \in (0,1)$ and that the market has reached its stationary limit.[6]

Buyers with experience i are in contact with n_i sellers, where n_i is distributed as a Poisson with coefficient λ_i. Among the buyers with experience i that purchase the good, the distribution of surplus is given by

$$\begin{aligned} F_i(s) &= \left[\sum\nolimits_{k=1}^{\infty} \frac{e^{-\lambda_i}\lambda_i^k}{k!} F^*(s)^k\right] \Big/ \left[1 - e^{-\lambda_i}\right] \\ &= \left[e^{-\lambda_i(1-F^*(s))} \sum\nolimits_{k=0}^{\infty} \frac{e^{-\lambda_i F^*(s)}}{k!} (\lambda_i F^*(s))^k - e^{-\lambda_i}\right] \Big/ \left[1 - e^{-\lambda_i}\right] \\ &= \frac{e^{-\lambda_i(1-F^*(s))} - e^{-\lambda_i}}{1 - e^{-\lambda_i}}. \end{aligned} \tag{2.43}$$

The numerator in the first line of (2.43) is the product between the probability that a buyer has k contacts and the probability that all of its contacts offer less surplus than s, summed over $k = 1, 2, \ldots$. The denominator in the first line of (2.43) is the probability that the buyer has at least one contact and, hence, purchases the good. The second line in (2.43) is obtained by algebraic manipulations. The third line is obtained by recognizing that the summation in the second line is equal to 1.

Buyers with experience $i+1$ are in in contact with n_{i+1} sellers, where n_{i+1} is distributed as a Poisson with coefficient λ_{i+1}. Among the buyers with experience $i+1$ that purchase

[6] We focus on the stationary limit because, in such limit, the difference in outcomes across buyers with different experience i are the same as the change in outcomes for an individual buyer j as its experience i increases. In an arbitrary period t, the differences in outcomes across buyers with different experience i are qualitatively identical to those in the stationary limit. In an arbitrary period t, however, the difference in outcomes across buyers with different experience i are not the same as the change in outcomes for an individual buyer j as its experience i grows, since the latter would have to include changes in market-wide outcomes between t and $t+1$.

the good, the distribution of surplus is similarly given by

$$F_{i+1}(s) = \frac{e^{-\lambda_i(1-F^*(s))} - e^{-\lambda_i}}{1 - e^{-\lambda_i}}. \tag{2.44}$$

Using the same results as in the proof of Proposition 3, it is easy to show that $\lambda_{i+1} > \lambda_i$ implies $F_{i+1}(s) < F_i(s)$. Therefore, the average surplus for a buyer with experience i is strictly increasing with respect to the buyer's experience. It is useful to point out that, while the buyers' average surplus increases with experience, the surplus of a specific buyer may decrease, remain constant, or increase depending on how many contacts the buyer has, how many contacts the buyer loses, and how many new contacts the buyer adds. For instance, if some buyer j has $n_{i,j}$ contacts, loses the best of its contacts, and does not acquire any new contacts, its surplus falls. If buyer j has $n_{i,j}$ contacts, does not lose any of its contacts and does not acquire any new contacts, its surplus remains unchanged. If buyer j reaches a seller that is better than any of its previous contacts, the buyer's surplus increases. Moreover, all of these three events have a strictly positive probability of occurring.

3 Word of mouth

In this section, we construct a version of the search-theoretic model of Butters (1977), Varian (1980) and Burdett and Judd (1983) in which buyers learn about sellers not only by directly searching the market but also by talking to buyers that were previously active in the market. We construct the model so that, even though sellers are long-lived, the seller's problem of choosing what price to post remains static. We establish that the equilibrium exists and is unique, and we characterize the evolution of equilibrium outcomes over time. We show that memory and word of mouth generate similar dynamics along some dimensions (i.e., concentration), but not along other dimensions (i.e., competition and prices).

3.1 Environment

We consider the market for some consumer good. The market is populated by a continuum b of infinitely-lived sellers with measure 1. Sellers are heterogeneous with respect to the quality y of their variety of the good. The distribution of sellers with respect to y is given by a cumulative distribution function $\Phi(y)$, with support $[y_\ell, y_h]$, $0 < y_\ell < y_h$. A seller produces the good at a constant marginal cost, which we set to 0. In every period t, a seller posts a price p.[7] If a seller trades q units of the good at the price p, its periodical profit is qp.

[7] We assume that sellers are not allowed to post price schedules. In particular, a seller is not allowed to post a price schedule that depends on the calendar time t, or on the purchasing history h_t of an individual buyer.

The market is also populated by a double continuum of short-lived buyers with measure b per seller.[8] New buyers enter the market in period t. If one of these buyers purchases a variety of the good of quality y at the price p, its periodical utility is $y - p$. If one of these buyers does not purchase the good, its periodical utility is 0. Whether it purchases the good or not, the buyer exits the market at the end of period t.

The product market is frictional, in the sense that a buyer cannot purchase the good from any seller in the market, but only from those sellers with which it is in contact. In every period t, a buyer comes into contact with sellers through two channels: search and word-of-mouth. Through search, a buyer comes into contact with m_t sellers, where m_t is a random variable distributed as a Poisson with coefficient $\mu > 0$. Through word-of mouth, a buyer comes into contact with r_t sellers, where r_t is a random variable distributed as a Poisson with coefficient $\rho > 0$. Each contact obtained through search is randomly selected from the seller's distribution $\Phi(y)$. Each contact obtained through word-of-mouth is randomly selected from the distribution $\hat{\Phi}_t(y)$, where $\hat{\Phi}_t(y)$ denotes the distribution of the highest quality seller contacted by each buyer in the previous period. Since the market opens in period 1, the word-of-mouth channel is initially inactive.

The environment described above is quite natural. A buyer that is new to the market can find out about sellers by searching the market directly. The outcome of this direct search process is a number m of meetings with randomly-selected seller. This is the same search process as in Butters (1977), Varian (1980) or Burdett and Judd (1983). A buyer that is new to the market, however, can also find out about sellers indirectly by looking for buyers that were in the market in the previous period. The outcome of this indirect search process is a number r of meetings with previous buyers. In the meeting, the new buyer learns about the highest quality seller with which the old buyer had been in contact in the previous period.[9]

We assume that an old buyer refers a new buyer to the one seller that, among its contacts, carries the variety of the good with the highest quality, rather than to the seller that, among its contacts, offered the highest surplus in the previous period. The assumption simplifies the analysis. In fact, if old buyers refer new buyers to the highest-quality seller of which they know, the seller's price in the current period does not affect the measure and type of buyers that reach the seller in the future. Therefore, if old buyers refer new buyers to the highest-quality seller of which they know, the seller's pricing problem is static. The assumption is also easy to rationalize as an equilibrium. If sellers do not expect the current price to affect their future demand, higher quality sellers always choose to offer more surplus to their buyers. In turn, if higher quality sellers are expected to offer more surplus to their buyers, new buyers ask old buyers about the highest-quality

[8] We assume that there is a continuum of buyers per seller. The assumption guarantees that an individual seller cannot learn anything about the demand that it faces from the quantity of the good that it has sold in the past, since such quantity is a deterministic function of the seller's quality and price.

[9] It seems quite natural that a new buyer would only ask an old buyer about the best seller of which it knows. Information about additional sellers does not improve the new buyer's payoffs.

seller of which they know, since that seller is the one expected to offer the highest surplus to the new buyer.

3.2 Equilibrium

Consider some arbitrary period $t = 1, 2, \ldots$. The distribution of sellers across qualities is given by the twice differentiable distribution function $\Phi(y)$. This is the distribution from which buyers sample when they contact a seller through the search channel. The distribution of the highest quality seller contacted by buyers in period $t-1$ is given by $\hat{\Phi}_t(y)$, which we conjecture (and later verify) is twice differentiable. This is the distribution from which buyers sample when they contact a seller through word of mouth. In period 1, buyers do not contact any sellers through word of mouth. We denote as $F_t(s)$ the distribution of surplus offered by sellers. For the same reasons as in Section 2, $F_t(s)$ does not have any mass points. We denote as $\hat{F}_t(s)$ the distribution of surplus offered by sellers reached through word of mouth, which also cannot have any mass points.

Consider a seller of quality y that offers the surplus $s \geq 0$ in some period $t = 2, 3 \ldots$. The seller is reached by buyers through both search and word of mouth. Let us first consider the buyers that reach the seller through search. There is a measure $b_{k,t}$ of buyers that reach the seller through search and that contacted k additional sellers through search, where $b_{k,t}$ is given by

$$b_{k,t} = b(k+1)\frac{e^{-\mu}\mu^{k+1}}{(k+1)!}. \tag{3.1}$$

Let us explain the expression in (3.1). Per seller, there is a measure $b\exp(-\mu)\mu^{k+1}/(k+1)!$ of buyers that have $k+1$ contacts through search. Therefore, per seller, there is a measure $b(k+1)\exp(-\mu)\mu^{k+1}/(k+1)!$ of search contacts that are generated by buyers that have met $k+1$ sellers through search. Since every seller has the same probability of being contacted through search, the measure of buyers that reach a particular seller through search and that have met k additional sellers through the search channel is given by the expression in (3.1).

It is easy to compute the probability that one of the $b_{k,t}$ buyers purchases the good from the seller. A fraction $\exp(-\rho)$ of the $b_{k,t}$ buyers did not contact any sellers through word of mouth. These buyers purchase the good from the seller with probability $F_t(s)^k$, where $F_t(s)$ denotes the distribution of surplus offered by sellers. A fraction $\exp(-\rho)\rho$ of the $b_{k,t}$ buyers did contact one seller through word of mouth. These buyers purchase the good from the seller with probability $F_t(s)^k\hat{F}_t(s)$, where $\hat{F}_t(s)$ is the distribution of surplus offered by sellers contacted through word of mouth. More generally, a fraction $\exp(-\rho)\rho^j/j!$ of the $b_{k,t}$ buyers did contact j sellers through word of mouth, with $j = 2, 3, \ldots$. These buyers purchase the good from the seller with probability $F_t(s)^k\hat{F}_t(s)^j$. Overall, the probability that one of the $b_{k,t}$ buyers purchases the good from the seller is given by

$$\pi_{k,t}(s) = \sum\nolimits_{j=0}^{\infty} \frac{e^{-\rho}\rho^{j}}{j!} F_t(s)^k \hat{F}_t(s)^j. \tag{3.2}$$

Next, consider the buyers that reach the seller through word of mouth. There is a measure $\hat{b}_{k,t}$ buyers that reach the seller through word of mouth and that reached k other sellers through word of mouth, where $\hat{b}_{k,t}$ is given by

$$\hat{b}_{k,t} = b(k+1)\kappa_t(y)\frac{e^{-\rho}\rho^{k+1}}{(k+1)!} \tag{3.3}$$

and $\kappa(y)$ is given by

$$\kappa_t(y) = \frac{\hat{\Phi}_t'(y)}{\Phi'(y)}. \tag{3.4}$$

Let us explain the expression in (3.3). Per seller, there is a measure $b\exp(-\rho)\rho^{k+1}/(k+1)!$ of buyers that have $k+1$ contacts through word of mouth. Therefore, per seller, there is a measure $b(k+1)\exp(-\rho)\rho^{k+1}/(k+1)!$ of word-of-mouth contacts that are generated by buyers that are in contact with $k+1$ sellers through word of mouth. If every seller has the same probability of being contacted through word of mouth, the measure of buyers that reach a particular seller through word of mouth and that have contacted an additional k sellers through the word-of-mouth channel would be $b(k+1)\exp(-\rho)\rho^{k+1}/(k+1)!$. But some sellers are more likely than others to be contacted through word of mouth. In particular, the fraction of word-of-mouth contacts that reaches sellers with quality $\hat{y} \in [y, y+\epsilon]$ is $\hat{\Phi}(y+\epsilon) - \hat{\Phi}(y)$, while the measure of sellers with quality $\hat{y} \in [y, y+\epsilon]$ is $\Phi(y+\epsilon) - \Phi(y)$. Therefore, the measure of buyers that reach a seller of quality $\hat{y} \in [y, y+\epsilon]$ through word of mouth and that contacted an additional k sellers through word of mouth is $b(k+1)\exp(-\rho)\rho^{k+1}/(k+1)!$ multiplied by $[\hat{\Phi}(y+\epsilon) - \hat{\Phi}(y)]/[\Phi(y+\epsilon) - \Phi(y)]$. Taking the limit for $\epsilon \to 0$ yields (3.3).

It is easy to compute the probability that one of the $\hat{b}_{k,t}$ buyers purchases the good from the seller. A fraction $\exp(-\mu)$ of the $\hat{b}_{k,t}$ buyers did not contact any sellers through search. These buyers purchase the good from the seller with probability $\hat{F}_t(s)^k$. A fraction $\exp(-\mu)\mu$ of the $\hat{b}_{k,t}$ buyers did contact one seller through search. These buyers purchase the good from the seller with probability $\hat{F}_t(s)^k F_t(s)$. More generally, a fraction $\exp(-\mu)\mu^j/j!$ of the $\hat{b}_{k,t}$ buyers did contact j sellers through search. These buyers purchase the good from the seller with probability $\hat{F}_t(s)^k F_t(s)^j$. Overall, the probability that one of the $\hat{b}_{k,t}$ buyers purchases the good from the seller is given by

$$\hat{\pi}_{k,t}(s) = \sum\nolimits_{j=0}^{\infty} \frac{e^{-\mu}\mu^{j}}{j!} F_t(s)^j \hat{F}_t(s)^k \tag{3.5}$$

We can now write the profit $V_t(y, s)$ for a seller with quality y offering the surplus

$s \geq 0$ is given by

$$\begin{aligned} V_t(y,s) &= \left[\sum\nolimits_{k=0}^{\infty} b_{k,t}\pi_{k,t}(s)\right](y-s) + \left[\sum\nolimits_{k=0}^{\infty} \hat{b}_{k,t}\hat{\pi}_{k,t}(s)\right](y-s) \\ &= \left[\sum\nolimits_{k=0}^{\infty} b(k+1)\frac{e^{-\mu}\mu^{k+1}}{(k+1)!}\left(\sum\nolimits_{j=0}^{\infty} \frac{e^{-\rho}\rho^j}{j!}F_t(s)^k\hat{F}_t(s)^j\right)\right](y-s) \\ &+ \left[\sum\nolimits_{k=0}^{\infty} b(k+1)\kappa_t(y)\frac{e^{-\rho}\rho^{k+1}}{(k+1)!}\left(\sum\nolimits_{j=0}^{\infty} \frac{e^{-\mu}\mu^j}{j!}F_t(s)^j\hat{F}_t(s)^k\right)\right](y-s). \end{aligned} \tag{3.6}$$

Collecting terms, we can rewrite (3.6) as

$$\begin{aligned} V_t(y,s) &= \left[\sum\nolimits_{k=0}^{\infty} b\mu e^{-\mu}\frac{\mu^k F_t(s)^k}{k!}e^{-\rho(1-\hat{F}_t(s))}\left(\sum\nolimits_{j=0}^{\infty} \frac{e^{-\rho\hat{F}_t(s)}\rho^j\hat{F}_t(s)^j}{j!}\right)\right](y-s) \\ &+ \left[\sum\nolimits_{k=0}^{\infty} b\rho\kappa_t(y)e^{-\rho}\frac{\rho^k\hat{F}_t(s)^k}{k!}e^{-\mu(1-F_t(s))}\left(\sum\nolimits_{j=0}^{\infty} \frac{e^{-\mu F_t(s)}\mu^j F_t(s)^j}{j!}\right)\right](y-s) \\ &= \left[\sum\nolimits_{k=0}^{\infty} b\mu e^{-\mu}\frac{\mu^k F_t(s)^k}{k!}e^{-\rho(1-\hat{F}_t(s))}\right](y-s) \\ &+ \left[\sum\nolimits_{k=0}^{\infty} b\rho\kappa_t(y)e^{-\rho}\frac{\rho^k\hat{F}_t(s)^k}{k!}e^{-\mu(1-F_t(s))}\right](y-s), \end{aligned} \tag{3.7}$$

where the second equality makes use of the fact that the summations over j in lines 1 and 3 are both equal to 1.

Collecting terms once more, we can rewrite (3.7) as

$$\begin{aligned} V_t(y,s) &= b\mu e^{-\mu(1-F_t(s))}e^{-\rho(1-\hat{F}_t(s))}\left[\sum\nolimits_{k=0}^{\infty} e^{-\mu F_t(s)}\frac{\mu^k F_t(s)^k}{k!}\right](y-s) \\ &+ b\rho\kappa_t(y)e^{-\mu(1-F_t(s))}e^{-\rho(1-\hat{F}_t(s))}\left[\sum\nolimits_{k=0}^{\infty} e^{-\rho\hat{F}_t(s)}\frac{\rho^k\hat{F}_t(s)^k}{k!}\right](y-s) \\ &= b\left(\mu+\rho\kappa_t(y)\right)e^{-\mu(1-F_t(s))}e^{-\rho(1-\hat{F}_t(s))}(y-s). \end{aligned} \tag{3.8}$$

where the second equality makes use of the fact that the summations over k in the second and third lines are both equal to 1.

Using the same arguments as in Lemma 4, it is straightforward to show that the support of the surplus distribution $F_t(s)$ is some interval $[s_{\ell,t}, s_{h,t}]$, with $s_{\ell,t} = 0$. In turn, using the same arguments as in Lemma 5, it is easy to show that the surplus offered by a seller is a strictly increasing function $s_t(y)$ of the quality of the seller's variety y. From the strict monotonicity of $s_t(y)$, it follows that the fraction of sellers offering a surplus smaller than $s_t(y)$ is equal to the fraction of sellers with quality smaller than y, i.e.

$$F_t(s_t(y)) = \Phi(y). \tag{3.9}$$

Similarly, in the distribution of sellers reached through word of mouth, the fraction offering a surplus smaller than $s_t(y)$ is equal to the fraction of sellers with quality smaller than y, i.e.

$$\hat{F}_t(s_t(y)) = \hat{\Phi}_t(y) \tag{3.10}$$

Differentiating both (3.9) and (3.10) with respect to y yields

$$F_t'(s_t(y)) = \frac{\Phi'(y)}{s_t'(y)}, \quad \hat{F}_t'(s_t(y)) = \frac{\hat{\Phi}'(y)}{s_t'(y)}. \tag{3.11}$$

The necessary condition for the optimality of the surplus $s_t(y)$ offered by a seller with quality y is

$$\begin{aligned} & \left[\mu F_t'(s_t(y)) + \rho \hat{F}_t'(s_t(y))\right] b\left(\mu + \rho\kappa_t(y)\right) e^{-\mu(1-F_t(s))} e^{-\rho(1-\hat{F}_t(s))} (y - s_t(y)) \\ = \quad & b\left(\mu + \rho\kappa_t(y)\right) e^{-\mu(1-F_t(s))} e^{-\rho(1-\hat{F}_t(s))} \end{aligned} \tag{3.12}$$

The left-hand side of (3.12) is the benefit to the seller from offering an extra unit of surplus to the buyers, which is given by the increase in the seller's volume multiplied by the seller's profit per trade. The right-hand side of (3.12) is the cost to the seller from offering an extra unit of surplus, which is given by the seller's volume. Using (3.9), (3.10) and (3.11), we can rewrite (**??**) as

$$s_t'(y) = \Phi'(y) \left[\mu + \rho\kappa_t(y)\right] (y - s_t(y)). \tag{3.13}$$

We can now compute the distribution of transactions in period t. Since buyers purchase from the contacted seller that offers the highest surplus and sellers with higher quality offer higher surplus, a buyer purchases the good from the contacted seller that carries the variety of the good with highest quality. The fraction of buyers that purchase the good from a seller with quality non-greater than y is

$$\begin{aligned} & H_t(y) \\ = \quad & \left\{ b e^{-\mu} \sum\nolimits_{j=1}^{\infty} \frac{e^{-\rho}\rho^j}{j!} \hat{\Phi}_t(y)^j \right\} \Big/ \left\{ b\left[1 - e^{-(\mu+\rho)}\right] \right\} \\ + \quad & \left\{ \sum\nolimits_{k=1}^{\infty} b \frac{e^{-\mu}\mu^k}{k!} \left[\sum\nolimits_{j=0}^{\infty} \frac{e^{-\rho}\rho^j}{j!} \Phi(y)^k \hat{\Phi}_t(y)^j \right] \right\} \Big/ \left\{ b\left[1 - e^{-(\mu+\rho)}\right] \right\} \end{aligned} \tag{3.14}$$

Let us explain (3.14). The first term at the numerator is the measure of buyers that do not reach any seller through search and reach j sellers through word of mouth, with $j = 1, 2, 3 \ldots$. Each one of these buyers purchases the good from a seller with quality non-greater than y with probability $\hat{\Phi}_t(y)^j$. The second term at the numerator is the measure of buyers that reach k sellers through search, with $k = 1, 2, \ldots.$, and reach j sellers through word of mouth, with $j = 0, 1, 2, \ldots$ Each one of these buyers purchases the good from a seller with quality non-greater than y with probability $\Phi(y)^k \hat{\Phi}_t(y)^j$. Overall, the numerator is the measure of buyers that purchase the good from a seller with quality non-greater than y. The denominator is the measure of buyers who purchase the good, which is equal to the measure of buyers multiplied by the probability that a buyer contacts at least one seller either through search or word of mouth.

We can rewrite the first term at the numerator of (3.14) as

$$\begin{aligned} & be^{-\mu}\sum\nolimits_{j=1}^{\infty}\frac{e^{-\rho}\rho^{j}}{j!}\hat{\Phi}_t(y)^j \\ = \quad & be^{-\mu}\left[e^{-\rho(1-\hat{\Phi}_t(y))}\sum\nolimits_{j=0}^{\infty}\frac{e^{-\rho\hat{\Phi}_t(y)}\rho^j}{j!}\hat{\Phi}_t(y)^j - e^{-\rho}\right] \\ = \quad & be^{-\mu}\left[e^{-\rho(1-\hat{\Phi}_t(y))} - e^{-\rho}\right]. \end{aligned} \tag{3.15}$$

The first equality in (3.15) is obtained by collecting terms. The second equality in (3.15) is obtained by recognizing that the summation over j is equal to 1.

We can rewrite the second term at the numerator of (3.14) as

$$\begin{aligned} & \sum\nolimits_{k=1}^{\infty} b\frac{e^{-\mu}\mu^k}{k!}\left[\sum\nolimits_{j=0}^{\infty}\frac{e^{-\rho}\rho^j}{j!}\Phi(y)^k\hat{\Phi}_t(y)^j\right] \\ = \quad & \sum\nolimits_{k=1}^{\infty} b\frac{e^{-\mu}\mu^k\Phi(y)^k}{k!}e^{-\rho(1-\hat{\Phi}_t(y))}\left[\sum\nolimits_{j=0}^{\infty}\frac{e^{-\rho\hat{\Phi}_t(y)}\rho^j\hat{\Phi}_t(y)^j}{j!}\right] \\ = \quad & be^{-\rho(1-\hat{\Phi}_t(y))}\left[e^{-\mu(1-\Phi(y))}\sum\nolimits_{k=0}^{\infty}\frac{e^{-\mu\Phi(y)}\mu^k\Phi(y)^k}{k!} - e^{-\mu}\right] \\ = \quad & be^{-\rho(1-\hat{\Phi}_t(y))}\left[e^{-\mu(1-\Phi(y))} - e^{-\mu}\right]. \end{aligned} \tag{3.16}$$

The first equality in (3.16) is obtained by collecting terms. The second equality in (3.16) is obtained by recognizing that the summation over j is equal to 1. The last equality in (3.16) is obtained by recognizing that the summation over k is equal to 1.

Replacing the expressions in (3.15) and (3.16) with the first and the seocnd term at the numerator of (3.14) yields the following expression for the transaction distribution

$$H_t(y) = \frac{e^{-\mu(1-\Phi(y))}e^{-\rho(1-\hat{\Phi}_t(y))} - e^{-(\mu+\rho)}}{1-e^{-(\mu+\rho)}}. \tag{3.17}$$

The analysis above leaves us with a unique candidate equilibrium in any period $t \geq 2$. In the candidate equilibrium, the surplus function $s_t(y)$ is given by the solution to the differential equation

$$s_t'(y) = \left[\mu + \rho\kappa_t(y)\right]\Phi'(y)(y - s_t(y)), \tag{3.18}$$

together with the boundary condition $s_t(y_\ell) = 0$. In the candidate equilibrium, the transaction distribution $H_t(y)$ is given by

$$H_t(y) = \frac{e^{-\mu(1-\Phi(y))}e^{-\rho(1-\hat{\Phi}_t(y))} - e^{-(\mu+\rho)}}{1-e^{-(\mu+\rho)}}. \tag{3.19}$$

Since the transaction distribution is equal to the distribution of the highest quality seller contacted by a buyer, the sampling distribution $\hat{\Phi}_{t+1}(y)$ for buyers that reach a seller through word of mouth in period $t+1$ is $H_t(y)$. As we did in Section 2, it is easy to verify that the unique candidate equilibrium is, indeed, an equilibrium since (3.18) is not only necessary but also sufficient for profit maximization.

The unique candidate equilibrium in period 1 is similar, except that the word-of-mouth channel is not active. In the candidate equilibrium, the surplus function $s_1(y)$ is the solution to the differential equation

$$s_1'(y) = \mu\Phi'(y)(y - s_1(y)), \tag{3.20}$$

together with the boundary condition $s_1(y_\ell) = 0$. In the candidate equilibrium, the transaction distribution $H_1(y)$ is

$$H_1(y) = \frac{e^{-\mu(1-\Phi(y))} - e^{-\mu}}{1 - e^{-\mu}}. \tag{3.21}$$

Since the transaction distribution is equal to the distribution of the highest quality seller contacted by a buyer, the sampling distribution $\hat{\Phi}_2(y)$ for buyers that reach a seller through word of mouth in period 2 is $H_1(y)$. Again, it is easy to verify that (3.20) is not only necessary for profit maximization but also sufficient and, hence, the candidate equilibrium is an equilibrium.

Theorem 2: (Existence and uniqueness of equilibrium with word of mouth) *An equilibrium exists and is unique. The equilibrium is such that:*

(i) *In period 1, the surplus function $s_1(y)$ is the solution to the differential equation (3.20) that satisfies the boundary condition $s_1(y_\ell) = 0$, and the transaction distribution $H_1(y)$ is given by (3.21).*

(ii) *In period $t \geq 2$, the sampling distribution $\hat{\Phi}_t(y)$ for buyers that reach a seller through word of mouth is given by $H_{t-1}(y)$. The surplus function $s_t(y)$ is the solution to the differential equation (3.18) that satisfies the boundary condition $s_t(y_\ell) = 0$, and the transaction distribution $H_t(y)$ is given by (3.19).*

3.3 Concentration and competition dynamics

The dynamics of the market are determined by the dynamics of the transaction distribution $H_t(y)$, which affects the distribution from which buyers sample when they contact a seller through word of mouth. Let $x(y)$ denote the fraction $\Phi(y)$ of sellers with quality smaller than y, and let $y(x)$ denote the inverse of $x(y)$. Let $z_t(y)$ denote the fraction $H_t(y)$ of transactions made by sellers with quality smaller than y in period t. From (3.21), it follows that $z_1(y)$ is given by

$$z_1(y) = \frac{e^{-\mu(1-x(y))} - e^{-\mu}}{1 - e^{-\mu}} \tag{3.22}$$

For $t = 1, 2, ...$, it follows from (3.19) that $z_{t+1}(y)$ is given by

$$z_{t+1}(y) = \frac{e^{-\mu(1-x(y))}e^{-\rho(1-z_t(y))} - e^{-(\mu+\rho)}}{1 - e^{-(\mu+\rho)}}. \tag{3.23}$$

The fraction $z_1(x)$ of transactions made by sellers with quality non-greater than $y(x)$ is the solution with respect to z to the equation

$$f(z, x) \equiv z\left(1 - e^{-\mu}\right) - e^{-\mu(1-x)} - e^{-\mu} = 0. \tag{3.24}$$

Consider any $x \in (0, 1)$. The function $f(z, x)$ is strictly increasing in z, and such that $f(0, x) < 0$ and $f(1, x) > 0$. Therefore, there exists one and only one $z_1(x)$ that solves the equation $f(z_1(x), x) = 0$ and $z_1(x) \in (0, 1)$. The function $g(x) \equiv f(x, x)$ is strictly concave and such that $g(0) = 0$ and $g(1) = 0$. Therefore, $g(x) = f(x, x) > 0$ for all $x \in (0, 1)$. Since $f(x, x) > 0$, $f(z_1(x), x) = 0$, and $f(z, x)$ is strictly increasing in x, it follows that $z_1(x) < x$. For $x = 0$, the unique solution to $f(z, 0) = 0$ is $z_1(0) = 0$. For $x = 1$, the unique solution to $f(z, 1) = 0$ is $z_1(1) = 1$.

The fraction $z_{t+1}(x)$ of transactions made by sellers with quality non-greater than $y(x)$ is given by

$$f_+(z, x) = \frac{e^{-\mu(1-x)}e^{-\rho(1-z)} - e^{-(\mu+\rho)}}{1 - e^{-(\mu+\rho)}}, \tag{3.25}$$

where z denotes the fraction of transactions made by sellers with quality non-greater than $y(x)$ in period t. Consider any $x \in (0, 1)$. The function $f_+(z, x)$ is strictly increasing and strictly convex in z. The function $f_+(z, x)$ is such that $f_+(0, x) > 0$ and $f_+(1, x) < 1$. Therefore, there exists one and only one $z^*(x)$ that solves the equation $f_+(z, x) = z$ with respect to z. Moreover, $f_+(z, x) \in (z, z^*(x))$ for every $z \in [0, z^*(x))$, and $f_+(z, x) \in (z^*(x), z)$ for every $z \in (z^*(x), 1]$. For $x = 0$, the unique solution to $f_+(z, x) = z$ is $z^*(0) = 0$. For $x = 1$, the relevant solution to $f_+(z, x) = z$ is $z^*(1) = 1$

It is easy to see that $f_+(x, x) < x$ for any $x \in (0, 1)$. Since $f_+(z, x) > z$ for every $z \in [0, z^*(x))$, and $f_+(z, x) < z$ for every $z \in (z^*(x), 1]$, $f_+(x, x) < x$ implies $z^*(x) < x$. In turn, $z^*(x) < x$ implies

$$\begin{aligned} z^*(x) &= \frac{e^{-\mu(1-x)}e^{-\rho(1-z^*(x))} - e^{-(\mu+\rho)}}{1 - e^{-(\mu+\rho)}} \\ &< \frac{e^{-\mu(1-x)}e^{-\rho(1-x)} - e^{-(\mu+\rho)}}{1 - e^{-(\mu+\rho)}} \\ &< \frac{e^{-\mu(1-x)} - e^{-\mu}}{1 - e^{-\mu}} = z_1(x). \end{aligned} \tag{3.26}$$

We are now in the position to describe the dynamics of the transaction distribution $H_t(y)$. Consider any $y \in (y_\ell, y_h)$. The fraction of sellers with quality lower than y is $x(y) \in (0,1)$. In period 1, the fraction of transactions $z_1(x(y))$ at sellers with quality lower than y is given by $f(z_1(x(y)), x(y)) = 0$. From the analysis of $f(z,x)$, it follows that $z_1(x(y))$ is strictly smaller than $x(y)$. From (3.26), it follows that $z_1(x(y))$ is strictly greater than $z^*(x(y))$. In period 2, the fraction of transactions $z_2(x(y))$ at sellers with quality lower than y is given by $f_+(z_1(x(y)), x(y))$ where $z_1(x(y)) > z^*(x(y))$. From the analysis of $f_+(z,x)$, it follows that $z_2(x(y))$ is strictly smaller than $z_1(x(y))$ and strictly greater than $z^*(x(y))$, since $z_1(x(y)) \in (z^*(x(y)), 1]$. Iterating the argument, we obtain that, in any period t, the fraction of transactions $z_t(x(y))$ is strictly smaller than $z_{t-1}(x(y))$ and strictly greater than $z^*(x(y))$. In every period $t = 1, 2, \ldots$, the fraction of transactions $z_t(x(y_\ell))$ at sellers with quality less than y_ℓ is equal to 0. In every period $t = 1, 2, \ldots$, the fraction of transactions $z_t(x(y_h))$ at sellers with quality less than y_h is equal to 1. Overall, the fraction of transactions taking place at sellers with quality less than $y \in (y_\ell, y_h)$ is strictly decreasing over time, which means that the transaction distribution $H_t(y)$ is strictly decreasing in t.

We have established the following proposition.

Proposition 7: (Transaction dynamics) *For any $\rho > 0$, the distribution of transactions across sellers of different quality, $H_t(y)$, decreases over time.*

Notice that the transaction distribution improves over time in the model with word of mouth just as it does in the model with memory. The reason why the transaction distribution improves over time is, however, different in the model with word of mouth and in the model with memory. In the model with memory, the transaction distribution improves over time because the buyers' choice sets become larger and larger and, for this reason, buyers can purchase the good from better and better sellers. In the model with word of mouth search, the transaction distribution improves over time not because the buyers' choice sets become larger and larger, but because the buyers' choice sets include better and better sellers. In period t, buyers sample, through word of mouth, the highest-quality sellers sampled by buyers in period $t-1$ and they sample, through direct search, some additional sellers. For this reason, the highest-quality sellers sampled by buyers in period t are better than the highest-quality sellers sampled by buyers in period $t-1$.

Next, we examine the dynamics of trade concentration. In period 1, the quantity of the good traded by a seller with a variety of quality y is given by

$$q_1(y) = b\mu e^{-\mu(1-\Phi(y))}. \tag{3.27}$$

In period $t \geq 2$, the quantity of the good traded by a seller with a variety of quality y is

$$q_t(y) = b\left(\mu + \rho\kappa_t(y)\right) e^{-\mu(1-\Phi(y))} e^{-\rho(1-\hat{\Phi}_t(y))}. \tag{3.28}$$

Clearly, the quantity $q_1(y)$ is strictly increasing in y. The quantity $q_t(y)$ is also strictly increasing in y because the term $\kappa_t(y)$ is strictly increasing in y. To see why this is the case, notice that (3.21) implies

$$\kappa_2(y) \equiv \frac{\hat{\Phi}_2(y)}{\Phi(y)} = \frac{H_1'(y)}{\Phi(y)} = \frac{\mu e^{-\mu(1-\Phi(y))}}{1-e^{-\mu}}. \tag{3.29}$$

For $t \geq 2$, (3.19) implies

$$\kappa_{t+1}(y) \equiv \frac{\hat{\Phi}_{t+1}(y)}{\Phi(y)} = \frac{H_t'(y)}{\Phi(y)} = \frac{(\mu + \rho\kappa_t(y))e^{-\mu(1-\Phi(y))}e^{-\rho(1-H_t(y))}}{1-e^{-(\mu+\rho)}}. \tag{3.30}$$

It is evident from (3.29) that $\kappa_2(y)$ is strictly increasing in y. Since $k_2(y)$ is strictly increasing in y, it is clear from (3.30) that $\kappa_3(y)$ is strictly increasing in y. Repeating the argument establishes that $\kappa_{t+1}(y)$ is strictly increasing in y for any t.

Since the quantity of the good traded by a seller is a strictly increasing function $q_t(y)$ of the seller's quality y, it follows that the largest sellers are the sellers with the highest quality. Hence, the fraction of $Q_t(x)$ of transactions made by the x fraction of largest sellers is given by

$$Q_t(x) = 1 - H_t(y(1-x)). \tag{3.31}$$

Since $H_t(y)$ is strictly deceasing in t, (3.31) implies that $Q_t(x)$ is strictly increasing in t.

Proposition 8: (Concentration dynamics) *For any $\rho > 0$, the fraction $Q_t(x)$ of sales made by the x fraction of largest sellers is strictly increasing over time.*

As in the model with memory, trade becomes more and more concentrated over time. In the model with memory, trade becomes more concentrated over time because the choice sets of buyers become larger and, hence, buyers can purchase the good from higher quality sellers, which are the sellers that offer more surplus and that are larger. In the model with word of mouth, trade becomes more concentrated over time because the sellers in the choice sets of buyers have higher quality, and higher quality sellers are those that offer more surplus and that are larger.

We now examine the dynamics of the surplus offered by sellers. In period 1, the surplus function $s_1(y)$ is the solution to the differential equation

$$s_1'(y) = \Phi'(y)\mu\,(y - s_1(y)) \tag{3.32}$$

that satisfies the boundary condition $s_1(y_\ell) = 0$. In period $t \geq 2$, the surplus function $s_t(y)$ is the solution to the differential equation

$$s_t'(y) = \Phi'(y)\,(\mu + \rho\kappa_t(y))\,(y - s_t(y)) \tag{3.33}$$

that satisfies the boundary conditions $s_t(y_\ell) = 0$. Since $\kappa_t(y)$ is strictly positive, the surplus function $s_t(y)$ is strictly greater than the surplus function $s_1(y)$ for every $y \in (y_\ell, y_h]$. Since the integral of $k_\tau(y) \equiv H_{\tau-1}'(y)/\Phi'(y)$ with respect to the seller's distribution $\Phi(y)$ is equal to 1 for all $\tau = 1, 2, ...$, $\kappa_{t+1}(y)$ cannot be greater than $\kappa_t(y)$ for all y. For this reason, the surplus function $s_{t+1}(y)$ need not be strictly greater than the surplus function $s_t(y)$. Indeed, it is easy to find examples in which the surplus function evolves in a non-monotonic fashion.

Proposition 9 (Surplus and price dynamics). *For any $\rho > 0$:*

(i) *The surplus offered by sellers is strictly higher in any period $t \geq 2$ than in period 1, in the sense that $F_t(s) < F_1(s)$. The prices posted by sellers are strictly lower in any period $t \geq 2$ than in period 1, in the sense that $G_t(p) > G_1(p)$.*

(ii) *The surplus offered by sellers need not be increasing over time, in the sense that $F_t(s)$ need not be strictly decreasing in t. The prices posted by sellers need not be decreasing over time, in the sense that $G_t(p)$ need not be strictly increasing in t.*

The above proposition highlights a critical difference between the model with memory and the model with word of mouth. In the model with memory, the choice set of buyers expands over time. The expansion of the buyers' choice sets increases the extent of competition and, in turn, leads to a decline in prices. Moreover, the expansion of the buyers' choice sets allows buyers to purchase from higher quality sellers and, in turn, leads to higher concentration. In the model with word of mouth, the choice set of buyers does not expand but it does improve, in the sense that it replaces lower-quality sellers with higher-quality sellers. The improvement in the buyers' choice sets directly leads to an increase in concentration. The improvement in the buyers' choice set, however, does not directly increase competition between sellers and, for this reason, it does not necessarily lead to a decline in prices.

It is useful to dive deeper in the comparison between the effect of memory and the effect of word of mouth on concentration and competition. Consider the model with memory. In equilibrium, the fraction x of the largest sellers trades a fraction $Q_t(x)$ of the good, where $Q_t(x)$ given by

$$Q_t(x) = 1 - \frac{e^{-\lambda_t x} - e^{-\lambda_t}}{1 - e^{-\lambda_t}}. \tag{3.34}$$

For x small and λ_t large, $Q_t(x)$ is approximately equal to

$$\begin{aligned} Q_t(x) &\simeq Q_t(0) - Q_t'(0)(0 - x) \\ &= (\lambda_t x) / (1 - e^{-\lambda_t}) \\ &\simeq \lambda_t x, \end{aligned} \tag{3.35}$$

The first line in (3.35) is a first-order approximation of $Q_t(x)$ around 0. The second line in (3.35) is obtained by noting that $Q_t(0) = 0$ and by computing the derivative of $Q_t(x)$ with respect to x and evaluating it at $x = 0$. The last line in (3.35) makes use of the fact that $\exp(-\lambda_t)$ is approximately equal to 0 for λ_t large.

Now, consider the model with word of mouth. In equilibrium, the fraction x of the largest sellers trades a fraction $Q_t(x)$ of the good, where $Q_t(x)$ is given by

$$Q_t(x) = 1 - \frac{e^{-\mu x} e^{-\rho(1-H_{t-1}(y(1-x)))} - e^{-(\mu+\rho)}}{1 - e^{-(\mu+\rho)}}. \tag{3.36}$$

For x small and $\mu + \rho$ large, $Q_t(x)$ is approximately equal to

$$\begin{aligned} Q_t(x) &\simeq Q_t(0) - Q_t'(0)(0 - x) \\ &= (\mu + \rho\kappa_t(y_h))\, x/(1 - e^{-(\mu+\rho)}) \\ &\simeq (\mu + \rho\kappa_t(y_h))x, \end{aligned} \tag{3.37}$$

The first line in (3.37) is a first-order approximation of $Q_t(x)$ around 0. The second line in (3.37) is obtained by noticing that $Q_t(0) = 0$ and by computing the derivative of $Q_t(x)$ and evaluating it at $x = 0$. The last line in (3.37) makes use of the fact that $\exp(-(\mu+\rho))$ is approximately equal to 0 for $\mu + \rho$ large.

Imagine a situation in which the concentration of trade at the x largest sellers is the same in the equilibrium of the model with memory as in the equilibrium of the model with word of mouth, for $x \simeq 0$ and λ_t and $\mu + \rho$ large. From (3.35) and (3.37), it follows that the concentration of sales at the largest sellers is the same in the two models if and only if

$$\lambda_t = \mu + \rho\kappa_t(y_h). \tag{3.38}$$

Even though the concentration of sales is identical in the equilibrium of the two models, the equilibrium distribution of surplus and prices is not. Indeed, in the equilibrium of the model with memory, the surplus function is given by the solution to the differential equation

$$\begin{aligned} s_t'(y) &= \Phi'(y)\lambda_t(y - s_t(y)) \\ &= \Phi'(y)\,(\mu + \rho\kappa_t(y_h))\,(y - s_t(y)) \end{aligned} \tag{3.39}$$

with the boundary condition $s_t(y_\ell) = 0$. In the equilibrium of the model with word of mouth, the surplus function is given by the solution to the differential equation

$$s_t'(y) = \Phi'(y)\,(\mu + \rho\kappa_t(y))\,(y - s_t(y)) \tag{3.40}$$

with the boundary condition $s_t(y_\ell) = 0$. Since $\kappa_t(y)$ is strictly increasing in y, it follows from (3.39) and (3.40) that the surplus function $s_t(y)$ is strictly greater in the model with memory than in the model with word of mouth. In turn, this implies that the surplus offered by sellers is strictly greater in the model with memory than in the model with word of mouth. That is, $F_t(s)$ is strictly smaller in the model with memory than in the model with word of mouth. Similarly, the prices posted by sellers are strictly lower in the model with memory than in the model with word of mouth. That is, $G_t(p)$ is strictly greater in the model with memory than in the model with word of mouth.

The findings above reveal that, when word of mouth and memory generate the same degree of market concentration, word of mouth generates less competition than memory. The findings are intuitive. Concentration depends on the distribution of the best seller in the buyers' choice set. Competition depends not only on the best seller in the buyers' choice set, but also on the second-best seller. Memory expands choice sets and, in doing so, it improves the best and the second-best seller in the buyers' choice sets. Word of mouth does not expand the buyers' choice sets. Memory tilts the content of part of the buyers' choice sets towards better sellers. For this reason, word of mouth improves the best seller in the buyers' choice sets relatively more than the second-best seller.

Lastly, we want to compare the equilibrium of the model with word of mouth ($\rho > 0$) with the equilibrium of a model without word of mouth ($\rho = 0$). In the model without word of mouth, buyers always come into contact with m_t sellers through search, where m_t is distributed as a Poisson with coefficient μ. In the model with word of mouth, buyers in period 1 come into contact with m_1 sellers through search, where m_1 is distributed as a Poisson with coefficient μ. In period $t \geq 2$, buyers come into contact with m_t sellers though search, where m_t is distributed as a Poisson with coefficient μ, and with r_t sellers through word of mouth, where r_t is distributed as a Poisson with coefficient ρ. Therefore, the equilibrium of the model without word of mouth in every period t is equal to the equilibrium of the model with word of mouth in period 1. In turn, this implies that the comparison between the equilibrium of the model with and without word of mouth follows directly from the analysis of the dynamics of the equilibrium of the model with word of mouth.

Proposition 10: (Word of mouth). *Let* $\rho_1 = 0$ *and* $\rho_2 > 0$.

(i) *In period* 1*, equilibrium surplus, prices and transactions are the same for* ρ_1 *and* ρ_2*, in the sense that* $F_1(s|\rho_1) = F_1(s|\rho_2)$*,* $G_1(p|\rho_1) = G_1(p|\rho_2)$*, and* $H_1(y|\rho_1) = H_1(y|\rho_2)$*. In period* $t \geq 2$*, equilibrium surplus is lower, prices are higher, and transactions are worse for* ρ_1 *than for* ρ_2*, in the sense that* $F_t(s|\rho_1) > F_t(s|\rho_1)$*,* $G_t(p|\rho_1) < G_t(p|\rho_2)$ *and* $H_t(y|\rho_1) > H_t(y|\rho_2)$*.*

(ii) *For* ρ_1*, equilibrium surplus, prices and transactions are constant over time, in the sense that* $F_t(s|\rho_1) = F_1(s|\rho_1)$*,* $G_t(p|\rho_1) = G_1(p|\rho_1)$ *and* $H_t(y|\rho_1) = H_1(y|\rho_1)$*.*

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